TRANSPORT THEORY AND STATISTICAL PHYSICS, 12(2), 129-155 (1983)

SOLUTIONS TO INVERSE PROBLEMS FOR THE BOLTZMANN-FOKKER-PLANCK EQUATION

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ABSTRACT

Equations are developed with which to evaluate the coefficients used in the one-speed Boltzmann-Fokker-Planck transport equation for strongly anisotropic scattering of photons or neutrons. The inverse equations, which are derived for a homogeneous slab target, require only measurements outside the medium. One set of equations is useful when the external illumination is nonuniformly incident over the surface, and the other set is valid for a uniform illumination of the target.

I. GENERAL INTRODUCTION

The Boltzmann equation that incorporates Fokker Planck terms has been developed by Ligou and co-workers to study the transport of charged particles in hot plasmas[1] and of very fast neutrons [2,3] and is also useful for radiative transfer with highly anisotropic scattering. The previous work has been directed at the solution of the direct transport problem, i.e., where the properties of the

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0041-1450/83/1202-0129$3.50/0

medium are known and the angular flux is to be calculated, whereas here we solve the inverse problem where the angular fluxes at the surfaces of a slab target are known and the coefficients that describe the anisotropy of single scattering are to be determined.

The number of equations needed to comprise an "inverse method" for characterizing a target depends upon the number of coefficients in an expansion of the scattering kernel. Any number of equations linear in the coefficients can be obtained by utilizing the information contained in the azimuthal dependence of the angular flux for a target that is uniformly illuminated in a non-normal direction,[4,5] or a complicated set of nonlinear equations can be developed if only the azimuthally-independent portion of the angular flux is utilized.[6-8] The difficulty with the former is that the angular flux must be measured in a large number of azimuthal directions in order to minimize the errors in the coefficients,[9] so it is desirable to reduce the number of unknowns (and equations and measurement directions) needed by developing inverse methods for the Boltzmann-Fokker-Planck (BFP) equation. In this equation the scattering kernel is broken up into a forward-scattered, quasi-singular contribution over a small, finite element of solid angle and a low-order Legendre polynomial expansion for the regular contribution to the scattering in all directions.

For the BFP equation we want to derive two inverse methods, one appropriate for nonuniform illumination of a homogeneous slab target and the other for uniform illumination. The nonuniform source problem recently has been solved for the ordinary Boltzmann (B) equation by Siewert and Dunn[10] who derived an equation that requires that angular moments of the spatial Fourier transform of the angular flux over the surfaces be calculated; it is worth mentioning that only the total flux, rather than the angular flux, need be measured if the scattering is known to be isotropic. We note that an inverse equation for the case of nonuniform illumination of the target also has been derived for the B equation by Larsen,[11] who did not restrict the target to have plane geometry, and we also examine this approach for the BFP equation.

In the next section we establish the notation for the BFP equation and review the general approach of Ref. 7 that we use for deriving the inverse equations. Then in Sec. III we derive the inverse equations for nonuniform illumination before developing the uniform source equations in Sec. IV. We conclude with a few comments about the inverse methods for the two problems.

Appendices A, B, and C treat, respectively, the derivation of the one-speed BFP equation, the evaluation of the scattering coefficients for this equation, and another inverse equation for the case of non-uniform illumination.

II. TECHNICAL INTRODUCTION

To keep the analysis as general as possible within the framework of particles of one speed v, we consider the time-dependent BFP equation

$$B\,\psi(z,\underset{\sim}{\rho},\mu,\phi,t) = 0 \ , \tag{1}$$

where the angular flux ψ is a function of distance z measured normal to the x-y plane and vector $\underset{\sim}{\rho}$ located on this plane. The direction of propagation $\underset{\sim}{\Omega}$ is described by $\mu = \underset{\sim}{\Omega}\cdot\hat{e}_z$ and ϕ, which is the azimuthal angle between $\hat{e}_x$ and the projection of $\underset{\sim}{\Omega}$ in the x-y plane; t is the time variable. The BFP operator

$$B = v^{-1}\partial_t + \mu\,\partial_z + \underset{\sim}{\Omega}\cdot\partial_{\underset{\sim}{\rho}} + K \tag{2}$$

consists of the time-change operator, the two streaming operations normal to and in the x-y plane, and the operator K which describes the interaction of radiation within the target,

