update: *23/02/2017*

September 9-11, ICSFA-2016
XVth Annual Conference of Society for Special Functions and their Applications
Department of Applied Sciences and Humanities
Faculty of Engineering and Technology,
Jamia Millia Islamia (Central University), New Delhi.
http://www.ssfaindia.webs.com/conf.htm

Irrationality and transcendence of values of special functions[1]

Michel Waldschmidt

Abstract

Given a complex function f and an algebraic point α where f is defined, we investigate the algebraic nature of $f(\alpha)$: is–it a rational number, an irrational algebraic number, or else a transcendental number? No general result exists, one needs to restrict the study to special functions f. We first consider the E–functions and then the G–functions introduced by C.L. Siegel in 1929. Next, we consider meromorphic functions satisfying differential equations as in the so–called Schneider–Lang criterion. Finally, we deal with analytic functions satisfying functional equations, where a powerful method has been introduced by K. Mahler.

1 Introduction: Emil Strauss question

After the proof by Hermite of the transcendence of the number e, the proof by Lindemann of the transcendence of the number π, and the Theorem of Hermite Lindemann stating that the only algebraic value α where the exponential function takes an algebraic value e^α is $\alpha = 0$, E. Strauss (1886) tried to prove that a transcendental function which is analytic in an open

[1]September 9, 2016, R. P. Agarwal Memorial Lecture, ICSFA-2016

domain D of $\mathbb{C}$ containing 0 cannot take rational values at all rational points of D. According to P. Stäckel [51], K. Weierstrass sent him a letter where he supplied him with a counterexample (see [38, §35]).

As pointed out in [57, Chap. 6], Hilbert's seventh problem include the following comment:

> *we expect transcendental functions to assume, in general, transcendental values for [...] algebraic arguments [...] we shall still consider it highly probable that the exponential function e^{iz}, [...] will [...] always take transcendental values for irrational algebraic values of the argument z.*

For a transcendental function f, denote by E_f the set of algebraic numbers α such that $f(\alpha)$ is algebraic.

Thanks to the work by A. Hurwitz (1891), P. Stäckel (1895), G. Faber (1904), C.G. Lekkerkerker (1949), A.O. Gel'fond (1965), K. Mahler (1965), it is known that for each countable subset A of $\mathbb{C}$ and each family of dense subsets $E_{\alpha,s}$ of $\mathbb{C}$ indexed by $(\alpha, s) \in A \times \mathbb{N}$, there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$ for each $(\alpha, s) \in A \times \mathbb{N}$. See [38, §35] and [25]. In 1968, A.J. van der Poorten showed that there exist entire functions mapping each number field into itself. The paper [53] by A. Surroca is related with some work by J. Pila, which was the origin of the introduction of tools from logic (o–minimality) which yielded a breakthrough towards main conjectures in Diophantine questions (Manin–Mumford and André–Oort conjectures, unlikely intersections).

In [29], it is proved that for each countable subset A of $\mathbb{C}$ and each family of dense subsets $E_{\alpha,s}$ of $\mathbb{C}$ indexed by $(\alpha, s) \in A \times \mathbb{N}$, there exists a transcendental entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f^{(s)}(\alpha) \in E_{\alpha,s}$ for each $(\alpha, s) \in A \times \mathbb{N}$.

Therefore we cannot expect general results on the arithmetic properties of the values of analytic functions, unless we restrict to special functions. Among many references on special functions, let us quote [30, 16].

2 E–functions

We first consider the E–functions and then the G–functions introduced by C.L. Siegel in 1929 [49], which includes Bessel's functions and hypergeometric functions. Siegel came back to this subject in 1949 [50].

update: *23/02/2017*

Consider a power series

$$f(z) = \sum_{n \geq 0} \frac{a_n}{n!} z^n \in \mathbb{Q}[[z]]$$

such that
• a_n increases at most exponentially in n (hence f is an entire function)
• f satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$
• The common denominator of $a_0, a_1, \ldots, a_n$ increases at most exponentially in n.

