ON CO-RECURSIVE ORTHOGONAL POLYNOMIALS

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1. **Introduction.** Let $P_n(x)$ be a polynomial of degree n $(n=0, 1, \cdots)$ with leading coefficient unity. Then (Favard [2]) a sufficient (as well as necessary) condition for the $P_n(x)$ to be real orthogonal polynomials is that they satisfy a recurrence formula of the form

$$(1.1) \qquad P_n(x) = (x + b_n)P_{n-1}(x) - \lambda_n P_{n-2}(x), \qquad n = 2, 3, \cdots$$

where $P_0(x)=1$, $P_1(x)=x+b_1$, b_n real, $\lambda_n>0$.

D. Dickinson [1] has shown that a relation of the form

$$(1.2) \quad P_n(x)P_{n-1}(-x) + P_n(-x)P_{n-1}(x) = a_n, \ (-1)^n a_1 a_n < 0, \qquad n = 1, 2, \cdots,$$

is equivalent to (1.1) with $b_n=0$ $(n \geqq 2)$ and $P_1(0) \neq 0$. The condition $b_n=0$ $(n \geqq 2)$ suggests the symmetric case, (i.e., $P_n(-x)=(-1)^n P_n(x)$) but this is denied by the condition $P_1(0) \neq 0$. (In fact, (1.2) shows that $P_n(-r) \neq 0$ whenever $P_n(r)=0$.) It then seems natural to ask what relations exist between a set of polynomials satisfying (1.2) and the corresponding symmetric polynomials which would be obtained from the equivalent relation (1.1) if the condition $P_1(0) \neq 0$ were replaced with $P_1(0)=0$.

We are thus led to consider the following more general situation. Given the $P_n(x)$ satisfying (1.1) (without the restriction $b_n=0$), let the "co-recursive" polynomials $P_n^*(x) \equiv P_n^*(x, c)$ be defined by

$$(1.3) \qquad P_n^*(x) = (x + b_n)P_{n-1}^*(x) - \lambda_n P_{n-2}^*(x), \qquad n = 2, 3, \cdots$$

where $P_0^*(x)=1$, $P_1^*(x)=P_1(x)-c$. We then seek to determine the properties of the $P_n^*(x)$ from those of the $P_n(x)$. In order to have orthogonal polynomials, we will restrict c to be real.

2. **Some formal relations.** Since $P_n(x)$ and $P_n^*(x)$ satisfy identical recurrence formulas, except for initial conditions, it follows that $P_n^*(x)-P_n(x)$ also satisfies (1.1) together with the initial conditions, $P_0^*(x)-P_0(x)=0$, $P_1^*(x)-P_1(x)=-c$. Therefore we have[1]

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[1] Sherman [3] noted a relation of this type for certain orthogonal polynomials associated with a Stieltjes continued fraction.

$$P_n^*(x) = P_n(x) - cQ_{n-1}(x), \qquad n = 0, 1, \cdots \tag{2.1}$$

where $Q_n(x)$ is a polynomial of degree n, satisfying

$$Q_n(x) = (x + b_{n+1})Q_{n-1}(x) - \lambda_{n+1}Q_{n-2}(x), \quad n = 1, 2, \cdots \tag{2.2}$$

where $Q_{-1}(x)=0$, $Q_0(x)=1$.

Comparison of the recurrence formulas shows that $Q_{n-1}(x)/P_n(x)$ and $Q_{n-1}(x)/P_n^*(x)$ are the nth convergents, respectively, of the continued fractions

$$K(z) = \frac{1|}{|z + b_1} - \frac{\lambda_2|}{|z + b_2} - \frac{\lambda_3|}{|z + b_3} - \cdots, \tag{2.3}$$

$$K^*(z) = \frac{1|}{|z + b_1 - c} - \frac{\lambda_2|}{|z + b_2} - \frac{\lambda_3|}{|z + b_3} - \cdots. \tag{2.4}$$

The $Q_n(x)$ are the "numerator polynomials" studied by Shohat and Sherman [4] and Sherman [3]; $Q_n(x)$ is itself the denominator of the nth convergent of the continued fraction

$$K_1(z) = \frac{1|}{|z + b_2} - \frac{\lambda_3|}{|z + b_3} - \frac{\lambda_4|}{|z + b_4} - \cdots. \tag{2.5}$$