$$K = \sigma_r - H_r - H_s \ . \tag{3}$$

Here

$$\sigma_r = \sigma - \sigma_s^s \ ,$$

where constant σ is the total cross section and

$$\sigma_s^s = 2\pi \int_{-1}^{1} \sigma_s^s(\mu)\,d\mu \ ,$$

where $\sigma_s^s(\mu)$ is the singular part of the scattering cross section that vanishes for $\mu<(1-\varepsilon)$ with $\varepsilon \ll 1$. Operator H_r accounts for the regular part of the scattering operator

$$H_r = \sigma \sum_{n\geq 0} f_n Q_n , \tag{4}$$

where Q_n is the orthogonal projection operator

$$(Q_n\psi)(\underset{\sim}{\Omega}) = \sum_{|m|<n} Y_n^m(\underset{\sim}{\Omega}) \int_{4\pi} Y_n^{-m}(\underset{\sim}{\Omega}')\psi(\underset{\sim}{\Omega}')d\underset{\sim}{\Omega}'$$

and the $Y_n^m(\underset{\sim}{\Omega})$ are spherical harmonics. The quasi-singular portion of the scattering operator is

$$H_s\psi = \beta\sigma\left[\partial_\mu(1-\mu^2)\partial_\mu + (1-\mu^2)^{-1}\ \partial_\phi^2\right]\psi , \tag{5}$$

where the Fokker-Planck constant β is

$$\beta\sigma = (\pi/2) \int_{-1}^{1} (1-\mu^2)\sigma_s^s(\mu)d\mu .$$

A somewhat novel derivation of the one-speed BFP equation is given in Appendix A and an example of the calculation of the coefficients f_n and β is given in Appendix B.

To derive a set of inverse equations for which measurements can be made over all times and over the external surfaces, but not within the homogeneous target $z_- \leq z \leq z_+$, we Laplace transform in time[8] and Fourier transform over the x-y plane[11] Eq. (1) to obtain

$$B_T\psi(z,\underset{\sim}{k},\mu,\phi,s) = 0 . \tag{6}$$

The transformed angular flux is

$$\psi(z,\underset{\sim}{k},\mu,\phi,s) = \int dt \int d\underset{\sim}{\rho}\ e^{-st + i\underset{\sim}{k}\cdot\underset{\sim}{\rho}}\ \psi(z,\underset{\sim}{\rho},\mu,\phi,t)$$

and the new BFP operator is

$$B_T = \mu\partial_z + K_T,$$

where

$$K_T = sv^{-1} - i\underset{\sim}{k}\cdot\underset{\sim}{\Omega} + K \ . \tag{7}$$

We note, by the way, that we do not hesitate to employ such transformations when developing inverse equations since, in contrast with direct transport problems, we will never be confronted with calculating any inverse transforms; rather, the use of transforms will ultimately give us more inverse equations since we can always choose different values of s and $\underset{\sim}{k}$.

In deriving the inverse equations, we specialize the general approach of Ref. 8 and introduce the inner product

$$(f,g) = \int_{z_-}^{z_+} dz \int_{4\pi} d\underset{\sim}{\Omega}\ fg \ , \tag{8}$$

where f and g are functions of z and $\underset{\sim}{\Omega}$. Integration by parts shows that

$$(f,B_T g) = (B_T^* f,g) + \langle f,g \rangle ,$$

where B_T^* is the adjoint operator

$$B_T^* = -\mu\partial_z + K_T \tag{9}$$

and the surface contribution is

$$\langle f,g \rangle = \int_{4\pi} \mu fg\ d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} \ . \tag{10}$$

Selection of $g = \psi$ and $f = L\psi$, where operator L is such that

$$(B_T^* L\psi,\psi) = 0 ,$$

leads to a general form of our inverse equations,

$$< L\psi,\psi > = 0 . \tag{11}$$

Again the basic idea when selecting L is to satisfy the equation $B_T^* L\psi = 0$ by constructing an operator $\tilde{L}$ such that

$$B_T^* L + \tilde{L}B_T = 0 . \tag{12}$$

Since we want an inverse method that does not require calculation of $\partial_z\psi$ on the surface, we select $\tilde{L}$ such that

