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Three-dimensional manifolds, Kleinian groups and Hyperbolic geometry

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This paper is the written version of a talk that Thurston gave at the AMS Symposium on the Mathematical Heritage of Henri Poincaré held at Bloomington in April 1980. It is a Research announcement.[1]

In his "Analysis Situs" paper, Poincaré introduces simplicial homology and the fundamental group, with the aim of finding invariants which could distinguish 3-manifolds. He concludes the Vth complement [88] discussing the question: Is any simply connected closed 3-manifold homeomorphic to the 3-sphere? This question, often understood as a conjecture, motivated much of the development of 3-dimensional topology. At the beginning of his paper Thurston proposes, "fairly confidently", his Geometrization Conjecture: *The interior of any compact 3-manifold has a canonical decomposition into pieces that carry a geometric structure.* By Kneser and Milnor any orientable compact 3-manifold is homeomorphic to the connected sum of manifolds which are either irreducible or homeomorphic to a sphere bundle over S^1. By Jaco-Shalen and Johannson any orientable irreducible 3-manifold can be decomposed along a disjoint union of embedded tori into pieces that are atoroidal or Seifert-fibered (see in [15] the corresponding statement for non-orientable manifolds). Saying that a manifold carries a *geometric structure* means that it has a complete Riemannian metric which is locally modelled on one of the eight geometries (which are introduced in Section 4 of the paper): the constant curvature ones $\mathbb{H}^3$, $\mathbb{S}^3$, $\mathbb{R}^3$, the Seifert-fibered ones $\widetilde{\mathrm{PSL}(2,\mathbb{R})}$, $\mathbb{H}^2 \times \mathbb{R}$, $\mathbb{S}^2 \times \mathbb{R}$, Nil and Sol. The pieces which are referred to in the Geometrization Conjecture are essentially the pieces of the Jaco-Shalen-Johannson decomposition; more precisely, all the pieces except that one needs to keep undecomposed any component of the Kneser-Milnor decomposition which is a torus bundle over S^1 with Anosov monodromy, since those bundles carry a Sol-geometry. This conjecture contains the Poincaré Conjecture as a very special case. But the main difference with the Poincaré Conjecture is that the Geometrization Conjecture proposes a global vision of *all* compact 3-manifolds. Furthermore, the geometric structure on a given 3-manifold when it exists is often unique, up to isometry; consequently, all the Riemannian invariants of the geometric structure on a 3-manifold like the volume, the diameter, the length of the shortest closed geodesic, the Chern-Simons invariant, etc. are then topological invariants.

The Geometrization Conjecture was proven in 2003 by Grigori Perelman, who followed an analytical approach that had been initiated by Richard Hamilton in [44]. This approach was entirely different from that of Thurston, but there is no doubt

[1]Before 1978 the Bulletin of the AMS did not accept Research announcements longer than 100 lines. For this reason, Thurston could not submit there his preprint about surface diffeomorphisms [102] which dates back to 1976 and which is written in the same informal style as this one. After a couple of refusals, he stopped trying to publish it and expected until 1988 to do so (see the introduction to [102]).

that the results announced in the present paper contributed to give a very strong evidence for the truth of the Geometrization Conjecture, and therefore for the truth of the Poincaré Conjecture (in the introduction of [73], John Morgan points out this apparent paradox that a conjecture might look less plausible than another one which is much stronger).

Among the results announced in this paper which gave strong evidence for the validity of the Geometrization Conjecture, the most important is certainly the Hyperbolization Theorem for Haken manifolds (Theorem 2.5): *Any Haken 3-manifold which is homotopically atoroidal is hyperbolic, except the particular case of the twisted I-bundle over the Klein bottle*. One corollary is that the Geometrization Conjecture is true for all Haken manifolds, and in particular for any irreducible 3-manifold with non-empty boundary (once accepted the Hyperbolization Theorem, the proof of this corollary reduces to checking that Seifert fibered spaces and T^2-bundles over the circle are geometric too).

Haken manifolds form a large class of 3-manifolds for which the Geometrization Conjecture is true. However, as Thurston writes after the statement of the Hyperbolization Theorem, even if the incompressible surface is an essential tool in the proof of the Hyperbolization Theorem for Haken manifolds, this surface seems to have little to do with the existence of the hyperbolic metric. And then he discusses the manifolds obtained by Dehn surgery on a 3-manifold with boundary M when the interior of M carries a finite volume hyperbolic metric. Theorem 2.6, *the Hyperbolic Dehn Surgery Theorem*, states that for all surgery data, except those involving a finite set of slopes on each boundary component, the resulting 3-manifold is hyperbolic. Furthermore, few of the resulting manifolds are Haken: Thurston had shown this when M is the figure-eight knot complement [98], and Hatcher-Thurston had shown this when M is a 2-bridge knot complement [43] (a paper which circulated as a preprint as early as 1979). So these manifolds form a new family of hyperbolic manifolds which are not covered by the Hyperbolization Theorem.

Thurston explains that he found using a computer program that most Dehn surgeries on certain punctured tori bundles over the circle were hyperbolic and that he constructed then by hand geometric structures for all the other Dehn surgeries. He insists on the beauty of the geometric structures "when you learn to see them" that is often revealed by computer pictures.

The following section "Applications" describes some properties of Haken manifolds and of hyperbolic manifolds.