This is a typical example of a E–function of Siegel. The general definition replaces the rational numbers a_n by algebraic numbers. Examples of E–functions are algebraic constants, polynomials with algebraic coefficients, the exponential function e^z, the trigonometric functions $\cos z$ and $\sin z$. Our next example is Bessel's function of index 0:

$$J_0(z) = \sum_{n \geq 0} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} = 1 - \left(\frac{z}{2}\right)^2 + \frac{1}{4}\left(\frac{z}{2}\right)^4 - \frac{1}{36}\left(\frac{z}{2}\right)^6 + \cdots$$

which is a solution of the Bessel differential equation

$$y'' + \frac{1}{z} y' + y = 0.$$

More generally, for $\lambda \in \mathbb{C}$, if we define

$$K_\lambda(z) = \sum_{n \geq 0} \frac{(-1)^n}{(\lambda+1)_n n!} \left(\frac{z}{2}\right)^{2n},$$

then the function

$$J_\lambda(z) = \sum_{n \geq 1} \frac{(-1)^n (z/2)^{2n+\lambda}}{n! \Gamma(n+1+\lambda)} = \frac{1}{\Gamma(\lambda+1)} \left(\frac{z}{2}\right)^\lambda K_\lambda(z),$$

is a solution of the differential equation

$$z^2 y'' + z y' + (z^2 - \lambda^2) y = 0,$$

and $J_{-\lambda}(z)$ is a solution of the same differential equation. The modified Bessel functions of the first kind are

$$I_\lambda(z) = \sum_{n \geq 1} \frac{(z/2)^{2n+\lambda}}{n! \Gamma(n+1+\lambda)} = i^{-\lambda} J_\lambda(iz).$$

See [30, Chap. 6]. Among the results proved by Siegel in 1929 is the transcendence of the so–called *Continued Fraction Constant*

$$\frac{I_1(2)}{I_0(2)} = [0; 1, 2, 3, \dots] = 0.697774658\dots$$

Other similar continued fractions, like

$$I_{1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e + e^{-1}}{2}, \quad I_{1/2}(1) = \frac{\sqrt{2}}{\pi} \cdot \frac{e - e^{-1}}{2},$$

$$[1; 3, 5, 7 \dots] = \frac{e^2 + 1}{e^2 - 1} = \frac{I_{-1/2}(1)}{I_{1/2}(1)}, \quad [2; 6, 10, 14 \dots] = \frac{e + 1}{e - 1} = \frac{I_{-1/2}(1/2)}{I_{1/2}(1/2)},$$

are considered in [54].

An interesting connection between Bessel's functions and Ramanujan's work is quoted by H. Cohen [18].

Further examples of E–functions are Siegel hypergeometric E–functions. Let $a_1, \dots, a_\ell$, $b_1, \dots, b_m$ be rational numbers with $m > \ell$ and $b_1, \dots, b_m$ not in $\{0, -1, -2, \dots\}$ and $b_m = 1$. Define

$$c_n = \frac{(a_1)_n \cdots (a_\ell)_n}{(b_1)_n \cdots (b_m)_n}$$

where $(x)_n$ denotes Pochhammer symbol: (rising factorial power)

$$(x)_n = x(x+1)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}\cdot$$

Set $t = m - \ell$. Then

$$f(z) = \sum_{n \geq 1} c_n z^{tn}$$

is an E–function.

For an introduciton to hypergeometric functions, see [10, 11].

After the work by Siegel, and later by Shidlovski and his school, the theory of E–functions has now reached a satisfactory state: to prove the algebraic independence of values, it suffices to prove the algebraic independence of the functions. Among many references on this topic [23, Chap. 2 § 4], [46, Chap. V], [34, Chap. VII], [38], [5, § 11], [48], [19, Chap. 5], [7, § 1.4, Th. 1.10], [56, Chap. 5].

Here is an example of a result achieved by this theory. *Let K be a number field, $E_1, E_2, \dots, E_n$ be E–functions which are algebraic independent over*

$K(z)$ and satisfy a system of linear differential equations

$$y_i' = \sum_{j=1}^{n} f_{ij}(x) y_j \quad (i = 1, \dots, n)$$

with $f_{ij} \in K[z]$ and α be a non-zero algebraic number in K not pole of the E_i. Then $E_1(\alpha), E_2(\alpha), \dots, E_n(\alpha)$ are algebraically independent.

3 G–functions

Consider a power series

$$g(z) = \sum_{n \geq 0} a_n z^n \in \mathbb{Q}[[z]]$$

such that
• g has a positive radius of convergence
• g satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$
• The common denominator of $a_0, a_1, \dots, a_n$ increases at most exponentially in n.

This is a typical example of a G–function of Siegel. The general definition replaces the rational numbers a_n by algebraic numbers. References on G–functions are [49], [19, Chap. 5, § 7], [2, 3]. The recent manuscript **??** by S. Fischler and T. Rivoal includes a survey on Diophantine properties of values of G–functions together with new results.

From the definitions, it follows that $\sum_{n \geq 0} a_n z^n$ is a G–function if and only if $\sum_{n \geq 0} (a_n/n!) z^n$ is an E–function.