$$-\mu\partial_z L + \tilde{L}\mu\partial_z = 0 .$$

Then, by requiring that L not depend upon z, it follows that

$\tilde{L} = \mu L \mu^{-1}$. Using this result in Eq. (12) gives

$$K_T L + \mu L \mu^{-1} K_T = 0 \tag{13}$$

and it is from this equation that operator L must be selected to obtain the desired inverse equations. The general solution of this equation is of the form $L = PX$, where P is a particular solution with a two-sided inverse and X is any operator commuting with $K_T^{-1}\mu$. That is,

$$L \in PA(K_T^{-1}\mu) ,$$

where $A(K_T^{-1}\mu)$ denotes the algebra of commutators of $K_T^{-1}\mu$. We choose to construct our inverse methods from the abelian subalgebra generated by integer powers of $K_T^{-1}\mu$ and its inverse and write our solutions as

$$L_\ell = P(K_T^{-1}\mu)^\ell \ . \tag{14}$$

This leads to a family of possible inverse equations of the form

$$<\psi, L\psi> = \int_{4\pi} \mu\psi\, P\,(K_T^{-1}\mu)^\ell\ \psi d\underset{\sim}{\Omega}\Big|_{z_-}^{z_+} = 0 \ , \tag{15}$$

from which we must select the particular set(s) that are appropriate for the boundary conditions.

III. INVERSE METHOD FOR NONUNIFORM ILLUMINATION

For the nonuniform illumination boundary condition, we choose the reflection operator $P = R = R^{-1}$ defined by $(Rf)(\mu) = f(-\mu)$. With this choice, not all of the values of ℓ in Eq. (15) will lead to inverse equations. This can be shown by observing that R, K_T, and μ are self adjoint with respect to integration over μ, so that

$$<\psi, L\psi> = \int \psi\ (\mu K_T^{-1})^\ell R\mu\,\psi\, d\underset{\sim}{\Omega}\Big|_{z_-}^{z_+}$$

$$= (-1)^{\ell-1} <\psi, L\psi> ,$$

and hence, for ℓ even, $<\psi, L\psi> = 0$ irrespective of the value of ψ. Also, since integrals in Eq. (15) for $\ell < -1$ could result in singularities, we restrict our attention to the values $\ell = -1, 1, 3$, etc.

For the case of $\ell = -1$, we just have to evaluate

$$\int \psi\, R\, K_T\ \psi d\underset{\sim}{\Omega}\Big|_{z_-}^{z_+} = 0 \ .$$

After use of Eqs. (3) and (7) and the fact that $RK_T = K_T R$, the inverse equation can be written as

$$(\sigma_r + sv^{-1}) \int \psi\, \psi_-\, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} - i\, \underset{\sim}{k} \cdot \int \underset{\sim}{\Omega}\, \psi\, \psi_-\, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = \int \psi (H_r + H_s) \psi_-\, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} . \quad (16)$$

Here, for notational convenience, we have defined

$$\psi_- = R\psi = \psi(z, \underset{\sim}{k}, -\mu, \phi, s).$$

To evaluate the first integral on the right hand side of Eq. (16), we first observe from Eq. (4) that

$$H_r\, \psi = (2\pi)^{-1} \sigma \sum_{n=0}^{N} \sum_{m=0}^{n} f_n (2 - \delta_{m0})\, \phi_n^m(\mu)\, x$$

$$\int_0^{2\pi} d\phi' \cos m(\phi - \phi') \int_{-1}^{1} d\mu'\, \phi_n^m(\mu')\, \psi(z, \underset{\sim}{k}, \mu', \phi', s), \quad (17)$$

where N is the order of anisotropy of the regular part of the scattering and $\phi_n^m(\mu)$ is the normalized associated Legendre function

$$\phi_n^m(\mu) = P_n^m(\mu)/N_n^m , \quad n \geq m ,$$

$$(N_n^m)^2 = \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} .$$

Thus it follows that

$$\int \psi H_r \psi_-\, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = (2\pi)^{-1} \sigma \sum_{n=0}^{N} \sum_{m=0}^{n} (-1)^{n-m} f_n (2 - \delta_{m0}) [C_n^m + S_n^m] \Big|_{z_-}^{z_+} , \quad (18)$$

for

$$C_n^m(z,\underset{\sim}{k},s) = \left[\int_0^{2\pi} d\phi \cos m\phi \int_{-1}^{1} d\mu\ \phi_n^m(\mu)\psi(z,\underset{\sim}{k},\mu,\phi,s)\right]^2$$

with S_n^m defined identically except that sin $m\phi$ replaces cos $m\phi$.