We will denote by $\psi(x)$, $\psi^*(x) \equiv \psi^*(x, c)$ and $\psi_1(x)$, respectively, solutions of the moment problems associated with the above continued fractions and write $F(z)$, $F^*(z)$ and $F_1(z)$ for the corresponding Stieltjes transforms:

$$K(z) \sim F(z) = \int_{-\infty}^{\infty} (z - u)^{-1} d\psi(u) (z \text{ nonreal}), \text{ etc.} \tag{2.6}$$

3. **Zeros.** By Favard's theorem, the sets $\{P_n(x)\}$, $\{P_n^*(x)\}$ and $\{Q_n(x)\}$ are orthogonal with respect to the distributions $\psi(x)$, $\psi^*(x)$ and $\psi_1(x)$, respectively; hence the zeros of the individual polynomials are real and simple.

If we multiply (1.1) by $P_{n-1}^*(x)$ and (1.3) by $P_{n-1}(x)$ and subtract, we obtain

$$P_{n-1}^*(x)P_n(x) - P_n^*(x)P_{n-1}(x) = \lambda_n[P_{n-2}^*(x)P_{n-1}(x) - P_{n-1}^*(x)P_{n-2}(x)].$$

Iterating this relation yields

$$P_n^*(x)P_{n-1}(x) - P_{n-1}^*(x)P_n(x) = -c\prod_{k=1}^{n} \lambda_k \qquad (n \geqq 1, \lambda_1 = 1).$$

If $x_{n,j}$ $(j=1, \cdots, n)$ denotes the zeros of $P_n(x)$ in ascending order of magnitude, then $P_n^*(x_{n,j})P_{n-1}(x_{n,j}) = -c\prod_{k=1}^{n} \lambda_k$, so $P_n^*(x_{n,j})$

alternates in sign with $P_{n-1}(x_{n,j})$. By the well known separation of the zeros of consecutive orthogonal polynomials, it follows that the zeros of $P_n(x)$ and $P_n^*(x)$ are mutually separated. Moreover, $P_{n-1}(x_{n,n}) > 0$, hence sgn $P_n^*(x_{n,n}) = -\text{sgn } c$. Therefore if $x_{n,j}^*$ $(j=1, \cdots, n)$ denotes the zeros of $P_n^*(x)$ in ascending order of magnitude, we have

$$(3.1) \qquad x_{n,j-1} < \overset{*}{x}_{n,j-1} < x_{n,j} < \overset{*}{x}_{n,j} \qquad (j = 2, \cdots, n; c > 0),$$

with the roles of $x_{n,j}$ and $x_{n,j}^*$ interchanged for $c<0$.

Finally, let $x'_{n-1,j} \equiv y_{n,j}$ $(j=1, \cdots, n-1)$ denote the zeros of $Q_{n-1}(x)$ in ascending order. Then it is well known (e.g. [6]) that

$$(3.2) \quad x_{n,j-1} < y_{n,j-1} < x_{n,j}; \qquad \overset{*}{x}_{n,j-1} < y_{n,j-1} < \overset{*}{x}_{n,j} \quad (j = 2, \cdots, n).$$

Combining (3.1) and (3.2), we have

THEOREM 1. *The zeros of $P_n(x)$, $P_n^*(x, c)$ and $Q_{n-1}(x)$ $(n \geqq 2)$ are mutually separated in the following manner:*

$$x_{n,j-1} < \overset{*}{x}_{n,j-1} < y_{n,j-1} < x_{n,j} < \overset{*}{x}_{n,j} \qquad (j = 2, \cdots, n; c > 0),$$

with the roles of $x_{n,j}$ and $x_{n,j}^*$ reversed for $c<0$.

COROLLARY. *The zeros of $P_n^*(x, c)$ are increasing functions of c and the zeros of any two distinct co-recursive orthogonal polynomials of the same degree $n (n \geqq 1)$ are mutually separated.*

PROOF. Let $P_n^*(x, c_1)$ and $P_n^*(x, c_2)$ be any two-recursive polynomials. Let $P_n^*(x, c_1) = P_{n,1}(x)$ and $P_n^*(x, c_2) = P_{n,2}(x)$. Then $P_{n,2}(x, c_1 - c_2) = P_{n,1}(x)$.

