To appear in: SIAM Journal on Numerical Analysis

FINITE ELEMENT METHODS FOR CONFORMAL MAPPINGS

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Dedicated to Prof. Dr. Jim Douglas, Jr. on His 60th Birthday

Abstract: Two finite element methods for the evaluation of singular integral operators typical in the theory of conformal mappings are analyzed.

Key words. finite element methods, conformal mappings, singular integral equations

AMS(MOS) subject classifications. 30C30, 65R20

1.

The facts, properties etc. concerning conformal mappings mentioned in this paper can be found in the book of Gaier, D. (1964), the references made are indicated by (G, p. ··). In the theory of conformal mappings the integral operator

$$\text{(1.1)}\qquad \begin{aligned} v &= Au \\ v(s) &= \tfrac{1}{2\pi}\oint \mathrm{ctg}\tfrac{1}{2}(s-t)\,u(t)\,dt \end{aligned}$$

plays a central role. Here u is a 2π-periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. We mention one application only:

The integral equation of Theodorsen (G., pp. 64, 65)
Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with star-shaped boundary $\partial\Omega$ in the complex w-plane with the representation $\varrho = \varrho(\vartheta)$ in polar-coordinates. Further let $w = f(z)$ be the conformal mapping of the unit disc in the z-plane onto Ω normed by $f(0) = 0$, $f'(0) > 0$ and let (r, φ) denote the polar-coordinates in the z-plane. Then $f(z)$ is uniquely defined by the correspondence $\varphi \mapsto \vartheta(\varphi)$ and the function ϑ is the solution of the equation

$$\text{(1.2)}\qquad \vartheta(\varphi) = \varphi + A[\log \varrho(\vartheta(\varphi))] \quad .$$

The operator A has some remarkable properties. It is skew-symmetric in the space $L_2(0, 2\pi)$. Further A maps the space $H = L_2(0, 2\pi) \setminus \mathbb{R}$ isometrically onto itself (G., p. 64):

$$(1.3) \qquad \| Au \| = \| u \| \quad .$$

If $u = u(s)$ are the boundary values of a function U harmonic in the unit disc then by (1.1) the boundary values v of the conjugate function V are given (G., p. 63). The Hilbert inverse relations may be written in the form

$$(1.4) \qquad A^2 = -I \quad .$$

This shows that the evaluation of v for given u resp. of u for given v is more or less equivalent.

Subsequently we will consider besides $H = H_0$ the space H_1 consisting of those functions $w \in H_0$ such that the derivative w' is an element of H_0. The space H_{-1} is the dual of H_1 with respect to H_0. Parallel to these spaces we will consider also the spaces C^0 resp. $C^{0.\lambda}$ (for $0 < \lambda < 1$) of periodic, continuous functions resp. functions fulfilling a Hölder condition according to the exponent λ modulo $\mathbb{R}$. The norms in these spaces are defined in the natural way. The last property of the operator A to be mentioned is: A is a bijective mapping of $C^{0.\lambda}$ onto itself - see §18, 19 in Muskhelishvili, N. I. (1953).

Much effort was given to the question of the solution of equation (1.2). In our context the convergence of numerical procedures is of special interest. Usually the spaces T_n of trigonometric polynomials of degree n are used in order to get appropriate approximations. We mention - slightly simplifying in mixing discrete and continuous norms - three typical results: Let $e = e_n$ denote the error of the numerical method. Under the hypotheses $g \in C^{1.1}$ resp. $g \in C^{\infty}$ the error estimates hold true (G., p. 95):

$$(1.5_2) \qquad \| e \|_{L_2} \leq B(\varepsilon)\, n^{-2} \quad ,$$

$$(1.5_\infty) \qquad \| e \|_{L_\infty} \leq B(\varepsilon)\, n^{-9/8} \quad ,$$

resp.

$$(1.6) \qquad \| e \| \leq B(\varepsilon)\, R^{-n}$$

with some $R > 1$. In all these cases $B(\varepsilon)$ are numerical constants depending on ε, the deviation of $g(\vartheta)$ from 1 in appropriate norms.