The second integral on the right hand side of Eq. (16) is

$$\int \psi H_s \psi_- d\underset{\sim}{\Omega}\Big|_{z_-}^{z_+} = \beta\sigma \int \psi\left[\partial_\mu(1-\mu^2)\partial_\mu + (1-\mu^2)^{-1}\partial_\phi^2\right]\psi_- d\underset{\sim}{\Omega}\Big|_{z_-}^{z_+} , \qquad (19)$$

and this integral requires that second-order angular derivatives of ψ_- on the surfaces be calculated directly. If the incident distribution is singular, then a modified form of Eq. (19) is useful. An example of such a distribution is a pulsed monodirectional illumination at $z = z_-$ and $t = 0$ that is incident only at $\underset{\sim}{\rho} = 0$; in this case,

$$\psi(z_-,\underset{\sim}{k},\mu,\phi,s) = \delta(\mu-\mu_o)\delta(\phi-\phi_o) \ , \quad \mu \geq 0 \ .$$

Then the integral is broken up into two parts,

$$\int \psi H_s \psi_-\ d\underset{\sim}{\Omega} = I_{+-} + I_{-+} \ ,$$

where

$$I_{+-} = \int_0^{2\pi} d\phi \int_0^1 d\mu\ \psi H_s \psi_-$$

and I_{-+} has the form as I_{+-} except that ψ and ψ_- are reversed. From the definition of H_s in Eq. (5),

$$I_{-+} = \beta\sigma \left[\int_0^1 \psi_- \partial_\mu (1-\mu^2)\partial_\mu \psi\, d\mu\Big|_{\phi=\phi_o} + (1-\mu_o^2)^{-1}\int_0^{2\pi} \psi_-\ \partial_\phi^2 \psi d\phi\Big|_{\mu=\mu_o}\right]$$

and, once each term is integrated by parts twice to show that $I_{-+} = I_{+-}$, then it follows in this special case that

$$\int \psi H_s \psi_- d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = 2\beta\sigma\left[\partial_{\mu_o}(1-\mu_o^2)\partial_{\mu_o} + (1-\mu_o^2)^{-1}\partial_{\phi_o}^2\right] \psi(z_-,\underset{\sim}{k},-\mu_o,\phi_o,s).$$

Consequently, in this case one would need to determine the angular derivatives only in the direction of the incident illumination.

Equations (16), (18), and (19) comprise an inverse equation for the (N+4) unknowns: σ_r, v^{-1}, $\beta\sigma$, and σf_n, $n=0$ to N; the equation is linear in the unknowns and contains a source term,

$$-i\underset{\sim}{k} \cdot \int \underset{\sim}{\Omega}\, \psi \psi_- \, d\Omega \Big|_{z_-}^{z_+} .$$

Since the equation has both real and imaginary parts, an inverse method for determining the unknowns can be obtained, in principle, from (N+4)/2 combinations of $\underset{\sim}{k}$ and s, provided $|\underset{\sim}{k}| > 0$[12] and no two values of $\underset{\sim}{k}$ are complex conjugates. In the special case that there is no time-dependence, then v^{-1} is no longer an unknown and cannot be determined; thus, (N+3)/2 different values of $\underset{\sim}{k}$ are needed for the inverse method unless one value is $\underset{\sim}{k}=0$, in which case (N+4)/2 values are required.

Another possibility is that β is identically zero, so that the BFP equation becomes the ordinary B equation. If s is also zero, then we recover the inverse method of Siewert and Dunn;[10] from this method, in principle, the unknowns σ and σf_n, $n=0$ to N, could be recovered using (N+2)/2 equations obtained by using different values of $\underset{\sim}{k}$, $|\underset{\sim}{k}| > 0$. We note that if σ were known, so that the value of $\underset{\sim}{k}$ could be measured in dimensionless units, then only the (N+1) unknown f_n-values would need to be determined, as reported by Siewert and Dunn.

We note that inverse methods developed from Eq. (15) for $\ell \geq 1$, ℓ odd, lead to complicated nonlinear equations and these appear to have little interest for practical applications.

It is interesting to compare the features of inverse method (16) with the inverse equation that can be obtained without first performing a spatial Fourier transform of the BFP equation. This analysis is done in Appendix C.