Next let $[a, b]$, $[a^*, b^*]$ and $[a', b']$ denote the "true" intervals of orthogonality of $\{P_n(x)\}$, $\{P_n^*(x)\}$ and $\{Q_n(x)\}$, respectively; that is [4], $a = \lim_{n\to\infty} x_{n,1}$, $b = \lim_{n\to\infty} x_{n,n}$, etc. Then by Theorem 1, for $c>0$, $a \leqq a^* \leqq a' < b' \leqq b \leqq b^*$, and for $c<0$, $a^* \leqq a \leqq a' < b' \leqq b^* \leqq b$. Further, $a = -\infty$ $(b = +\infty)$ if and only if $a' = -\infty$ $(b' = +\infty)$ [3], hence a, a^*, a' (b, b^*, b') are all finite or all infinite. If a or b is finite, Theorem 1 shows that $P_n^*(x)$ has at most one zero exterior to $[a, b]$, while all zeros of $Q_n(z)$ (for all n) are interior to $[a, b]$.

THEOREM 2. *The zeros of $P_n^*(x, c)$ are all interior to $[a, b]$ for all n if and only if*

$$(3.3) \qquad \lim_{n\to\infty} \frac{P_n(a)}{Q_{n-1}(a)} \equiv A \leqq c \leqq B \equiv \lim_{n\to\infty} \frac{P_n(b)}{Q_{n-1}(b)} \cdot$$

Here $A (B)$ must be replaced by $-\infty (+\infty)$ in case $a = -\infty$ $(b = +\infty)$.

PROOF. Suppose a is finite. Then it follows from the "determinant formula" for continued fractions [6] that for $x \leqq a$, $P_n(x)/Q_{n-1}(x) < P_{n+1}(x)/Q_n(x) < 0$ (as is known). Hence A exists and by (2.1)

$$P_n^*(a)/Q_{n-1}(a) = P_n(a)/Q_{n-1}(a) - c < A - c \qquad (n \geqq 2).$$

Thus $P_n^*(a)/Q_{n-1}(a) < 0$ and $P_n^*(x)$ does not change sign for $x \leqq a$ (for all n) if and only if $A \leqq c$.

4. **On the distribution functions.** Sherman [3] has shown that the complete convergence of (2.3) implies the complete convergence of (2.5). By a theorem of Carleman (cf. [5]) or by direct verification, the complete convergence of (2.3) is seen to imply the complete convergence of (2.4) also. Thus if $\psi(x)$ is a solution of a determined moment problem, and hence is uniquely determined up to an additive constant at all points of continuity [5], then so is $\psi^*(x)$ (as well as $\psi_1(x)$). Moreover, if we define

$$q_n(x) = (\lambda_1 \cdots \lambda_{n+1})^{-1/2} Q_{n-1}(x), \quad p_n(x) = (\lambda_1 \cdots \lambda_{n+1})^{-1/2} P_n(x),$$

$$p_n^*(x) = (\lambda_1 \cdots \lambda_{n+1})^{-1/2} P_n^*(x), \qquad n = 0, 1, 2, \cdots,$$

then $\sum_{n=0}^{\infty} |p_n(x)|^2$ diverges at all points of continuity of $\psi(x)$ and converges at a discontinuity of $\psi(x)$ [5]. Similar interpretations hold for $\sum_{n=0}^{\infty} |p_n^*(x)|^2$ and $\sum_{n=0}^{\infty} |q_n(x)|^2$. But if (2.3) corresponds to a determined moment problem, then $\sum_{n=0}^{\infty} |p_n(x)|^2$ and $\sum_{n=0}^{\infty} |q_n(x)|^2$ cannot converge for a common value of x [7]. It follows from (2.1) that no two of the above three series can converge for a common value of x, hence we have

THEOREM 3. *If $\psi(x)$ is a solution of a determined moment problem, then so are $\psi^*(x)$ and $\psi_1(x)$. In this case, no two of the three distribution functions can have a common discontinuity.*

The above argument shows that if, say, b is finite and $B \neq 0$, then $\sum_{n=0}^{\infty} |p_n(b)|^2$ and $\sum_{n=0}^{\infty} |q_n(b)|^2$ must diverge together, hence $\psi(x)$ and $\psi_1(x)$ must be continuous at b. Similarly, $\psi^*(x)$ is continuous at b if $c \neq B$.