2.

Obviously the main step in order to get an approximation on the solution of (1.2) is the numerical treatment of (1.1). In this section we discuss an approach based on the Ritz method using finite elements resp. in the present case spline spaces. We will consider only the simplest cases of splines. For $n \in \mathbb{N}$ let $h = 2\pi/n$ and $x_i = hi$. The space $S_h = S_h^{k,t}$ with $0 \le k < t$ consists of the functions χ with the properties

$$(2.1) \qquad \begin{aligned} \chi &\in C^{k-1} , \\ \chi_{[x_i,\, x_{i+1}]} &\in \mathbb{P}^{t-1} . \end{aligned}$$

Moreover we will work with the spaces $\dot{S}_h = S_h \cap H$. A finite element approximation $v_h = R_h v \in \dot{S}_h$ on the function v in (1.1) may be defined by

$$(2.2) \qquad (v_h, \chi) = (Au, \chi) \qquad \text{for } \chi \in \dot{S}_h .$$

If P_h denotes the orthogonal projection of H onto S_h then the solution of (2.2) may be written in the form

$$(2.3) \qquad v_h = P_h v = P_h Au .$$

The orthogonal projection P_h has norm 1 in H. Since A is an isometry we get at once from (2.3) the error estimate

$$(2.4) \qquad \| Au - v_h \| \le \inf \{ \| Au - \chi \| \mid \chi \in \dot{S}_h \} .$$

Corresponding error estimates in the norms of the Sobolev spaces H_k are direct consequences. Sobolev space methods are not adequate in treating the corresponding finite element counterpart of (1.2)

$$(2.5) \qquad \vartheta_h(\varphi) = \varphi + P_h A [\log\{ g(\vartheta_h(\varphi))\}]$$

iteratively. Estimates in the norms of Hölder spaces will be advantageous. Haar, A. (1910) was the first to observe that the orthogonal projection in L_2 onto step functions, i. e. onto the space $S_h^{0,1}$ has bounded norm in L_∞. The boundedness of the L_2-projection in L_∞ also in higher dimensions was proved by Douglas, J. et al. (1975), we refer also to Nitsche, J. (1969).

For the solution of (2.3) we get (here and in the following c denotes a generic numerical constant which may differ at different places)

(2.6) $$\| v_h \|_{L_\infty} \leq c \| v \|_{L_\infty}$$

resp.

(2.7) $$\| P_h \|_{L_\infty \cdot L_\infty} \leq c \quad .$$

A consequence is

> Assume $u \in C^{\tau.\lambda}$ with $\tau \leq t-1$ and $0 < \lambda < 1$. Then the error estimate is valid

(2.8) $$\| P_h v - Au \|_{L_\infty} \leq c h^{\tau+\lambda} \| u \|_{C^{\tau.\lambda}} \quad .$$

Now spline spaces admit in case of $k \geq 1$ the following direct and inverse properties:

(D) Let $w \in \dot{C}^{0.\lambda}$ be given. There exists an element $\omega \in \dot{S}_h$ according to

(2.9) $$\| w - \omega \|_{L_\infty} \leq c h^{\lambda} \| w \|_{C^{0.\lambda}} \quad ,$$
$$\| \omega \|_{C^{0.\lambda}} \leq c \| w \|_{C^{0.\lambda}} \quad .$$

(I) For $\chi \in \dot{S}_h$ the inverse relation holds true

(2.10) $$\| \chi \|_{C^{0.\lambda}} \leq c h^{-\lambda} \| \chi \|_{L_\infty} \quad .$$

We get using the argument in Nitsche, J. (1970) at once from (2.6)

(2.11) $$\| v_h \|_{C^{0.\lambda}} \leq c \| v \|_{C^{0.\lambda}}$$

or since the norm of v is bounded by that of u

(2.12) $$\| v_h \|_{C^{0.\lambda}} \leq c \| u \|_{C^{0.\lambda}} \quad .$$

Now we go back to the iterative solution of (2.5). Due to limitations in space we cannot give a detailed analysis. For short we introduce the parameter ε by

(2.13) ε is a bound of $g - 1 = g(\vartheta) - 1$ in the norm of $C^{1.\lambda}$.