Examples of G–functions are algebraic functions, Gauss hypergeometric functions ${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| z\right)$ with rational parameters a, b, c [11], solutions of Picard–Fuchs equations over $\mathbb{Q}(z)$. The hypergeometric function

$${}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix} \middle| z\right) = \sum_{n \geq 0} \frac{(a)_n (b)_n}{(c)_n n!} z^n$$

satisfies the second order linear differential equation

$$z(z-1)y'' + \big((a+b+1)z - c\big)y' + aby = 0.$$

Special cases of hypergeometric functions are

$${}_2F_1\left(\begin{matrix} 1 & 1 \\ & 2 \end{matrix} \middle| z\right) = -\frac{1}{z} \log(1-z)$$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1\\ & 1\end{matrix}\middle| z\right) = (1-z)^{-1/2} $$

$$ {}_2F_1\left(\begin{matrix}1/2 & 1/2\\ & 1\end{matrix}\middle| z\right) = \frac{2}{\pi}\int_0^1 \frac{\mathrm{d}t}{\sqrt{(1-t^2)(1-zt^2)}}. $$

The G–function

$$ \sum_{n\ge 0}(-1)^n \frac{\binom{5n}{n}}{4n+1} z^{4n+1} = z - z^5 + 10\frac{z^9}{2!} - 15\cdot 14\frac{z^{13}}{3!} + \cdots, $$

which converges for $|z| < 5^{-5/4}$, is a solution of the quintic equation $x^5+x = z$. (see [11, 52]).

Apéry's proof of the irrationality of $\zeta(3)$ is related with the theory of G–functions [20, 32].

In [21], Fischler and Rivoal introduce the set $\mathbb{G}$ of all values taken by any analytic continuation of any G–function at any algebraic point. Using a theorem of André–Chudnovski–Katz, they prove that $\mathbb{G}$ is a countable subring of $\mathbb{C}$ which contains the field $\overline{\mathbb{Q}}$ of algebraic numbers and the logarithms of algebraic numbers. Conjecturally, $\mathbb{G}$ is not a field. According to a conjecture of Bombieri and Dwork, $\mathbb{G}$ should coincide with the set of periods of algebraic varieties defined over $\overline{\mathbb{Q}}$. See also [31].

It is expected (see [33]) that Euler constant

$$ \gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n\right) = 0.577\,215\,664\,901\ldots $$

does not belong the the field of fractions of $\mathbb{G}$.

The group of units of $\mathbb{G}$ contains $\overline{\mathbb{Q}}^\times$ and the values $\mathrm{B}(a,b)$, (a, b in $\mathbb{Q}$) of Euler Beta function. The numbers $\Gamma(a/b)^b$, $a/b \in \mathbb{Q}\setminus\{0,-1,-2,\ldots\}$, are units in the ring $\mathbb{G}$. For instance, $\pi = \Gamma(1/2)^2$ is a unit:

$$ \pi = \sum_{n\ge 1}\frac{4(-1)^n}{2n+1}, \quad \frac{1}{\pi} = \sum_{n\ge 1}\frac{(42n+5)\binom{2n}{n}^3}{2^{12n+4}}. $$

The formula

$$ \frac{16}{\pi} = \sum_{n\ge 0}(42n+5)\frac{(1/2)_n^3}{n!^3 2^{6n}} $$

appeared in the Walt Disney film *High School Musical* – see [9].

Adapting Strauss's question (see § 1) to the context of G–functions, it is natural to ask:

Let g be a G–function which is not algebraic. Is–it true that $g(\alpha)$ is algebraic for at most finitely many algebraic α?

For Gauss hypergeometric functions, the answer is given by a result of Wolfart [60]: *Let $f(z) = {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right)$ with a, b, c in $\mathbb{Q}$. Let Δ be the monodromy group and*

$$\mathbb{E} = \left\{\alpha \in \overline{\mathbb{Q}} \ \mid \ f(\alpha) \in \overline{\mathbb{Q}}\right\}.$$