Henceforth, we assume (2.3) corresponds to a determined moment problem so that by [3], the continued fractions (2.3), (2.4) and (2.5) converge completely for all nonreal z to $F(z)$, $F^*(z)$ and $F_1(z)$, respectively, with $F(z) = [z + b_1 - \lambda_2 F_1(z)]^{-1}$ and $F^*(z) = [z + b_1 - c - \lambda_2 F_1(z)]^{-1}$. Also,

$$F^*(z) = F(z)/[1 - cF(z)], \qquad (z \text{ nonreal}). \tag{4.1}$$

We further assume that $\psi(x)$, $\psi^*(x)$ and $\psi_1(x)$ have been normalized: $\psi(x)=[\psi(x+0)+\psi(x-0)]/2$, etc. Then by the Stieltjes inversion formula

$$\psi^*(x) - \psi^*(x_0) = \lim_{y\to 0^+} \frac{1}{\pi}\int_x^{x_0} \operatorname{Im} \frac{F(u+iy)}{1-cF(u+iy)}\,du. \tag{4.2}$$

Sherman [3] has shown that if the analytic continuation of $F(z)$ is such that $1/F(z)$ is regular on a segment, $[x_0, x]$, of the real axis, then

$$\psi_1(x) - \psi_1(x_0) = \frac{1}{\pi}\int_x^{x_0} \operatorname{Im}\,[F(u)]^{-1}du. \tag{4.3}$$

If the analytic continuation of $F^*(z)$ is regular on $[x_0, x]$, then by the same procedure used by Sherman, we can show that

$$\psi^*(x) - \psi^*(x_0) = \frac{1}{\pi}\int_x^{x_0} \operatorname{Im} \frac{F(u)}{1-cF(u)}\,du. \tag{4.4}$$

If $\psi(x)$ has an interval of constancy, (α, β), then (2.6) defines a single analytic function which is regular on the upper and lower half-planes and (α, β). If $F(\xi)=1/c$, $\alpha<\xi<\beta$, then $F^*(z)$ has a pole at ξ and, in general, $\psi^*(x)$ will have a jump at ξ which can be determined as follows.

We can choose $x_0<\xi<x$ so that $[x_0, x]\subset(\alpha, \beta)$ and $F(u)\neq 1/c$ for $u\neq\xi$, $u\in[x_0, x]$. On the closed rectangular contour γ, whose vertices are $x+iy$, x_0+iy, x_0-iy, $x-iy$ $(y>0)$, we have by the residue theorem

$$\int_x^{x_0}[F^*(u+iy) - F^*(u-iy)]du + i\int_{-y}^{y}[F^*(x+iv) - F^*(x_0+iv)]dv = 2\pi i \operatorname*{Res}_{z=\xi} F^*(z),$$

$$\frac{1}{\pi}\int_x^{x_0} \operatorname{Im} F^*(u+iy)du = \operatorname*{Res}_{z=\xi} F^*(z) + \frac{1}{\pi}\int_0^{y} \operatorname{Re}\,[F^*(x+iv) - F^*(x_0+iv)]dv.$$

Since $F^*(z)$ is bounded on γ, we obtain on letting $y\to 0+$,

$$\psi^*(x) - \psi^*(x_0) = \operatorname*{Res}_{z=\xi} F^*(z). \tag{4.5}$$

If follows that if $\psi(x)$ is stepwise constant in an interval (α, β), then so is $\psi^*(x)$ (and $\psi_1(x)$ since a similar procedure clearly applies to

$F_1(z)$). Moreover, if $c \notin [A, B]$, say $B < c < \infty$, then $b < b^*$ by Theorem 2 and $x^*_{n,n-1} < b$, $\lim_{n\to\infty} x^*_{n,n} = b^*$. Since every point of increase of $\psi^*(x)$ is a limit point of zeros of $\{P_n^*(x)\}$ [5], it follows that $\psi^*(x)$ is constant for $b < x < b^*$, hence $\psi^*(x)$ must have a jump at b^* which can be determined by (4.5).