– When M is Haken and atoroidal, the group of isotopy classes of homeomorphisms of M is finite and it lifts to a group of homeomorphisms of M.

– The fundamental group of a Haken manifold is *residually finite*.

– Theorem 3.4: the Gromov-Thurston theorem on the behavior of volume under a non-zero degree map between hyperbolic manifolds $M \to N$: $\mathrm{vol\,M} \geq \mathrm{degree} \cdot \mathrm{vol\,N}$ with equality only when the covering map is homotopic to a regular cover.

– Theorem 3.5: the Jørgensen-Thurston theorem that says that the set of volumes of hyperbolic manifolds is well-ordered. In particular, there is a finite volume manifold with smallest volume.

– Thurston suggests that the volume and the eta invariant of a hyperbolic 3-manifold should be considered as the real and imaginary parts of a complex number.

Then Thurston describes concrete approaches "to make the calculation of hyperbolic structures routine" and he explains how Robert Riley used computers for constructing hyperbolic structures [90]. Riley was maybe the first person who tried to "hyperbolize" a given topological 3-manifold, namely the complement in $\mathbb{S}^3$ of certain knots (see [91]).

The last section addresses two theorems which are directly related to the proof of the hyperbolization of Haken manifolds. Theorem 5.7, *the Double Limit Theorem*, is the main step of the hyperbolization for manifolds which fiber over the circle; it gives a condition for a sequence (ρ_i) of quasi-Fuchsian groups to contain a subsequence which converges up to conjugacy in the space of representations of $\pi_1(S)$ in $\mathrm{PSL}(2,\mathbb{C})$. The condition, formulated in terms of the Ahlfors-Bers coordinates of ρ_i is that the two coordinates diverge to laminations whose reunion fills up the surface. This theorem led to beautiful objects, in particular to *the Cannon-Thurston map* which is a sphere-filling curve $S^1 \to \partial\mathbb{H}^3$ which conjugates the action of the surface group on S^1 as a Fuchsian group to its action on $\partial\mathbb{H}^3$ as the limiting Kleinian group; furthermore this map can be described in geometric terms. Theorem 5.8 is *the Compactness Theorem of the space of discrete and faithful representations into* PSL(2,$\mathbb{C}$) *of the fundamental group of acylindrical 3-manifolds*, one of the key ingredients of the proof in the non-fibered case.

The paper contains neither proofs nor hints of proofs. But the Princeton lecture notes [98] were circulating widely since the end of the 70's. These notes were not aimed to give a proof of the Hyperbolization Theorem, but they did contain important material which is relevant to the proof (in particular in the notoriously difficult Chapters 8 and 9 which discuss algebraic versus geometric convergence of hyperbolic 3-manifolds). Also Thurston had already lectured about the Hyperbolization Theorem at several conferences (see [99] and [100]). The first versions of [101] (proving the Compactness Theorem 5.8 above) and [103] (the Double Limit Theorem 5.7) were circulating since 1980. The preprint [104] which contains a refinement of the main result of [101] was distributed later, in 1986. In 1982, the most detailed presentation of the proof of the Hyperbolization Theorem in the non-fibered case, written by John Morgan, was published in the Proceedings of the Smith Conjecture conference [72], a conference held in 1979; this presentation follows the proof described in [100]. Dennis Sullivan had reported about the proof in the fibered case at the Bourbaki seminar in 1980 [97].

The Hyperbolization Theorem for Haken manifolds was sometimes named *the Monster Theorem*, because of the length of the proof and the unusual amount of different techniques that were required. It became a challenge to find other approaches. John Hubbard observed a deep relation between the main step of the proof for the non-fibered case – *the fixed point theorem for the skinning map* – and a conjecture of Irwin Kra in Teichmüller theory saying that the classical Theta operator acting on holomorphic quadratic differentials is a contraction for the L^1-norm. This conjecture was solved by Curt McMullen who showed how to apply it to the fixed point problem [61], [62] (see also [84]). A proof of the fibered case is contained in [83]; it is based on a proof of the Double Limit Theorem, different from Thurston's and uses $\mathbb{R}$-trees instead of the delicate "Uniform injectivity of doubly incompressible pleated surfaces" Theorem (see also [50]).

Thurston concludes his paper with a list of problems about 3-manifolds and about Kleinian groups which contains many of the most difficult problems of these fields that had already been posed. Any mathematician can appreciate the importance and the impact of this paper through the fact that among these 24 problems, 22 were solved by 2012 (and indeed the two remaining ones, Problem 19 – on properties of arithmetic hyperbolic manifolds – is more a research theme than a problem and Problem 23 – on the rational independence of volumes of hyperbolic manifolds – leads to difficult conjectures in number theory). Any topologist knows how much the new perspective presented in this paper influenced and inspired research in 3-dimensional topology, in hyperbolic geometry and even much beyond: it contributed in a lot of ways to the developments of geometric group theory, of complex dynamics, to the study of spaces of representations, to the study of the Weil-Petersson geometry of Teichmüller space etc. Reporting about these influences is not the purpose of this review and I will end by trying to describe each problem and, for all but two, by giving references to the papers which brought the solution. As the copious references to be found in Zentralblatt or in MathSciNet testify, this work has inspired a vast literature. For reasons of space, I have been unable to cite many important papers which solved significant intermediate steps.