(1) *If f is algebraic (Δ finite), then $E = \overline{\mathbb{Q}}$.*
(2) *If f is arithmetic, then $\mathbb{E}$ is dense in $\overline{\mathbb{Q}}$.*
(3) *Otherwise, $\mathbb{E}$ is finite.*
For the definitions of the monodromy group and of arithmetic, see [55]. An example is given by taking $a = 1/12$, $b = 5/12$, $c = 1/2$. Then the monodromy group is $\mathrm{SL}(2, \mathbb{Z})$. It can be shown that

$${}_2F_1\left(\begin{matrix} 1/12 & 5/12 \\ & 1/2 \end{matrix}\middle| 1 - \frac{1}{J(\tau)}\right)^4 = \frac{E_4(\tau)}{E_4(i)}$$

where

$$E_4(\tau) = 1 + 240 \sum_{n \geq 1} \frac{n^3 q^n}{1 - q^n}, \quad \Delta(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}$$

with $q = e^{2i\pi\tau}$ and $J(\tau) = E_4(\tau)^3/1728\Delta(\tau)$. So

$$\begin{aligned} J(q) &= q^{-1}\left(1 + 240 \sum_{m=1}^{\infty} m^3 \frac{q^m}{1 - q^m}\right)^3 \prod_{n=1}^{\infty} (1 - q^n)^{-24} \\ &= \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots \end{aligned}$$

In particular $J(i) = 1$. From the theory of Complex Multiplication, it follows that if $\tau_0 \in \mathbb{Q}(i)$, $\mathrm{Im}(\tau_0) > 0$, then both $J(\tau_0)$ and $E_4(\tau_0)/E_4(i)$ are algebraic.

Other examples (Beukers–Wolfart [12]) are

$${}_2F_1\left(\begin{matrix} 1/12 & 5/12 \\ & 1/2 \end{matrix}\middle| \frac{1323}{1331}\right) = \frac{3}{4}\sqrt[4]{11},$$

$${}_2F_1\left(\begin{matrix} 1/4 & 1/2 \\ & 3/4 \end{matrix}\middle| \frac{80}{81}\right) = \frac{9}{5},$$

$${}_2F_1\left(\begin{matrix} 1/3 & 2/3 \\ & 5/6 \end{matrix}\middle| \frac{27}{32}\right) = \frac{8}{5},$$

as well as (Akihito Ebisu, 2014)

$$ {}_2F_1\left(\begin{matrix}1/12 & 1/4\\ & 5/6\end{matrix}\middle|\frac{135}{256}\right) = \frac{2}{5}\sqrt[6]{270} $$

and also (Yifan Yang, 2015)

$$ {}_2F_1\left(\begin{matrix}1/24 & 7/24\\ & 5/6\end{matrix}\middle| -\frac{2^{10}3^3 5}{11^4}\right) = \sqrt{6}\sqrt[6]{\frac{11}{5^5}}. $$

Non arithmetic examples, due to Beukers, are

$$ {}_2F_1\left(\begin{matrix}1-3a & 3a\\ & a\end{matrix}\middle|\frac{1}{2}\right) = 2^{3-2a}\cos\pi a, $$

$$ {}_2F_1\left(\begin{matrix}2a & 1-4a\\ & 1-a\end{matrix}\middle|\frac{1}{2}\right) = 4^{a}\cos\pi a $$

and

$$ {}_2F_1\left(\begin{matrix}7/48 & 31/48\\ & 29/24\end{matrix}\middle| -\frac{1}{3}\right) = 2^{5/24}3^{-11/12}5\cdot\sqrt{\frac{\sin\pi/24}{\sin 5\pi/24}}. $$

4 Schneider–Lang criterion

One of the very powerful tools for proving the transcendence of values of meromorphic functions is the so–called Schneider–Lang criterion see for instance [46, Chap. II § 3], [34, Chap. III], [58, Th. 1.1.1], [5, § 6.1], [19, Chap. 3 § 3 Th. 3.18], [42, Chap. 9], [7, § 2.3], [39, Th. 19.1], [57, Th. 6.3], [56, Chap. 2].

Theorem 4.1 (Schneider–Lang)**.** *Let $d \geq 2$ be an integer and $f_1, \dots, f_d$ be meromorphic functions of finite order of growth. Assume f_1 and f_2 are algebraically independent. Let K be a number field. Assume that for $1 \leq i \leq d$, the derivative f_i' of f_i belongs to the ring $K[f_1, \dots, f_d]$. Then the set of $w \in \mathbb{C}$ which are not pole of $f_1, \dots, f_d$ and such that $f_i(w) \in K$ for $i = 1, 2, \dots, d$ is finite.*

Corollaries of this results are the Theorem of Hermite and Lindemann on the transcendence of e^α, the Theorem of Gel'fond and Schneider on the transcendence of α^β, as well as elliptic analogues due to Schneider [46, Chap. II § 4], [34, Chap. III], [58, Chap. 3], [5, Chap. 6], [19, Chap. 6], [42, Chap. 11–18], [39, Chap. 20], [57, Chap. 6].