IV. INVERSE METHOD FOR UNIFORM ILLUMINATION

The usual advantage of the uniform illumination experiment is that the B equation that depends upon azimuthal angle ϕ can be separated into a set of B equations,[13] one for each coefficient of a Fourier expansion in ϕ, and the same feature holds true in developing inverse equations.[4] This suggests that the single inverse BFP equation (15) can be expressed as a set of inverse BFP equations. Indeed, because of the uniform illumination, the BFP equation does not contain the transverse gradient term $\underset{\sim}{\Omega}\cdot\nabla_{\underset{\sim}{\rho}}$ and therefore the BFP operator is invariant with respect to the group of orthogonal transformations on the variable ϕ. This fact allows us to construct a subalgebra A_ϕ of geometrical transformations which commute with the BFP operator. Selecting the complete, orthogonal basis $\{e^{im\phi},\ m=0, \pm 1, \ldots\}$ in $L_2[0,\pi]$ it can be shown that A_ϕ consists of all linear combinations of elements of the form[14]

$$e^{im\phi} e^{in\phi} * \ , \quad |m| = |n| \ ,$$

where the asterisk denotes the integration over $0 \leq \phi \leq 2\pi$.

We now construct the inverse solutions from

$$L \in P\, A_\phi A(K_T^{-1}\mu) \ ,$$

where P is a particular solution, and observe that Eq. (14) is valid if $P = RT$, where again $(Rf)(\mu) = f(-\mu)$, and T is an element of A_ϕ of the form

$$T = j^m(\phi)k^m(\phi) * \ . \qquad (20)$$

Here, for example, j^m and k^m are selected from the set of functions $\cos m\phi$ and $\sin m\phi$. As for the non-uniform illumination case, it can be proved that only the operators in Eq. (14) with ℓ odd provide inverse methods.

For the case of $\ell = -1$, we now have to evaluate

$$\int \psi R T K_T \psi \, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = 0 \ ,$$

where K_T from Eq. (8) is

$$K_T = sv^{-1} + K = sv^{-1} + \sigma_r - H_r - H_s$$

since uniform illumination corresponds to the case where $\underset{\sim}{k} = 0$. The analog of Eq. (16) is

$$(\sigma_r + sv^{-1}) \int \psi \, T \psi_- d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = \int \psi \, T(H_r + H_s) \psi_- \, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} . \qquad (21)$$

We next introduce the notation

$$\psi_j^m = \psi_j^m(z,\mu,s) = \int_0^{2\pi} j^m(\phi) \ \psi(z,\mu,\phi,s) d\phi \ ,$$

where the lower index j refers to the particular function used in the integral. Once we recall the form of H_r and H_s from Eqs. (17) and (5), respectively, we observe that

$$k^m * (H_r + H_s) = (H_r^m + H_s^m) k^m * \ , \qquad (22)$$

where

$$H_r^m \psi_k^m = \frac{1}{2} \sigma \sum_{n=m}^{N} f_n \ \phi_n^m(\mu) \int_{-1}^{1} \phi_n^m(\mu') \psi_k^m(z,\mu',s) d\mu'$$

and

$$H_s^m \psi_k^m = \beta\sigma \ [\partial_\mu (1-\mu^2) \partial_\mu - m^2 (1-\mu^2)^{-1}] \psi_k^m \ .$$

The general form of the inverse equations thus becomes

$$\int_{-1}^{1} \psi_j^m (\psi_k^m)_- d\mu \Big|_{z_-}^{z_+} = \frac{1}{2} \sum_{n=m}^{N} (-1)^{n-m} \hat{f}_n \psi_{j,n}^m \psi_{k,n}^m \Big|_{z_-}^{z_+}$$

$$+ \hat{\beta} \int_{-1}^{1} \psi_j^m [\partial_\mu (1-\mu^2) \partial_\mu - m^2 (1-\mu^2)^{-1}] (\psi_k^m)_- \, d\mu \Big|_{z_-}^{z_+} , \qquad (23)$$

where we have defined

$$\psi_{j,n}^m = \int_{-1}^{1} \phi_n^m \psi_j^m \, d\mu \qquad (24)$$

and the dimensionless coefficients

$$\hat{f}_n = f_n \sigma / (\sigma_r + sv^{-1}) ,$$

$$\hat{\beta} = \beta \sigma / (\sigma_r + sv^{-1}) .$$

For $m = 0,1,2,...(N+1)$, this is a heterogeneous, triangular system of equations for determining the unknowns $\hat{\beta}$ and $\hat{f}_n$, $n = 0$ to N. In practice, the functions $j^m(\phi)$ and $k^m(\phi)$ are selected to be either cos $m\phi$ or sin $m\phi$. Therefore in general we have three different sets of inverse equations, the cosine-cosine, the sine-sine, and the cosine-sine sets. We observe that for the cosine-sine and sine-sine sets, the $m = 0$ equation identically vanishes, so that the coefficient $\hat{f}_0$ cannot be determined from these two sets.