In order to simplify the presentation we omit the subscript "$C^{0,\lambda}$" in the norms for the rest of this section.

For any two functions $\hat{\vartheta}(\varphi)$, $\tilde{\vartheta}(\varphi) \in C^{0,\lambda}$ we have

$$\| \log g(\hat{\vartheta}) - \log g(\tilde{\vartheta}) \| \leq c\,\varepsilon \| \hat{\vartheta} - \tilde{\vartheta} \| \quad . \tag{2.14}$$

Now let us consider the iteration procedure

$$\begin{aligned} \vartheta_h^0(\varphi) &= \varphi \quad , \\ \vartheta_h^{\nu+1}(\varphi) &= \varphi + P_h A[\log\{g(\vartheta_h^\nu(\varphi))\}] \quad \text{for } \nu \geq 0 \quad . \end{aligned} \tag{2.15}$$

Because of (2.12), (2.14) we get

$$\| \vartheta_h^{\nu+1} - \vartheta_h^\nu \| \leq c\,\varepsilon \| \vartheta_h^\nu - \vartheta_h^{\nu-1} \| \quad . \tag{2.16}$$

Applying the Banach fixpoint theorem we see: For $\varepsilon \leq \varepsilon_0 < c^{-1}$ - with c being the constant in (2.16) - the iterates ϑ_h^ν converge in the norm of $C^{0,\lambda}$ to the solution ϑ_h of (2.5). Corresponding error estimates for the difference e_h = $\vartheta - \vartheta_h$ follow from the representation

$$\begin{aligned} \vartheta - \vartheta_h &= P_h A[\log\{g(\vartheta)\} - \log\{g(\vartheta_h)\}] + \\ &\qquad (I - P_h) A[\log\{g(\vartheta)\}] \quad . \end{aligned} \tag{2.17}$$

The counterpart of (1.5_∞) is

$$\| e_h \|_{L_\infty} \leq B(\varepsilon)\, h^2 \quad . \tag{2.18}$$

Of course for spline approximations an estimate of the type (1.6) is not valid.

3.

Let $\Omega \subset \mathbb{R}^2$ be a simple connected bounded domain with boundary $\partial\Omega$ sufficiently smooth and - for simplicity - normed such that the arc length of $\partial\Omega$ equals 2π. The boundary value problem

$$(3.1) \qquad \begin{aligned} \Delta U &= 0 \quad \text{in} \quad \Omega \\ U &= v \quad \text{on} \quad \partial\Omega \end{aligned} ,$$

may be solved using a simple layer potential. Without loss of generality we assume at least $v \in H$. Let z, ζ denote complex variables and let $\zeta(s)$ be the complex parametrization of $\partial\Omega$. Further let $\Gamma(z)$ be the fundamental solution of the Laplacian defined by

$$(3.2) \qquad \Gamma(z) = -\tfrac{1}{\pi} \log |z| .$$

The function

$$(3.3) \qquad U(z) = \oint \Gamma(z-\zeta(t))\, u(t)\, dt$$

is harmonic in Ω. In order that U is the solution of the boundary value problem (3.1) the function u has to be the solution of

$$(3.4) \qquad \begin{aligned} v &= B\,u \\ v(s) &= \oint \gamma(s-t)\, u(t)\, dt \end{aligned}$$

with

$$(3.5) \qquad \gamma(s-t) = \Gamma(\zeta(s)-\zeta(t)) .$$

The Ritz method applied to (3.4) using spline spaces sometimes is called boundary finite element method.

In Plemelj, J. (1911) it was suggested to replace (3.4) by

$$(3.6) \qquad v(s) = \oint \gamma(s-t)\, du(t) .$$

The argument of Plemelj (§8 of the paper cited) is

> Bisher war es üblich, für das Potential die Form (3.4) zu nehmen. *Eine solche Annahme erweist sich aber als eine derart folgen - schwere Einschränkung, daß dadurch dem Potentiale der größte Teil seiner Leistungsfähigkeit hinweg genommen wird.* Für tiefergehende Untersuchungen erweist sich das Potential nur in der Form (3.6) verwendbar.