An extension of the Schneider–Lang Theorem in several variables includes a number of further results, in particular Baker's Theorem on linear independence of logarithms of algebraic numbers (as noticed by Bertrand and Masser – see [59, Chap. 4]), as well as a number of results related with abelian functions and algebraic groups [34, Chap. IV], [58, Chap. 5]. Among these results is the Theorem of Schneider (1941) [46, Chap. II § 4], [58, Th. 5.2.8] [61] on the transcendence of $\mathrm{B}(a, b)$ for rational numbers a and b such that $a + b$ is not a negative integer.

A number of far reaching recent results are included in the very recent book [56]. Among the topics included in this book are modular functions and criteria for complex multiplication, periods of 1–forms on complex curves and Abelian varieties, transcendence criterion for complex multiplication on $K3$ surfaces, Hodge structures of higher level.

A remarkable development of this classical method has been achieved by the so–called *Théorème Stéphanois* [8], [7, § 1.5], which answers a question of Mahler in the complex case and of Manin in the p–adic case: *for q a complex or p–adic number satisfying $0 < |q| < 1$, one at least of the to numbers q, $J(q)$ is transcendental.*

The proof in [8], based on the Gel'fond–Schneider transcendence method, also involves ideas close to the ones which occur in the method of Mahler (see § 6 below).

A modern presentation of the proof of the Schneider–Lang Theorem, using the slope method of J–B. Bost based on Arakelov theory (developed also by A. Chambert–Loir, C. Gasbarri, É. Gaudron, Ph. Graftieaux, M. Herblot [28], G. Rémond, E. Viada and others – see [17]) and using Falting's height, is given in [24].

5 Linear and algebraic independence

After the pioneer work of A. Baker on the linear independence of logarithms of algebraic numbers (see [5, Chap. 2], [7, § 2.1], [42, Chap. 19–20]) his method has been extensively developed, in particular in the realm of algebraic groups (see [7, S 5.5: The Wüstholz Theory]). Strong tools are now available to prove the linear independence of transcendental numbers – see for instance [61].

The method used by Gel'fond and Schneider to solve Hilbert's seventh problem on the transcendence of α^β [57, Chap. 6] has been extended by Gel'fond to obtain results of algebraic independence [23, Chap. 3 § 4]. This approach has been developed by many a mathematicians, including

W.D. Brownawell, G.V. Chudnovskii and Yu. V. Nesterenko – see [5, Chap. 12], [44, Chap. 3, 4,13 and 14], [13], [43], [39, Chap. 20].

The transcendence of $\Gamma(a/b)$ for $a/b \in \mathbb{Q}$ is known only for a restricted set of values of a/b: in the interval $(0,1)$, we know that the numbers

$$\Gamma(1/6),\quad \Gamma(1/4),\quad \Gamma(1/3),\quad \Gamma(1/2),\quad \Gamma(2/3),\quad \Gamma(3/4),\quad \Gamma(5/6)$$

are transcendental, but we do not know any further irrational value of Γ. It is conjectured by Deligne, Rohrlich and Lang [35, 41, 26][2] that any algebraic relations among Gamma values at rational points should belong to the ideal of relations generated by the standard relations, namely
Translation:

$$\Gamma(a+1) = a\Gamma(a),$$

Reflection:

$$\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$$

and
Multiplication: for any positive number n,

$$\prod_{k=0}^{n-1}\Gamma\left(a+\frac{k}{n}\right) = (2\pi)^{(n-1)/2} n^{-na+(1/2)}\Gamma(na).$$

An example [4] is

$$\frac{\Gamma(1/24)\Gamma(11/24)}{\Gamma(5/24)\Gamma(7/24)} = \sqrt{3}\sqrt{2+\sqrt{3}},$$

a root of $X^4 - 12X^2 + 9$.

It is not know whether the three numbers $\Gamma(1/5)$, $\Gamma(2/5)$ and $e^{\pi\sqrt{5}}$ are algebraically independent: this would follow from the above mentioned conjecture of Deligne–Rohrlich–Lang, as shown by F. Adiceam, who also deduces from Nesterenko's result that each of the three numbers

$$\Gamma(1/20)\Gamma(3/20)\Gamma(7/20)\Gamma(9/20),$$

[2] For the history of this conjecture, see [41, §16.3]. As pointed out by P. Deligne, the transcendance conjecture is probably a consequence of Grothendieck period conjecture, which is a good way to say that if we have a family of motives over $\mathbb{Q}$, giving rise to the (Betti) motivic Galois group G, to the G-torsor P of isomorphisms between the Betti and de Rham realisations (G and P are defined over $\mathbb{Q}$), and to the element "can" of $P(\mathbb{C})$, the canonical isomorphism between the complexifications of the Betti and de Rham realization, then this element "can" is Zariski dense in P.

$$\Gamma(1/5)\Gamma(7/20)\Gamma(9/20),$$
$$\Gamma(1/5)^{-1}\Gamma(1/20)\Gamma(3/20)$$

is transcendental over the field $\mathbb{Q}(\pi, e^{\pi\sqrt{5}})$.