The form of inverse equations for the case of $\ell = 1$ is

$$\int \tilde{\psi} \, RT \, K_T^{-1} \tilde{\psi} \, d\underset{\sim}{\Omega} \Big|_{z_-}^{z_+} = 0 ,$$

where $\tilde{\psi}$ is defined by $\tilde{\psi} = \mu\psi$. Again the ϕ-integration can be performed to yield a set of m-dependent equations

$$\int_{-1}^{1} \tilde{\psi}_j^m \, (K_T^m)^{-1} (\tilde{\psi}_k^m)_- d\mu \Big|_{z_-}^{z_+} ,$$

where we have used the fact that

$$k^m * K_T^{-1} = (K_T^m)^{-1} k^m * ,$$

where

$$K_T^m = \sigma_r + sv^{-1} - H_r^m - H_s^m .$$

To calculate $(K_T^m)^{-1}$, we use the same orthonormal basis functions $\{\phi_n^m(\mu), \; n \geq m\}$ and observe that

$$H_s^m \phi_n^m = -\beta\sigma n(n+1)\phi_n^m .$$

Hence the operator K_T^m is diagonal and its inverse is

$$(K_T^m)^{-1} = (\sigma_r + sv^{-1})^{-1} \sum_{n \geq m} [1 - \hat{\alpha}_n]^{-1} \; \phi_n^m \phi_n^m \cdot ,$$

where we have introduced the scalar product

$$f \cdot g = \int_{-1}^{1} f \, g \, d\mu$$

and the coefficient

$$\hat{\alpha}_n = \frac{1}{2} \hat{f}_n - n(n+1)\hat{\beta} .$$

The first form for the inverse equations is

$$- \int_{-1}^{1} \tilde{\psi}_j^m (\tilde{\psi}_k^m)_- d\mu \Big|_{z_-}^{z_+} = \sum_{n \geq m} \left(\frac{\hat{\alpha}_n}{1 - \hat{\alpha}_n} \right) (-1)^{n-m} \, \tilde{\psi}_{j,n}^m \, \tilde{\psi}_{k,n}^m \Big|_{z_-}^{z_+} , \tag{25}$$

where we have extracted the source term by using the fact that

$$(1-\hat{\alpha}_n)^{-1} = 1+\hat{\alpha}_n/(1-\hat{\alpha}_n) \ .$$

The inverse BFP methods for $\ell = -1$ and $\ell = 1$ in Eqs. (23) and (25) both reduce to the usual sets of inverse B methods when $\beta = 0$.[15] In the case of the most general boundary conditions, there are the cosine-cosine, sine-sine, and cosine-sine sets of equations for both inverse methods. For the $\ell = -1$ sets, for example, there are (N+2) cosine-cosine equations, (N+1) sine-sine equations, and (N+1) cosine-sine equations (since in the latter two cases the equations for $m = 0$ identically vanish). However, if the incident illumination can be made symmetric with respect to some azimuthal reference angle, say $\phi = 0$, then there are only the (N+2) cosine-cosine equations.[4] Furthermore, if the incident distribution is azimuthally symmetric, so that $m = 0$, then only a single cosine-cosine equation can be used.

Inverse equations (25) for $\ell = 1$ would be more difficult to use in practice than the $\ell = -1$ equations. This is because, in principle, an infinite number of moments for $n \geq m$ would be required, rather than just a finite number as for the $\ell = -1$ equations. This arises because operator K_T includes the differential operator H_s which is not of finite rank.