5. **Examples.** The Tchebicheff polynomials furnish simple examples for which $P_n^*(x)$ can be determined explicitly and for which $\psi^*(x)$ is of "mixed type," that is, $\psi^*(x)$ is absolutely continuous on part of the orthogonality interval and is a step function (with a single jump) on the remainder.

(a) $P_n(x) = 2^{-n} U_n(x) = 2^{-n} \sin (n+1)\theta / \sin \theta$, $x = \cos \theta$. We have $b_1 = b_n = 0$, $\lambda_n = 1/4$ $(n \geqq 2)$; $F(z) = 2/(z + (z^2-1)^{1/2})$ (cf. [3]). Clearly[2] $Q_n(x) = P_n(x)$ so by (2.1), (4.1) and (4.4),

$$P_n^*(x, c) = 2^{-n}[U_n(x) - 2cU_{n-1}(x)]$$

and[3]

$$\psi^*(x) - \psi^*(x_0) = \frac{2}{\pi} \int_{x_0}^{x} \frac{(1 - t^2)^{1/2}}{4c^2 + 1 - 4ct} dt, \qquad -1 < x_0 < x < 1.$$

Here $-A = B = 1/2$ so $\psi^*(x)$ is constant for all $x \notin (-1, 1)$ except for $|c| > 1/2$ when it has a jump at $\xi = c + 1/4c$ which is found by (4.5) to be $1 - 1/4c^2$. $\psi^*(x)$ is continuous at ± 1 except possibly when $c = \pm 1/2$. However, in this case, $P_n^*(x, c)$ reduces to a Jacobi polynomial:

$$P_n^*(x, c) = 2^n (n!)^2/(2n)! P_n^{(c,-c)}(x), \qquad c = \pm 1/2.$$

(b) $P_n(x) = 2^{1-n} T_n(x) = 2^{1-n} \cos n\theta$. Here $b_n = 0$ $(n \geqq 1)$, $\lambda_n = 1/4$ for $n \geqq 3$ but $\lambda_2 = 1/2$; $F(z) = (z^2 - 1)^{-1/2}$ [3]. Thus $Q_n(x) = 2^{-n} U_n(x)$ and we have

$$P_n^*(x, c) = 2^{1-n}[T_n(x) - cU_{n-1}(x)],$$

$$\psi^*(x) - \psi^*(x_0) = \frac{1}{\pi} \int_{x_0}^{x} \frac{(1 - t^2)^{1/2}}{1 + c^2 - t} dt, \qquad -1 < x_0 < x < 1.$$

$A = B = 0$, hence for all $c \neq 0$, $\psi^*(x)$ is continuous at ± 1 and constant for $x \notin (-1, 1)$ except for a jump at $\xi = (\operatorname{sgn} c)(1 + c^2)^{1/2}$ of magnitude $|c|(1 + c^2)^{-1/2}$.

[2] Sherman has shown that this is essentially the only case when $Q_n(x) = P_n(x)$, as is also evident from the recurrence formula.

[3] The constant factor is determined in accordance with $\int_{-\infty}^{\infty} d\psi^*(x) = \lambda_1 = 1$.

The modified Lommel polynomials, $P_n(x)=[2^n(\nu)_n]^{-1}h_{n,\nu}(x)$ (cf. [1]) furnish another simple example where $P_n^*(x)$ can be explicitly given. Since $b_1=b_n=0$, $\lambda_n\equiv\lambda_{n,\nu}=[4(n+\nu-1)(n+\nu-2)]^{-1}=\lambda_{n-1,\nu+1}$ $(n\geqq 2)$, we have $Q_n(x)=[2^n(\nu+1)_n]^{-1}h_{n,\nu+1}(x)$, $P_n^*(x, c)=[2^n(\nu)_n]^{-1}\cdot[h_{n,\nu}(x)-2c\nu h_{n-1,\nu+1}(x)]$. $\psi^*(x)$ is a step function whose jumps can be obtained from (4.1) and (4.5) with $F(z)=J_\nu(1/z)/J_{\nu-1}(1/z)$ where $J_\nu(z)$ is the Bessel function of order ν.

REMARK. It may be noted that in the preceding examples, $P_n^*(x, c)$ satisfies Dickinson's relation (1.2) (since $b_n=0$ $(n\geqq 1)$).

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