We refer to the book [42] by R. Murty and P. Rath for a large collection of further results concerning elliptic integrals and hypergeometric series, Dirichlet series and L-functions, values of modular forms, periods, multiple zeta functions and $\zeta(3)$. See also [27].

There are also many recent papers dealing with special values of the Riemann zeta function, they would deserve a specific survey updating [20].For a reference to MZV, see [15]

6 Mahler's method

The name *Fredholm series* is often wrongly attributed to the power series

$$\chi_2(z) = \sum_{n\geq 0} z^{2^n}$$

(for instance in [7, § 1.5]). According to [1, Notes on chapter 13 p. 403], Fredholm studied rather the theta series

$$\sum_{n\geq 0} z^{n^2}.$$

It is ironical that both series occur in connection with modular functions and paper folding [40, 47].

The transcendence of $\chi_2(1/2)$ was proved by A.J. Kempner in 1916. Much more general results were achieved by K. Mahler in 1930, and then in 1969, an example of which is the transcendence of the values at algebraic points of the function

$$\chi_d(z) = \sum_{n\geq 0} z^{d^n}$$

for $d \geq 2$. The point is that this function satisfies a functional equation

$$\chi_d(z) = z + \chi_d(z^d) \qquad (|z| < 1).$$

The theory of Mahler (see for instance [6, Chap. 15], [45], [44, Chap. 12], [39, § 11]) has now reached a satisfactory state. In order to prove the algebraic independence of values, it suffices to prove the algebraic independence of the functions. Here is a recent example [14]. For $\ell > 1$, denote by Φ_ℓ the ℓ-th cyclotomic polynomial and set $\Phi_1(x) = 1 - x$. For $d \geq 2$, define

$$F_{d,\ell}(z) = \prod_{j \geq 0} \Phi_\ell(z^{d^j}).$$

Given positive integers $d \geq 2$ and $\ell \geq 1$, P. Bundschuh and K. Väänänen prove that the following statements are equivalent:
(i) d is composite or does not divide ℓ.
(ii) $F_{d,\ell}$ is hypertranscendental.
(iii) $F_{d,\ell}$ is not a rational function.
Using the functional equation

$$F_{d,\ell}(z) = \Phi_\ell(z) F_{d,\ell}(z^d)$$

together with Mahler's method, they deduce deep results of algebraic independence on the values of this infinite product and its derivatives.

References

[1] J.-P. Allouche and J. Shallit. *Automatic sequences.* Cambridge University Press, Cambridge, 2003. Theory, applications, generalizations.

[2] Y. André. *G-functions and geometry.* Aspects of Mathematics, E13. Friedr. Vieweg & Sohn, Braunschweig, 1989.

[3] Y. André. G-fonctions et transcendance. *J. Reine Angew. Math.*, 476:95–125, 1996.

[4] Y. André. Groupes de Galois motiviques et périodes. *Astérisque, to appear*, (Séminaire Bourbaki, n° 1104, Novembre 2015), 2016.

[5] A. Baker. *Transcendental number theory.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 1990.

[6] A. Baker and D. W. Masser, editors. *Transcendence theory: advances and applications.* Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1977.

[7] A. Baker and G. Wüstholz. *Logarithmic forms and Diophantine geometry*, volume 9 of *New Mathematical Monographs.* Cambridge University Press, Cambridge, 2007.

[8] K. Barré-Sirieix, G. Diaz, F. Gramain, and G. Philibert. Une preuve de la conjecture de Mahler-Manin. *Invent. Math.*, 124(1-3):1–9, 1996.

[9] N. D. Baruah, B. C. Berndt, and H. H. Chan. Ramanujan's series for $1/\pi$: a survey. *Amer. Math. Monthly*, 116(7):567–587, 2009.

[10] F. Beukers. Gauss' hypergeometric function. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 23–42. Birkhäuser, Basel, 2007.

[11] F. Beukers. Hypergeometric functions, how special are they? *Notices Amer. Math. Soc.*, 61(1):48–56, 2014.

[12] F. Beukers and J. Wolfart. Algebraic values of hypergeometric functions. In *New advances in transcendence theory (Durham, 1986)*, pages 68–81. Cambridge Univ. Press, Cambridge, 1988.