One possible aid for reducing the number of moments to be calculated is to observe that

$$\hat{\alpha}_n = -n(n+1)\hat{\beta} \ , \quad n > N \ ,$$

and hence that $\hat{\alpha}_n/(1-\hat{\alpha}_n) \to -1$ in the limit as n gets large. Thus the summation in Eq. (25) could be decomposed into two parts, written symbolically as

$$\sum_{n\geq m} = \sum_{n=m}^{\tilde{N}} + \sum_{n>\tilde{N}} ,$$

where $\tilde{N}$ is the number beyond which $\hat{\alpha}_n/(1-\hat{\alpha}_n) \approx -1$. Then

$$\sum_{n>\tilde{N}} \left(\frac{\hat{\alpha}_n}{1-\hat{\alpha}_n}\right) (-1)^{n-m}\, \tilde{\psi}^m_{j,n}\, \tilde{\psi}^m_{\cdot,n} \approx$$

$$\sum_{n=m}^{\tilde{N}} (-1)^{n-m}\, \tilde{\psi}^m_{j,n}\, \tilde{\psi}^m_{k,n} - \int_{-1}^{1} \tilde{\psi}^m_j (\tilde{\psi}^m_k)_-\, d\mu \, .$$

The approximate form of inverse equation (25) would then be the set of equations

$$-\sum_{n=m}^{\tilde{N}} (-1)^{n-m}\, \tilde{\psi}^m_{j,n}\, \tilde{\psi}^m_{k,n} = \sum_{n=m}^{\tilde{N}} \left(\frac{\hat{\alpha}_n}{1-\hat{\alpha}_n}\right) (-1)^{n-m}\, \tilde{\psi}^m_{j,n} \tilde{\psi}^m_{k,n} . \quad (26)$$

Of course, since the value of $\tilde{\beta}$ would not be known beforehand, the value of $\tilde{N}$ would have to be obtained by trial-and-error, but at least the approximate inverse method for $\ell = 1$ involves only the unknowns $\hat{\alpha}_n$, $n = 0$ to $\tilde{N}$, rather than an infinite number of unknowns.

V. CONCLUSIONS

We have shown that a set of inverse equations can be derived to obtain the coefficients for the one-speed Boltzmann-Fokker-Planck equation by making measurements of the ingoing and outgoing particle angular fluxes in all angular directions for a uniformly illuminated slab target and the same measurements over the entire surface of a non-uniformly illuminated slab. The advantage of the inverse BFP equations as compared to previous inverse transport equations is that fewer coefficients are needed to describe strongly anisotropic scattering; the disadvantage is that the angular fluxes must be measured accurately enough that second-order angular densities of these fluxes can be obtained with precision.

For the case of nonuniform illumination, a single inverse equation is obtained in terms of Fourier transformed surface angular fluxes. Selection of different values of the transform variable yields a set of equations for determining the unknown coefficients. For the case of uniform illumination, there are three sets of inverse equations that combine the cos $m\phi$ and sin $m\phi$ moments of the fluxes. The cosine-cosine set contains enough equations to calculate all the unknowns, while the cosine-sine and sine-sine sets permit calculation of all coefficients but one. If the incident illumination is azimuthally symmetric, then all inverse equations identically vanish except for the $m = 0$ cosine-cosine equation.

ACKNOWLEDGMENTS

We appreciate helpful discussions with E.W. Larsen about his general geometry inverse method, and L.O. Reynolds provided the coefficients in Table B-1. This work was supported in part by NSF grant CPE-8209908.

REFERENCES

1. J. Ligou, Proc. ANS-ENS Int. Topl. Mtg. Advances in Mathematical Methods for the Solution of Nuclear Engineering Problems, (Munich 1981), Fachinformationszentrum Energie, Physik, Mathematik GmbH, Karlsruhe, Vol. 2, p. 386 (1981).
2. K. Przybylski and J. Ligou, Nucl. Sci. Eng. 81, 92 (1982).
3. M. Caro and J. Ligou, Nucl. Sci. Eng. 83, 242 (1983).
4. N.J. McCormick, J. Math. Phys. 20, 1504 (1979).
5. N.J. McCormick and R. Sanchez, J. Math. Phys. 22, 199 (1981).
6. C.E. Siewert, J. Math. Phys. 19, 1587 (1978).
7. C.E. Siewert, J. Quant. Spectrosc. Rad. Transfer 22, 441 (1979).
8. R. Sanchez and N.J. McCormick, J. Math. Phys. 22, 847 (1981).
9. R. Sanchez and N.J. McCormick, J. Quant. Spectrosc. Rad. Transfer 28, 169 (1982).
10. C.E. Siewert and W.L. Dunn, J. Math. Phys. 23, 1376 (1982).
11. E.W. Larsen, J. Math. Phys. (in preparation).
12. If N were odd, then $(N+5)/2$ equations would be needed for $\underset{\sim}{k} > 0$.