The list of problems

I have classified the problems according to the following themes: Geometrization of 3-manifolds and of 3-orbifolds, Topology of 3-manifolds, Kleinian groups, Subgroups of the fundamental groups of 3-manifolds, Computer programs and tabulations, Arithmetic properties of Kleinian groups.

It appeared that there were connections between certain of the problems. For the exposition, I followed the chronological order of the solutions within a given theme, rather than the order in which Thurston presented the problems.

Geometrization of 3-manifolds and of 3-orbifolds: Problems 1, 3.

1. *The Geometrization Conjecture for* 3*-manifolds.*

This conjecture is proven by Perelman in [85, 86, 87]. The content of these preprints is explained with details by Bruce Kleiner and John Lott in [55]. John Morgan and Gang Tian give complete proofs of the Poincaré Conjecture and of the geometrization of 3-manifolds with finite fundamental groups in [74]. A complete proof of the Geometrization Conjecture for arbitrary 3-manifolds, including a simplification of the original argument (but assuming the Hyperbolization Theorem for Haken manifolds) is contained in [9].

3. *The Geometrization Conjecture for* 3*-orbifolds.*

Thurston makes the comment that this problem contains Problem 2 on the classification of finite group actions on 3-manifolds. Indeed the language of orbifold gives a way to "encode" non-free actions of finite groups (but it does cover much more examples). If G is a finite group of diffeomorphisms acting on a 3-manifold M, the quotient space M/G is naturally the underlying space of an orbifold whose singular locus is the image of the set of points in M which are fixed by some non-trivial element in G and labeled by the isotropy groups. Showing that the action of G on M is geometric is equivalent to showing that this orbifold is geometric.

The Geometrization Conjecture for 3-orbifolds can be formulated in a way close to the Geometrization Conjecture for 3-manifolds [15]. In a footnote added in proof, Thurston announces that he has proven this conjecture when the singular locus of the orbifold has dimension at least 1: this is the *Orbifold Theorem*. Thurston did not write anything about the Orbifold Theorem comparable to what he wrote about the Hyperbolization Theorem, although he planned to do so in 1986 (cf. the Introduction to [101]). In lectures at Durham in 1984, he outlined his proof. A complete proof of the Orbifold Theorem (for orientable orbifolds) is given by Michel Boileau, Bernhard Leeb and Joan Porti in [11]; previously Boileau and Porti had solved the case when the singular locus is a 1-dimensional submanifold [13]. The proofs in both papers differ from Thurston's at the delicate point of recognizing Seifert fibered pieces: those pieces are identified after a simplicial volume computation, using Gromov's vanishing theorem. Another reference, closer to Thurston's original proof, is [32].

Topology of 3-manifolds: Problems 2, 4, 24.

2. *Is any finite group action on a (geometric)* 3*-manifold equivalent to an isometric action?*

This problem is a broad extension of *the Smith conjecture*: *Any orientation preserving periodic diffeomorphism of* $\mathbb{S}^3$ *is conjugated to an orthogonal rotation* solved in [71] (see in particular the history at the end of [94]). When the geometry on M is $\widetilde{\mathrm{PSL}}(2,\mathbb{R})$, $\mathbb{H}^2 \times \mathbb{R}$, Nil, $\mathbb{R}^3$ and Sol, Problem 2 is solved by Williams Meeks and Peter Scott in [65] using minimal surfaces and topological techniques. The case of the $\mathbb{S}^2 \times \mathbb{R}$ geometry is solved by Meeks and Yau in [66], except for actions of the alternating group A_5. The case when the geometry is $\mathbb{S}^3$ or $\mathbb{H}^3$ is solved by Jonathan Dinkelbach and Bernhard Leeb in [35] by making equivariant Perelman's proof; it is important to observe that this more recent result applies to an *arbitrary* finite group action on $\mathbb{S}^3$ or $\mathbb{H}^3$ whereas those (non-orientable) actions which have only isolated fixed points cannot be covered by the Orbifold theorem.

4. *Develop a global theory of hyperbolic Dehn surgery.*

Let M be a finite volume hyperbolic 3-manifold with k cusps; choose disjoint horoball neighborhoods of these cusps which are bounded by tori denoted T_i. If for $i = 1, \dots, k$, s_i is *a slope*, i.e., a non trivial isotopy class of simple closed curves on T_i, $M(s_1, \dots, s_k)$ denotes the manifold obtained from M by Dehn filling along $s = (s_i)$. The *exceptional set* is the set of the s's such that $M(s)$ is not hyperbolic. Ian Agol and Mark Lackenby find independently conditions on the slopes s_i that imply that $M(s)$ is irreducible with a *Gromov hyperbolic* fundamental group ([1], [56]). When ∂M is connected, Lackenby and Meyerhoff prove that the exceptional set contains at most 10 slopes [58], as it was conjectured by Cameron Gordon; the figure-eight knot has 10 exceptional slopes ([98], §4) and is still conjectured to be the only knot with this property. A different approach to a quantitative version of the Dehn Surgery Theorem is due to Craig Hodgson and Steve Kerckhoff using deformations of hyperbolic cone-manifolds ([45], [46]). When ∂M is connected, other important universal properties of the exceptional set are also the *Cyclic Dehn Surgery Theorem* of Mark Culler, Cameron Gordon, John Luecke, and Peter Shalen

[34] which says that there are at most three slopes such that $M(s)$ has cyclic fundamental group, and the *Finite Dehn Surgery Theorem* of Steve Boyer and Xingru Zhang which says that there are at most six slopes such that $M(s)$ has a finite fundamental group [22].