[13] V. Bosser. Indépendance algébrique de valeurs de séries d'Eisenstein (théorème de Nesterenko). In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 119–178. Soc. Math. France, Paris, 2005.

[14] P. Bundschuh and K. Väänänen. Arithmetic properties of infinite products of cyclotomic polynomials. *Bull. Aust. Math. Soc.*, 93(3):375–387, 2016.

[15] J. I. Burgos Gil and J. Fresán. Multiple zeta values: from numbers to motives. *Clay Mathematics Proceedings*, to appear.

[16] K. Chakraborty, S. Kanemitsu, and H. Tsukada. *Vistas of special functions II.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2010.

[17] A. Chambert-Loir. Théorèmes d'algébricité en géométrie diophantienne (d'après J.-B. Bost, Y. André, D. & G. Chudnovsky). *Astérisque*, (282):Exp. No. 886, viii, 175–209, 2002. Séminaire Bourbaki, Vol. 2000/2001.

[18] H. Cohen. Some formulas of Ramanujan involving Bessel functions. In *Actes de la Conférence "Fonctions L et Arithmétique"*, Publ. Math. Besançon Algèbre Théorie Nr., pages 59–68. Lab. Math. Besançon, Besançon, 2010.
`http://pmb.univ-fcomte.fr/2010/Cohen.pdf`.

[19] N. I. Fel′dman and Y. V. Nesterenko. Transcendental numbers. In *Number theory, IV*, volume 44 of *Encyclopaedia Math. Sci.*, pages 1–345. Springer, Berlin, 1998.

[20] S. Fischler. Irrationalité de valeurs de zêta (d'après Apéry, Rivoal, ...). *Astérisque*, 294:vii, 27–62, 2004.

[21] S. Fischler and T. Rivoal. On the values of G-functions. *Comment. Math. Helv.*, 89(2):313–341, 2014.

[22] S. Fischler and T. Rivoal. Linear independence of values of G-functions. `https://arxiv.org/abs/1701.09051`

[23] A. O. Gel′fond. *Transcendental and algebraic numbers.* Translated from the first Russian edition by Leo F. Boron. Dover Publications, Inc., New York, 1960. Translation in English of *Transcendentnye i algebraičeskie čisla*, Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1952.

[24] P. Graftieaux. Théorème stéphanois et méthode des pentes. In *Formes modulaires et transcendance*, volume 12 of *Sémin. Congr.*, pages 179–213. Soc. Math. France, Paris, 2005.

[25] F. Gramain. Fonctions entières arithmétiques. In *Séminaire d'analyse 1985–1986 (Clermont-Ferrand, 1985–1986)*, pages Exp. No. 9, 9. Univ. Clermont-Ferrand II, Clermont-Ferrand, 1986.

[26] S. Gun, M. R. Murty, and P. Rath. Linear independence of digamma function and a variant of a conjecture of Rohrlich. *J. Number Theory*, 129(8):1858–1873, 2009.

[27] S. Gun, M. R. Murty, and P. Rath. Transcendental nature of special values of L-functions. *Canad. J. Math.*, 63(1):136–152, 2011.

[28] M. Herblot. Algebraic points on meromorphic curves. Submitted on 27 Apr 2012
`https://arxiv.org/abs/1204.6336`.

[29] J. Huang, D. Marques, and M. Mereb. Algebraic values of transcendental functions at algebraic points. *Bull. Aust. Math. Soc.*, 82(2):322–327, 2010.

[30] S. Kanemitsu and H. Tsukada. *Vistas of special functions.* World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2007.

[31] M. Kontsevich and D. Zagier. Periods. In *Mathematics unlimited—2001 and beyond*, pages 771–808. Springer, Berlin, 2001.

[32] C. Krattenthaler and T. Rivoal. Hypergéométrie et fonction zêta de Riemann. *Mem. Amer. Math. Soc.*, 186(875):x+87, 2007.

[33] J. C. Lagarias. Euler's constant: Euler's work and modern developments. *Bull. Amer. Math. Soc. (N.S.)*, 50(4):527–628, 2013.

[34] S. Lang. *Introduction to transcendental numbers.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1966. in Collected Papers I [36], p.396–506.

[35] S. Lang. Relations de distributions et exemples classiques. In *Séminaire Delange-Pisot-Poitou, 19e année: 1977/78, Théorie des nombres, Fasc. 2*, pages Exp. No. 40, 6. Secrétariat Math., Paris, 1978. in Collected Papers III [37], 59–65.