By partial integration we may rewrite (3.6) in the form

$$(3.7) \qquad v(s) = -\oint \partial_t \gamma(s-t)u(t)dt \quad .$$

For boundaries $\partial\Omega$ smooth we get

$$(3.8) \qquad -\partial_t \gamma(s-t) = \tfrac{1}{2\pi} \mathrm{ctg} \tfrac{1}{2}(s-t) + g(s,t)$$

with $g(\cdot,\cdot)$ corresponding regular. Thus the integral equation (3.7) has the structure

$$(3.9) \qquad v = Au + \tilde{A}u$$

with $\tilde{A}$ being an operator compact in H resp. in $C^{0,\lambda}$. In view of (1.4) the solution u of (3.9) solves

$$(3.10) \qquad u = -Av + A\tilde{A}u \quad .$$

The procedure discussed in section 2 leads to the finite element approximation defined by

$$(3.11) \qquad u_h = -P_h A v + P_h A \tilde{A} u_h \quad .$$

Since (3.6) possesses an unique solution also (3.11) is uniquely solvable for h small enough, see Schatz, A. (1974). It is worthwhile to mention that u_h defined by (3.11) may be considered as the least squares approximation on u according to (3.9).

The error estimates concerning the approximation on the solution of (3.11) are obvious and not discussed in detail here.

4.

Although in the present context there is no need for discussing the convergence of the Galerkin method for equations of the type (1.1) this question is still of interest. Let the Galerkin approximation $\varphi = G_h u \in \dot{S}_h$ on the solution u of (1 .1) be defined by

$$(4.1) \qquad (\chi, A\varphi) = (\chi, Au) \qquad \text{for } \chi \in \dot{S}_h \, .$$

In order to apply the theory developed by Babuska, I. (1971) the two properties have to hold:

Proposition 4.1: For $\psi \in \dot{S}_h$ the condition holds true with a constant m:

$$(4.2) \qquad \| \psi \| \quad \leq \quad m \cdot \sup\{ (\chi, A\psi) \mid \chi \in \dot{S}_h \wedge \| \chi \| \leq 1 \} \ .$$

Proposition 4.2: $\psi = 0$ is the consequence of $\psi \in \dot{S}_h$ and

$$(4.3) \qquad (\chi, A\psi) \quad = \quad 0 \qquad \text{for} \quad \chi \in \dot{S}_h \qquad .$$

At least for spline spaces subordinate to uniform subdivisions these conditions are valid. This leads to the boundedness of the Galerkin approximation in the norm of the space H:

$$(4.4) \qquad \| \varphi \| \quad \leq \quad m \cdot \| u \| \qquad .$$

By the argument analogue to that used in section 2 we get for spline spaces of the type $S_h^{k,t}$ with $k \geq 1$ the boundedness in H_1 and because of duality also in H_{-1}:

$$(4.5) \qquad \begin{aligned} \| \varphi \|_1 \quad &\leq \quad \bar{\bar{m}} \cdot \| u \|_1 \qquad , \\ \| \varphi \|_{-1} \quad &\leq \quad \bar{\bar{m}} \cdot \| u \|_{-1} \end{aligned}$$

with a constant $\bar{\bar{m}}$ independent of h. It is of interest and will be needed below that

Proposition 4.3: For $\psi \in \dot{S}_h$ the condition holds true

$$(4.6) \qquad \| \psi \|_1 \quad \leq \quad \bar{\bar{m}} \cdot \sup\{ (\chi, A\psi) \mid \chi \in \dot{S}_h \wedge \| \chi \|_{-1} = 1 \}$$

is equivalent to (4.5). This may be regarded as generalization of a condition given by Polskii, J (1955). (4.4), (4.5) guarantee almost best convergence of the Galerkin approximations in the norms of H_{-1}, H, and H_1.