24. *Show that most* 3*-manifolds with Heegaard diagrams of a given genus have hyperbolic structures.*

By the Geometrization Theorem, this problem amounts to show that most 3-manifolds with Heegaard diagrams of a given genus are irreducible and atoroidal. These properties can be checked on the curve complex of the Heegaard surface in terms of the distance between the two subcomplexes which are respectively generated by the meridian systems of each splitting [44]. But Thurston had maybe in mind a more concrete description of the hyperbolic metric, closer to the description that comes with the Hyperbolic Dehn Surgery Theorem. In [75] Hossein Namazi and Juan Souto consider 3-manifolds obtained by glueing 2 copies of a handlebody by a power ϕ^n of a pseudo-Anosov diffeomorphism which satisfies some genericity assumption. For all sufficiently high powers n, they construct a negatively curved Riemannian metric on the resulting manifold with curvature arbitrarily close to -1. Also in this case the Geometrization Theorem implies that those manifolds are hyperbolic. However, the advantage of the approach in [75] is to show the quasi-isometry type of the hyperbolic metric in terms of ϕ and n.

Kleinian groups: Problems 5–14.

In the 60's, Lipman Bers constructed an embedding of Teichmüller space as a bounded domain in $\mathbb{C}^n$. He deduced using a topological argument that most points in the frontier of this embedding did correspond to Kleinian groups which are *degenerate*, in the sense that they are not *geometrically finite* (see also [40]). Explicit examples of degenerate groups had been given by Troels Jørgensen in [47] (see also [48], a preprint from around 1975 which describes the fundamental domains for doubly degenerate once punctured torus groups). But before Thurston, no general geometric properties of degenerate Kleinian groups had been established. For proving the Hyperbolization Theorem, Thurston needed to consider those Kleinian groups and he began their study. Problems 5 to 14 address mostly those groups.

5. *Are all Kleinian groups geometrically tame?*

Let G be a finitely generated Kleinian group with infinite covolume that we will also suppose without parabolic elements for this exposition. By a Theorem of Peter Scott, the quotient manifold $N = \mathbb{H}^3/G$ has *a compact core*, i.e.,, a codimension 0 submanifold M such that the inclusion $M \hookrightarrow N$ is a homotopy equivalence. Each component of ∂M separates the interior of M from an *end* of N. The groups studied by Thurston in §8 and §9 of [98] have the following two properties: (1) M is boundary-incompressible (i.e., G does not split as a free product or as an HNN extension over the trivial group) and (2) G is a limit of *geometrically finite groups* (i.e., the inclusion homomorphism $G \to \mathrm{PSL}(2,\mathbb{C})$ is a limit of group embeddings $\rho_i : G \to \mathrm{PSL}(2,\mathbb{C})$ such that for all i, the manifold $\mathbb{H}^3/\rho_i(G)$ has a compact core with convex boundary). A consequence of (1) is that for any boundary component S of ∂M bounding an end E, the inclusion $S \hookrightarrow E$ is a homotopy equivalence. Thurston introduced the notion of *pleated surface* (also called *uncrumpled surface* in [98], §8). He showed that when G satisfies (2), then each end of N is *geometrically tame*, i.e., either E is geometrically finite, or there exists a sequence of pleated surfaces $f_i : S \to E$ homotopic to the inclusion map $S \hookrightarrow E$ and such that the maps f_i tend to ∞. The existence of this sequence has two important consequences. First N is *topologically tame*: it is diffeomorphic to the interior of a compact 3-manifold. Second, N is *analytically tame*, a property that implies in particular that G satisfies *the Ahlfors conjecture* which says that the Lebesgue measure of the limit set of any finitely generated Kleinian group is 0 or 1; even more, analytical tameness implies that the action of G on its limit set is ergodic when this limit set has full measure. The notion of "geometric tameness" which appears in Problem 5 is this one. As indicated before, Thurston had solved this problem for the Kleinian groups which satisfy (1) and (2) in [98].

The first breakthrough on Problem 5 is due to Francis Bonahon. He shows in [14] that any finitely generated Kleinian group is geometrically tame when it satisfies property (1). In fact, his theorem applies to groups G which satisfy a property weaker than (1), which requires that no parabolic element of G is conjugated to an element which is contained in a factor of a decomposition of G as a free product or as an HNN extension over the trivial group. One of the tools that Bonahon introduced in his proof are *the geodesic currents*; these are the transverse measures to the geodesic foliation on the unit tangent bundle of S. Those objects had been much studied before, in particular in the context of Anosov flows, but they were considered by Bonahon as a natural generalization of the notion of measured geodesic laminations.

The next important progress on Problem 5 after Bonahon is due to Richard Canary: he shows in his thesis, using a beautiful branched covering trick, that, when $N = \mathbb{H}^3/G$ is the interior of a compact 3-manifold, then it is geometrically tame [27]. In the 70's, Al Marden had conjectured that for any finitely generated Kleinian group G, $\mathbb{H}^3/G$ is *topologically tame*, meaning that it is the interior of a compact manifold [64]; this *Marden tameness conjecture* is Problem 9 from Thurston's list. Therefore, if the Marden conjecture is true, it will follow from Canary's theorem that any Kleinian group is geometrically tame.