[36] S. Lang. *Collected papers. I. 1952–1970.* Springer Collected Works in Mathematics. Springer, New York, 2013. Reprint of the 2000 edition.

[37] S. Lang. *Collected papers. III. 1978–1990.* Springer Collected Works in Mathematics. Springer, New York, 2013. Reprint of the 2000 edition.

[38] K. Mahler. *Lectures on transcendental numbers.* Lecture Notes in Mathematics, Vol. 546. Springer-Verlag, Berlin-New York, 1976.

[39] D. Masser. *Auxiliary polynomials in number theory.* Cambridge Tracts in Mathematics **207**; Cambridge University Press, 2016.

[40] M. Mendès France and A. Sebbar. Pliages de papiers, fonctions thêta et méthode du cercle. *Acta Math.*, 183(1):101–139, 1999.

[41] H. Movasati. A course in Hodge theory with emphasis on multiple integrals. To appear.
`http://w3.impa.br/%7Ehossein/myarticles/hodgetheory.pdf`

[42] M. R. Murty and P. Rath. *Transcendental numbers.* Springer, New York, 2014.

[43] Y. V. Nesterenko. *Algebraic independence.* Published for the Tata Institute of Fundamental Research, Bombay; by Narosa Publishing House, New Delhi, 2009.

[44] Y. V. Nesterenko and P. Philippon, editors. *Introduction to algebraic independence theory*, volume 1752 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001. Edited by Nesterenko and Philippon.

[45] K. Nishioka. *Mahler functions and transcendence*, volume 1631 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1996.

[46] T. Schneider. *Introduction aux nombres transcendants*. Traduit de l'allemand par P. Eymard. Gauthier-Villars, Paris, 1959. Translation in French of *Einführung in die transzendenten Zahlen*, Springer-Verlag, 1957.

[47] A. Sebbar. Paperfolding and modular functions. In *Exponential analysis of differential equations and related topics*, RIMS Kôkyûroku Bessatsu, B52, pages 97–126. Res. Inst. Math. Sci. (RIMS), Kyoto, 2014.

[48] A. B. Shidlovskii. *Transcendental numbers*, volume 12 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1989. Translated from the Russian by Neal Koblitz, With a foreword by W. Dale Brownawell.

[49] C. L. Siegel. Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]. In *On some applications of Diophantine approximations* (English Translation), volume 2 of *Quad./Monogr.*, pages 81–138. Ed. Norm., Pisa, 2014.

[50] C. L. Siegel. *Transcendental Numbers*. Annals of Mathematics Studies, no. 16. Princeton University Press, Princeton, N. J., 1949.

[51] P. Stäckel. Ueber arithmetische Eigenschaften analytischer Functionen. *Math. Ann.*, 46:513–520, 1895.

[52] J. Stillwell. Eisenstein's footnote. *Math. Intelligencer*, 17(2):58–62, 1995.

[53] A. Surroca. Valeurs algébriques de fonctions transcendantes. *Int. Math. Res. Not.*, pages Art. ID 16834, 31, 2006.

[54] B. Sury. Bessel's contain continued fractions of progressions. *Resonance*, 10(3):80–87, 2005.

[55] P. Tretkoff. *Complex ball quotients and line arrangements in the projective plane*, volume 51 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 2016.

[56] P. Tretkoff. *Periods and Special Functions in Transcendence*, World Scientific, Advanced Textbooks in Mathematics, 2017.

[57] R. Tubbs. *Hilbert's seventh problem*, volume 2 of *Institute of Mathematical Sciences Lecture Notes*. Hindustan Book Agency, New Delhi, 2016.

[58] M. Waldschmidt. Nombres transcendants et groupes algébriques. *Astérisque*, 69-70:218, 1987. With appendices by Daniel Bertrand and Jean-Pierre Serre.

[59] M. Waldschmidt. *Diophantine approximation on linear algebraic groups*, volume 326 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 2000. Transcendence properties of the exponential function in several variables.

[60] J. Wolfart. Werte hypergeometrischer Funktionen. *Invent. Math.*, 92(1):187–216, 1988.

[61] J. Wolfart and G. Wüstholz. Der Überlagerungsradius gewisser algebraischer Kurven und die Werte der Betafunktion an rationalen Stellen. *Math. Ann.*, 273(1):1–15, 1985.

Michel WALDSCHMIDT
Sorbonne Universités
UPMC Univ Paris 06
UMR 7586 IMJ-PRG
Institut de Mathématiques de Jussieu-Paris Rive Gauche
F – 75005 Paris, France
`michel.waldschmidt@imj-prg.fr`
`http://www.imj-prg.fr/~michel.waldschmidt`