With respect to L_∞-error estimates two results are to be mentioned. Stiller, S. (1979) treats the boundary value problem (3.1) using a double layer potential. The integral operator entering the corresponding singular integral equation of the second kind has properties similar to A in (1.1). Stiller uses step functions as approximations. The essential point in her work is a carefull analysis of the structure of the corresponding linear equations. In Rannacher, R. and Wendland, W. (1985) operator equations and the convergence in L_∞ of corresponding Galerkin approximations are discussed. In their terminology the operator A (1.1) is of order zero. Unfortunately the theory developed in the paper mentioned is based on the assumption that the underlying operator is positive definite or that at least a Garding type inequality will hold.

Analogue to the analysis in case of boundary value problems we will work with weighted norms (For details see e. g. Nitsche, J. and Wheeler, M. (1981)). Let the weight factor $\mu = \mu(s-s_0)$ be defined by

$$(4.7) \qquad \mu = h^2 + \sin^2(s-s_0)$$

with $s_0 \in [0, 2\pi)$ to be fixed later and let for $\alpha \in \mathbb{R}$ the weighted scalar products resp. norms be defined by

$$(4.8) \qquad \begin{aligned} ((v, w))_\alpha &= \oint \mu^{-\alpha} v w \, ds \quad , \\ |||v|||_\alpha &= ((v, v))_\alpha^{1/2} \quad . \end{aligned}$$

Actually we will work with $\alpha = 1$. In order to avoid any confusion with the norms in H_1 resp. H_{-1} we will still use the subscripts α resp. $-\alpha$. The connection of weighted norms and the L_∞-norm is given by the inequality (for $\alpha = 1$)

$$(4.9) \qquad |||v|||_\alpha \le c h^{-1/2} \| v \|_{L_\infty}$$

for any $v \in C^0$ on the one hand and by

$$(4.10) \qquad \| \chi \|_{L_\infty} \le c h^{1/2} \sup\{ |||\chi|||_\alpha \mid s_0 \in [0, 2\pi) \}$$

for $\chi \in \mathring{S}_h$ on the other hand. Therefore it suffices to get a bound for $\varphi = G_h u \in \mathring{S}_h$ in the α-norm. Let us introduce an auxiliary function w by

$$(4.11) \qquad Aw = \mu^{-1} \varphi$$

and let $\omega = G_h w \in \mathring{S}_h$ be its Galerkin approximation defined by:

$$(4.12) \qquad (\chi, A\omega) = (\chi, \mu^{-1} \varphi) \quad .$$

Using (4.1) we get

$$(4.13)\quad \begin{aligned} |||\varphi|||_\alpha^2 &= (\varphi, Aw) \\ &= (\varphi, A\omega) = (u, A\omega) \\ &= (u, A(\omega - w)) + (\!(u, \varphi)\!)_\alpha \quad . \end{aligned}$$

Therefore we derive

$$(4.14)\quad |||\varphi|||_\alpha^2 \leq |||u|||_\alpha \cdot |||\varphi|||_\alpha + |||u|||_\alpha \cdot |||A(w - \omega)|||_{-\alpha}$$

and further (with $\delta > 0$ arbitrary small)

$$(4.15)\quad |||\varphi|||_\alpha^2 \leq c_\delta |||u|||_\alpha^2 + \delta |||A(w-\omega)|||_{-\alpha}^2 \quad .$$

In order to analyze the second term on the right hand side of (4.15) we may rewrite it in the form

$$(4.16)\quad \begin{aligned} |||A(w-\omega)|||_{-\alpha}^2 &= (\mu(\mu^{-1}\varphi - A\omega), A(w - \omega)) \\ &= (\varphi - \mu A\omega, A(w - \omega)) \quad . \end{aligned}$$

Because of the defining relation for ω we can replace φ in (4.16) by any element $\psi \in \dot{S}_h$. In this way we get

$$(4.17)\quad \begin{aligned} |||A(w-\omega)|||_{-\alpha}^2 &= (\psi - \mu A\omega, A(w - \omega)) \\ &\leq |||A(w-\omega)|||_{-\alpha} \cdot |||\mu A\omega - \psi|||_\alpha \quad . \end{aligned}$$