9. *Are all Kleinian groups topologically tame?*

The Marden conjecture was proven in 2004 independently by Ian Agol [2] and by Danny Calegari and David Gabai [25] following distinct approaches. Teruhiko Soma gave a simplification of the argument of Calegari-Gabai [95] making use of the notion of "disk-busting" that Agol used. Brian Bowditch exposes a self-contained proof in [23] which simplifies the original approach and avoids in particular the use of the "end-reductions" as in the previous references. Canary surveys the history and applications of the tameness theorem in [30].

10. *The Ahlfors measure* 0 *problem.*

Lars Ahlfors showed in [5] that if G is a geometrically finite Kleinian group, then its limit set either has measure 0, or is equal to the whole sphere and he conjectured that the same holds for any finitely generated Kleinian group. Thurston had shown in [98] that when a Kleinian group is geometrically tame and indecomposable, then it satisfies the Ahlfors conjecture (and furthermore the action of G on the boundary of $\mathbb{H}^3$ is ergodic when its limit set has full measure). In [27] Canary shows how to adapt Thurston's argument to the decomposable case. Therefore the Ahlfors conjecture follows from the positive answer to the geometric tameness question (Problem 5) which follows from the truth of the Marden conjecture (Problem 9).

11. *Classify geometrically tame representations of a given group.*

This problem is the *Ending Lamination Conjecture*. It comes with

12. *Describe the quasi-isometry type of a given group.*

Let $N = \mathbb{H}^3/G$ be an hyperbolic 3-manifold with indecomposable fundamental group and let $M \hookrightarrow N$ be a compact core. When N is geometrically tame, Thurston defined *end invariants* which retain geometric information about the ends of N. By Bonahon's theorem, his definition applies to any hyperbolic manifold with indecomposable fundamental group. For simplicity, suppose again that G has no parabolic elements: then each component of ∂M faces exactly one end of N. If the end E facing the component S is geometrically finite, one classical invariant of E is the *conformal structure at infinity*, which is an element of the Teichmüller space of S. When the (geometrically tame) end facing S is not geometrically finite, Thurston defined an *ending lamination*: it is a measured geodesic lamination on S, well-defined up to the transversal measure. Problem 11 asks wether this set of invariants, the elements in Teichmüller space and the ending laminations are sufficient to reconstruct N. When N has indecomposable fundamental group, the problem can be reduced to the case when N has the homotopy type of a closed surface. Minsky solves the case when N has *bounded geometry* (the length of the closed geodesics is bounded from below by a positive constant) in [68]. His proof provides also a *geometric model* for N, i.e., a metric space constructed directly from the end invariants which is quasi-isometric to N (this is the "formula" sought for in Problem 12): in the *doubly degenerate case*, this model is the metric on $S \times \mathbb{R}$ such the metric on the slice $S \times \{t\}$ describes a Teichmüller geodesic between the two ending laminations. Understanding the case in the presence of short geodesics required a lot of efforts and the development of new techniques; one can say that much of the study from the geometric view point of the *curve complex of a surface* that was initiated by Howard Masur and Yair Minsky was motivated by this problem. In [69], Minsky shows that the property that N has bounded geometry in one end can be read on the ending lamination. In [70], he introduces a new model, which depends on the end invariants. This model is a metric on $S \times \mathbb{R}$ which predicts (in terms of the geodesic in the curve complex joining the end invariants) which should be the short geodesics of N and where they should be located; Minsky constructs also a map from this model to N which is Lipschitz on the thick part. In [20] Jeff Brock, Canary and Minsky show that this model is biLipschitz equivalent to N; they announce also the same result (the existence of a model) for any finitely generated Kleinian group. Another approach to the same result is also explained in [24]. Therefore, if two hyperbolic manifolds have the same end invariants, they are biLipschitz equivalent and then by Sullivan's No Invariant Line Fields Theorem, they are isometric.

6. *Is every Kleinian group a limit of geometrically finite groups?*

Let G be a finitely generated Kleinian group. The space of representations of G into $\mathrm{PSL}(2, \mathbb{C})$ that send each parabolic element to a parabolic element is an affine algebraic set; the faithful representations with discrete image form a closed subset $\mathcal{DF}(G)$ of this space. The geometrically finite representations without accidental parabolics form an open subset $\mathcal{GF}(G) \subset \mathcal{DF}(G)$. The question asks if this subset is dense in $\mathcal{DF}(G)$. When G is a surface group and for representations contained in a *Bers slice*, this had been conjectured by Bers in [7] and it was solved by Kenneth Bromberg in [19] using cone manifolds technics. A positive answer to Thurston's question in full generality is given independently by Hossein Namazi and Juan Souto in [76] and by Ken'ichi Ohshika in [80]; previously Brock and Bromberg solve the case of indecomposable groups without parabolic elements in [21] (see also [20]). The proofs in [76] and [80] depend on the Ending Lamination Theorem (Problem 11). By this theorem, any hyperbolic 3-manifold is determined up to isometry by its topology and by its end invariants. Therefore it suffices to show that for any $\rho \in \mathcal{DF}(G)$, there is a representation ρ_∞ in the closure of $\mathcal{GF}(G)$ which has the same end invariants. The existence of such a ρ_∞ comes from a generalization of the Double Limit Theorem. A different solution to Problem 6, which does not use the full strength of the Ending Lamination Theorem, has been announced by Bromberg and Souto.