From the last inequality we get at once an estimate for the second term on the right hand side in (4.15):

$$(4.18)\quad |||A(w-\omega)|||_{-\alpha} \leq \inf\{\, |||\mu A\omega - \psi|||_\alpha \mid \psi \in \dot{S}_h \,\} \quad .$$

In view of extensions to higher dimensions to be published elsewhere we denote the derivative w' of a function $w = w(s)$ subsequently by ∇w. By standard estimates we get from (4.18)

$$(4.19)\quad |||A(w-\omega)|||_{-\alpha} \leq c h |||\nabla(\mu A\omega)|||_\alpha \quad .$$

Our aim is to prove the inequality

$$(4.20)\quad |||\varphi|||_\alpha \leq c |||u|||_\alpha$$

which leads because of (4.9), (4.10) to the final estimate

(4.21) $$\| G_h u \|_{L_\infty} \leq c \| u \|_{L_\infty} .$$

Because of (4.15), (4.19) it is sufficient to prove

(4.22) $$|||\nabla(\mu A\omega)|||_\alpha \leq c h^{-1} ||| \varphi |||_\alpha .$$

It is

(4.23) $$\nabla(\mu A\omega) = (\nabla\mu)\cdot A\omega + \mu\cdot\nabla(A\omega) .$$

Therefore we get - remembering $\alpha = 1$

(4.24) $$\begin{aligned} |||\nabla(\mu A\omega)|||_\alpha &\leq c\|A\omega\| + c|||\nabla(A\omega)|||_{-\alpha} \\ &\leq \mathfrak{J}_1 + \mathfrak{J}_2 . \end{aligned}$$

$\mathfrak{J}_1$ may be estimated in the way

(4.25) $$\begin{aligned} \mathfrak{J}_1 &\leq c\| A\omega \| = c\| \omega \| \\ &\leq c\| w \| = c\| Aw \| \\ &\leq c\| \mu^{-1}\varphi \| \leq ch^{-1} ||| \varphi |||_\alpha . \end{aligned}$$

Next we write $\mathfrak{J}_2$ in the form (with the abbreviation $\sigma = \sigma(s) = \sin(s-s_0)$)

(4.26) $$\begin{aligned} \mathfrak{J}_2^2 &= ch^2\| \nabla(A\omega) \|^2 + c\| \sigma\nabla(A\omega) \|^2 \\ &= \mathfrak{J}_3^2 + \mathfrak{J}_4^2 . \end{aligned}$$

Parallel to (1.3) for any function $z \in H_1$

(4.27) $$\| \nabla(Az) \| = \| A\nabla z \| = \| \nabla z \|$$

holds true. Thus we get

(4.28) $$\mathfrak{J}_3 = ch\| \nabla\omega \| = ch\| \omega \|_1$$

and using inverse properties of the approximation spaces $\mathring{S}_h$

(4.29) $$\mathfrak{J}_3 \leq c\| \omega \| .$$

Thus $\mathfrak{J}_3$ can be estimated like $\mathfrak{J}_1$. Because of

(4.30) $$\sigma\nabla(A\omega) = \nabla(\sigma A\omega) - \cos(\cdot - s_0)A\omega$$

we get

$$(4.31)\qquad \begin{aligned} \mathfrak{J}_4 &\leq c\|\nabla(\sigma A\omega)\| + \|A\omega\| \\ &\leq \mathfrak{J}_5 + \mathfrak{J}_6 \end{aligned} .$$

The second term $\mathfrak{J}_6$ is bounded like $\mathfrak{J}_1$, $\mathfrak{J}_3$. Since

$$(4.32)\qquad [\sigma, A] = \sigma A - A\sigma$$

is a bounded operator in L_2 and thus $\|[\sigma, A]\omega\|$ is bounded like $\mathfrak{J}_1$ it remains to analyze

$$(4.33)\qquad \begin{aligned} \mathfrak{J}_5 &= c\|\nabla(A\sigma\omega)\| \\ &= c\|\sigma\omega\|_1 \end{aligned} .$$

At this place one remark is worthwhile to be made. Besides the estimate (4.10) all the derivations, estimates etc. of this section are independent of the structure of the underlying approximation spaces. In order to estimate $\mathfrak{J}_5$ we will make use of a certain super-approximability property typically for splines resp. finite elements (see Nitsche, J. and Schatz, A (1972)).