8. *Analyse limits of quasi-Fuchsian groups with accidental parabolics.*

One important and difficult step of Thurston's original proof of the Hyperbolization of Haken manifolds is the theorem which says that when a sequence of geometrically tame representations into $\mathrm{PSL}(2, \mathbb{C})$ of the fundamental group of a 3-manifold with incompressible boundary which preserves the type of the elements and converges algebraically, then it converges also geometrically (up to possibly extracting a subsequence) if the limit representation has *no accidental parabolics*. This last hypothesis is really necessary since Kerckhoff and Thurston give an example of a sequence of quasi-Fuchsian representations of a punctured torus group which converges algebraically but such the geometric limit contains a $\mathbb{Z}+\mathbb{Z}$ parabolic subgroup [54]. The geometric limit of an algebraically converging sequence of quasi-Fuchsian groups can even be non finitely generated and for many reasons ([16], [18]). Ohshika and Soma have announced a complete description of the topological type of the limit hyperbolic manifolds, and also a classification up to isometry of those limits in the spirit of the Ending Lamination Theorem [81].

7. *Develop a theory of Schottky groups and their limits.*

In [98], Thurston studies Kleinian groups which satisfy properties (1) and (2) (cf. the discussion of Problem 5), in particular surface groups which are limits of quasi-Fuchsian groups. For those groups he defines the end invariants in §8 and he shows in §9 the theorem mentioned in the last paragraph: “algebraic convergence without accidental parabolics implies geometric convergence”. Problem 7 is to develop a similar study for the Kleinian groups which are limits of *Schottky groups*, i.e., limits of geometrically finite free groups.

Brock, Canary and Minsky announce in [20] that using Canary's work on Problem 5, the solution of Marden's conjecture (Problem 9) and extending their work on the Ending lamination Theorem in the boundary incompressible case, they have constructed biLipschitz models for the geometry of $\mathbb{H}^3/G$ when G is an arbitrary finitely generated Kleinian group; they show indeed that each end of $\mathbb{H}^3/G$ is quasi-isometric to an end of a degenerate surface group (with quasi-isometric constants depending on G).

Jørgensen conjectured that for any finitely generated Kleinian group G, a sequence of representations $\rho_i : G \to \mathrm{PSL}(2,\mathbb{C})$ which converges algebraically to a representation ρ without accidental parabolics converges also geometrically to ρ up to extracting a subsequence. This is proven by Jim Anderson and Canary in [6] when the domain of discontinuity of ρ is non empty. When the domain of discontinuity is empty, then any end of $\mathbb{H}^3/\rho(G)$ is degenerate; it follows from the tameness of the algebraic limit and from the Thurston-Canary covering theorem [29] that ρ has finite index in the limit ρ_∞ of any subsequence which converges geometrically, and therefore $\rho = \rho_\infty$.

One could also interpet this problem as Thurston asking for a generalization of the Double Limit Theorem to the context of Schottky groups. He had conjectured a boundedness criterion for a sequence of geometrically finite representations of a free group that he formulated in terms close to those of the Double Limit Theorem but involving laminations in the *Masur domain*. First attempts to prove this conjecture are [82] and [54] (see also [28]). In its most general form, when the free group is replaced by an arbitrary finitely generated Kleinian group, the conjecture is solved by Namazi-Souto in [76] and by Ohshika in [80] (cf. also the announcement by Ingkang Kim, Cyril Lecuire and Ohshika of a slightly more general result in [52]).

13. *If the Hausdorff dimension of the limit set of a Kleinian group is* < 2*, is it geometrically finite?*

Bishop and Jones give a positive answer to this question using analytical tools in [10]. Another solution can be deduced from the geometric tameness of finitely generated Kleinian groups (which was proven after [10]). Canary observes in [26] that if N is a geometrically tame hyperbolic 3-manifold which is not geometrically finite, its Cheeger constant is 0 and therefore $\lambda_0(N)$, the lowest eigenvalue of the Laplacian on N is 0. By a result of Sullivan, for any hyperbolic 3-manifold $N = \mathbb{H}^3/G$, $\lambda_0(N) = \delta_G(2-\delta_G)$, when *the critical exponent* δ_G of G is ≥ 1 or $\lambda_0(N) = 1$. Therefore $\delta_G = 2$ when G is tame and not geometrically finite. By another result from [10] (true for any non-elementary discrete group of isometries of $\mathbb{H}^n$, for any dimension) δ_G equals the Hausdorff dimension of *the radial limit set* of G. Therefore the limit set of G has Hausdorff dimension 2.

14. *Existence of Cannon-Thurston maps.*

Let H be a finitely generated Kleinian group and let M be a compact core of $N = \mathbb{H}^3/H$. It follows from the Hyperbolisation Theorem that there exists a convex cocompact Kleinian group G such that M is homeomorphic to the quotient by G of the reunion $\mathbb{H}^3 \cup \Omega(G)$ of $\mathbb{H}^3$ and of the domain of discontinuity of G. The group G is not unique but its limit set $L(G)$ and the action of G on it are. They can be defined indeed in purely combinatorial terms, due to the fact that G is convex cocompact: $L(G)$ is equivariantly homeomorphic to *the Floyd boundary* of G. Denote by ρ the isomorphism between G and H induced by the homotopy equivalence $M \hookrightarrow N$. The problem asks if there is a continuous G-equivariant map from $L(G)$ to $L(\rho(G))$; such a map is called *a Cannon-Thurston map* since it had been shown to exist when N is the cyclic cover of an hyperbolic manifold which fibers over the circle by Cannon and Thurston in [31] (previously Bill Floyd had constructed the map when N is geometrically finite in [37]). Progress on this problem followed progress on the Ending lamination conjecture. Minsky solves the case when N has the homotopy type of a surface and has bounded geometry [68]. Mahan Mj solves the case of pared manifolds with incompressible boundary and bounded geometry in [59] (see also [53]). More recently, using the Minsky model for surface groups, Mahan Mj solves the general case when N has the homotopy type of a closed surface [60]. He also announces the solution for an arbitrary hyperbolic 3-manifold.

Subgroups of the fundamental group of 3-manifolds: Problems 15–18.

15. *Are finitely generated subgroups of a Kleinian group separable?*

A group G is LERF (*locally extended residually finite*) if each finitely generated subgroup $H \subset G$ is *separable*, i.e., H equals the intersection of the finite index subgroups of G which contain H. This algebraic property for a group has important geometric consequences: Peter Scott had shown in [93] that closed surface groups are LERF and deduced that any finitely generated subgroup of a surface group $\pi_1(S)$ is the fundamental group of a subsurface in a finite cover of S. Thurston asks the question because of its potential implications to the Geometrization Conjecture. Even if the Geometrization was obtained by quite different routes, the research on the LERF property has led to a vast literature which shows strong and profound connections between geometric group theory and hyperbolic manifolds. In particular, Daniel Wise developed a vast research program to attack this problem which is centered around the notion of *cube complexes* [106]. This program is an elaboration of Scott's approach (for proving that surface groups are LERF, Scott exploited the property that the fundamental group of the non-orientable closed surface of Euler characteristic -1 – which is a quotient of any closed surface – acts on $\mathbb{H}$ with a right-angled pentagon as fundamental domain).

A *cube complex* is a cell complex whose cells are isomorphic to cubes $[-1, 1]^n$ and such that the attaching maps between cells are isometries. One says that a group G is *cubulated* when it acts freely and cocompactly on a simply connected CAT(0) cube complex. In 2007, Frédéric Haglund and Daniel Wise introduced the notion of *special cube complex* which imposes some restrictions on the behavior of the *hyperplanes* [41]; they prove that if a compact cube complex X is special and has a Gromov hyperbolic fundamental group, then all the *quasi-convex subgroups* of $\pi_1(X)$ are separable. Wise conjectures in [106] that any compact cube complex which has a Gromov hyperbolic fundamental group has a finite cover that is *special.* This is precisely the conjecture that Agol solves in [4]. The solution of Problems 15 to 18 follows from this.

The separability of quasi-convex subgroups (i.e., convex cocompact subgroups) of a cocompact Kleinian group is a direct application of this result since Nicolas Bergeron and Daniel Wise proved in 2009 that any cocompact Kleinian group is cubulated [8] (subgroups which are not convex cocompact are virtual fibers as a consequence of the tameness and the Thurston-Canary covering theorem). Bergeron and Wise used a method introduced by Michah Sageev who constructed, from any finite collection of quasi-convex subgroups of a group G (which satisfy certain conditions) an action of G on a CAT(0) cube complex. This theorem of Bergeron and

Wise could be proven in such a generality thanks to the recent result of Jeremy Kahn and Vlad Markovic which says that any cocompact Kleinian group contains (many) quasi-Fuchsian subgroups [49].

16. *Is every irreducible* 3*-manifold with infinite fundamental group virtually Haken?*

This problem contains the question whether the fundamental group of an irreducible 3-manifold contains a surface group when it is infinite. For a cocompact Kleinian group, this is solved by Kahn and Markovic in [49]; they use in particular fine properties of the geodesic flow on the unit tangent bundle of hyperbolic manifolds such as the *exponential mixing with respect to the Liouville measure*. By the LERF property of Kleinian groups, any surface subgroup of a cocompact Kleinian group G provided by this theorem is the fundamental group of a closed surface which is embedded in an appropriate finite cover of $\mathbb{H}^3/G$. This solves Problem 16 for the case of hyperbolic manifolds which was the remaining case (by the Geometrization Theorem and since all the other geometries can be handled directly).

17. *Does every aspherical* 3*-manifold have a finite cover with positive first Betti number?*

18. *Does every finite volume hyperbolic* 3*-manifold have a finite cover which fibers over the circle?*

In [3], Agol introduces a new class of groups, the *residually finite rational solvable* (RFRS) groups and shows that this class contains all *right-angled Artin groups*. He observes that if the fundamental group of a 3-manifold M is not abelian but RFRS, then M has a first Betti number virtually infinite; he shows also that when M is irreducible with $\chi(M) = 0$ and $\pi_1(M)$ RFRS, then M virtually fibers over the circle. In [41], among the properties of special cube complexes they study, Haglund and Wise show that if a group is cubulated by a special cube complex, then some finite index subgroup of it embeds into a right-angled Artin group. Therefore, any cocompact Kleinian group has a finite index subgroup which is RFRS and therefore Problems 17 and 18 have positive answers.