To any $\psi \in \dot{S}_h$ there exists an element $\tilde{\psi} \in \dot{S}_h$ according to

$$(4.34)\qquad \|\sigma\psi - \tilde{\psi}\|_k \leq ch\|\psi\|_k \qquad \text{for } k = -1, 0, 1 .$$

Now let $\tilde{\omega}$ be an approximation on $\sigma\omega$ according to (4.34). Then we get

$$(4.35)\qquad \begin{aligned} c\|\sigma\omega\|_1 &\leq c\|\tilde{\omega}\|_1 + ch\|\omega\|_1 \\ &\leq \mathfrak{J}_7 + \mathfrak{J}_8 \end{aligned} .$$

The term $\mathfrak{J}_8$ is bounded like $\mathfrak{J}_3$. Now we make use of Proposition 4.3. Let $\psi \in \dot{S}_h$ be such that

$$(4.36)\qquad c\|\tilde{\omega}\|_1 \leq c(\psi, A\tilde{\omega})$$

and normed by $\|\psi\|_{L_1} = 1$. We may replace $\tilde{\omega}$ on the right hand side leading to

$$(4.37)\qquad \begin{aligned} c\|\tilde{\omega}\|_1 &\le c(\psi, A\sigma\omega) + ch\|\omega\|_1 \\ &\le \mathfrak{J}_9 + \mathfrak{J}_{10} \end{aligned}$$

$\mathfrak{J}_{10}$ corresponds $\mathfrak{J}_8$. Next we rewrite $\mathfrak{J}_9$ in the form

$$(4.38)\qquad \begin{aligned} (\psi, A\sigma\omega) &= (\sigma\psi, A\omega) - (\psi, [\sigma, A]\omega) \\ &= \mathfrak{J}_{11} + \mathfrak{J}_{12} \end{aligned}$$

$\mathfrak{J}_{12}$ may be estimated like $\mathfrak{J}_1$. With $\tilde{\psi}$ chosen according to (4.34) we get because of (4.12)

$$(4.39)\qquad \begin{aligned} (\sigma\psi, A\omega) &= (\sigma\psi - \tilde{\psi}, A\omega) + (\tilde{\psi}, A\omega) \\ &= (\sigma\psi - \tilde{\psi}, A\omega) + (\tilde{\psi}, Aw) \\ &= (\sigma\psi - \tilde{\psi}, A\omega) - (\sigma\psi - \psi, \mu^{-1}\varphi) + (\sigma\psi, \mu^{-1}\varphi) . \end{aligned}$$

Thus $\mathfrak{J}_9$ is bounded by

$$(4.40)\qquad \mathfrak{J}_9 \le ch\|\omega\|_1 + ch\|\mu^{-1}\varphi\|_1 + \|s\mu^{-1}\varphi\|_1 .$$

The first term on the right hand side is covered by $\mathfrak{J}_3$. Because of

$$(4.41)\qquad |\nabla(\mu^{-1}\varphi)| \le \mu^{-1}|\nabla\varphi| + 2\mu^{-3/2}|\varphi|$$

we get

$$(4.42)\qquad \begin{aligned} \|\mu^{-1}\varphi\|_1 &\le |||\nabla\varphi|||_{\alpha+1} + 2|||\varphi|||_{\alpha+2} \\ &\le ch^{-2}|||\varphi|||_\alpha \end{aligned}$$

The last term in $\mathfrak{J}_9$ can be estimated in the same way. Putting together the above estimates we have proven (4.22) and hence (4.21).

By the argument given in section 2 we have the final result:

The Galerkin operator G_h defined by (4.1) is bounded in the norm of $C^{0,\lambda}$.

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