Computer programs and tabulations: Problems 20–22.

In his PhD thesis, Jeff Weeks wrote the program `SnapPea` which computes the hyperbolic structure of a link complement in $\mathbb{S}^3$ when this structure exists (`http://www.geometrygames.org/SnapPea/`). This program provides a lot of useful information: it shows the Ford domain, it gives the shape of the cusps and shows the induced tessellation of those, it computes many invariants of the hyperbolic metric like the volume, the length of its shortest closed geodesic etc. [105]. It allows to tabulate many families of 3-manifolds like the Callahan-Hildebrand-Weeks census of non compact finite volume hyperbolic 3-manifolds, or the Hodgson-Weeks census of small volume closed hyperbolic 3-manifolds. It shows evidence for several problems in the list; in particular in [36], Nathan Dunfield and Thurston check the virtual Haken conjecture for all 3-manifolds from the Hodgson-Weeks census (all have indeed positive virtual Betti number). The *Weeks manifold* was observed to have the smallest volume among the known hyperbolic manifolds and was conjectured to be the closed hyperbolic manifold with smallest volume: this conjecture is now solved by David Gabai, Robert Meyerhoff and Peter Milley (see [38], [39]). Furthermore, SnapPea permits to "feel" the well ordering of hyperbolic manifolds by the volume as Thurston asks in Problem 22: it gives evidence that this ordering is compatible with the topological complexity given by the *Mom number* [38]. SnapPea is a powerful tool, which in addition to helping the intuition with many problems, suggests also new ones. For instance, its computation of the hyperbolic structure on a link complement exhibits in all cases a triangulation of this complement by *geodesic ideal tetrahedra*; however there are no theoretical proofs yet of the existence of such a triangulation.

Other software doing computations with 3-manifolds or with surface diffeomorphisms can be found at the CompuTop.org Software Archive website http://www.math.illinois.edu/~nmd/computop/index.html.

Arithmetic properties of Kleinian groups: Problems 19, 23.

19. *Find topological and geometric properties of quotient spaces of arithmetic subgroups of* $\mathrm{PSL}(2,\mathbb{C})$.

This problem is more a theme of research than a problem like the ones above. Certain topological problems from the list were first established for arithmetic hyperbolic 3-manifolds: Lackenby proves the Surface subgroup conjecture for arithmetic manifolds in [57] and Agol applies his criterion for virtual fibering to Bianchi groups in [3]. A geometric property of arithmetic hyperbolic manifolds that is often used is that each commensurability class of arithmetic 3-manifolds contains many representatives with a non-trivial isometry group. An example of this is the following property which characterizes the arithmetic manifolds among all hyperbolic finite volume 3-manifolds [33]: *any closed geodesic* γ *in an arithmetic hyperbolic 3-manifold* N *lifts to a geodesic* $\tilde{\gamma}$ *in some finite cover* N_γ *of* N *which admits an isometric involution whose fixed point set contains* $\tilde{\gamma}$. One central question, closely related to questions in number theory like the Lehmer conjecture, is the Short Geodesic Conjecture: it asks wether the injectivity radius of arithmetic hyperbolic 3-orbifolds is bounded from below by a global positive number (cf. [63], §12). For more information regarding this problem, see the survey by Alan Reid [89].

23. *Show that the volumes of hyperbolic* 3*-manifolds are not all rationally related.*

Today, one knows very little about arithmetic properties of volumes of hyperbolic 3-manifolds and this problem is far from being solved; one does not even know of one single hyperbolic 3-manifold for which one could decide whether its volume is rational or irrational. However, the algebraic framework for studying arithmetic properties of volumes is now well established (see [79]). Given a field $k \subset \mathbb{C}$, its *Bloch group* $\mathcal{B}(k)$ is defined as a certain subspace of a certain quotient of the free $\mathbb{Z}$-module generated by the elements of $k \setminus \{0,1\}$; there is also a *Bloch regulator map* $\rho : B(k) \to \mathbb{C}/\mathbb{Q}$. In [79], Walter Neumann and Jun Yang assign to any finite volume hyperbolic 3-manifold $N = \mathbb{H}^3/G$ an element $\beta(N) \in \mathcal{B}(k(N)) \subset \mathcal{B}(\mathbb{C})$, where $k(N)$ is the *invariant trace field of* N (i.e., the subfield of $\mathbb{C}$ generated by the squares of the traces of the elements of G). They show that, up to a constant multiple, the volume of N and its Chern-Simons invariant are respectively, the imaginary part and the real part of $\rho(\beta(N))$ (this is one realization of Thurston's hint that volume and Chern-Simons invariant should be considered simultaneously as the real and imaginary parts of the same complex number (see also [107])). It is conjectured that when $k = \overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$, the imaginary part of the Bloch regulator map is injective. If this was true, this would imply that two hyperbolic 3-manifolds with the same volume, have Dirichlet domains which are scissors congruent. See [77] and [78] for a detailed discussion and for applications to the study of Chern-Simons invariant.

See also [78] for a discussion of the conjecture that any number field $k \subset \mathbb{C}$ can appear as the invariant trace field $k(N)$ of some hyperbolic 3-manifold N, a conjecture which is directly relevant to Problems 19 and 23.

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