13. S. Chandrasekhar, Radiative Transfer, Oxford Univ. Press, 1950.

14. The operators with m = −n are the basis elements for the representation of the subgroup of rotations, while those with m = n are the elements for the subset of symmetries.

15. N.J. McCormick and R. Sanchez, J. Quant. Spectrosc. Rad. Transfer 29, (to be published).

APPENDIX A

DERIVATION OF THE ONE-SPEED BOLTZMANN-FOKKER-PLANCK TRANSPORT EQUATION

The Boltzmann-Fokker-Planck (BFP) equation provides a useful description of linear particle transport in media with highly anisotropic scattering in that it considerably reduces the number of coefficients needed to represent the scattering kernel.[1] The BFP equation is based on an approximate treatment of the forward-scattering collisions and can be viewed as an improvement of the classical transport correction in which a Dirac-delta term is added to a low-order expansion of the kernel. To derive this equation in one-speed theory, we decompose the scattering kernel into a regular part and a quasi-singular part,

$$\sigma_s(\underset{\sim}{r},\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega}) = \sigma_s^r(\underset{\sim}{r},\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega}) + \sigma_s^s(\underset{\sim}{r},\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega}) \ , \tag{A-1}$$

where σ_s^r is a low-order, Legendre polynomial expansion accounting for the regular contribution to the scattering in all directions, and σ_s^s is zero except in a small solid angle around the forward direction $\underset{\sim}{\Omega}$, i.e., $\sigma_s^s(\underset{\sim}{r},\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega}) = 0$ if $\underset{\sim}{\Omega}'\cdot\underset{\sim}{\Omega} < (1-\varepsilon)$.

For angular directions we will use spherical coordinates with unit vectors $\underset{\sim}{\Omega}$, $\underset{\sim}{e}_\theta$ and $\underset{\sim}{e}_\phi$ so that

$$\underset{\sim}{\Omega}' = \mu'\underset{\sim}{\Omega} + [1-(\mu')^2]^{1/2}\,(\cos\phi'\,\underset{\sim}{e}_\theta + \sin\phi'\,\underset{\sim}{e}_\phi).$$

In order to obtain an approximate expression for the quasi-singular scattering operator

$$S\,\psi = \int_{-1}^{1} d\mu'\sigma_s^s(\underset{\sim}{r},\mu')\int_0^{2\pi} d\phi'\psi(\underset{\sim}{r},\underset{\sim}{\Omega}',t) \;, \tag{A-2}$$

we expand the angular flux about $\underset{\sim}{\Omega}$:

$$\psi(\underset{\sim}{r},\underset{\sim}{\Omega}',t) = \left[1+(\underset{\sim}{\Omega}'-\underset{\sim}{\Omega})\cdot\nabla_{\underset{\sim}{\Omega}}+\frac{1}{2}(\underset{\sim}{\Omega}'-\underset{\sim}{\Omega})(\underset{\sim}{\Omega}'-\underset{\sim}{\Omega}):\nabla_{\underset{\sim}{\Omega}}\nabla_{\underset{\sim}{\Omega}}\right]\psi(\underset{\sim}{r},\underset{\sim}{\Omega},t)+\cdots \;,$$

where the colon symbol denotes the total tensor contraction.

After substituting this expansion into Eq. (A-2) and suppressing terms that vanish upon integration over ϕ', we obtain

$$S\psi \sim \int_{-1}^{1} d\mu'\sigma_s^s(\underset{\sim}{r},\mu')\int_0^{2\pi} d\phi'\Big(1-(1-\mu')\underset{\sim}{\Omega}\cdot\nabla_{\underset{\sim}{\Omega}}+\{(1-\mu')^2\underset{\sim}{\Omega}\underset{\sim}{\Omega}+$$
$$\frac{1}{4}\,[1-(\mu')^2 I_{\perp}\}:\nabla_{\underset{\sim}{\Omega}}\nabla_{\underset{\sim}{\Omega}}\Big)\;\psi(\underset{\sim}{r},\underset{\sim}{\Omega},t)\;.$$

Here $I_{\perp} = \underset{\sim}{e}_\theta\underset{\sim}{e}_\theta+\underset{\sim}{e}_\phi\underset{\sim}{e}_\phi$ is the projection operator onto the plane orthogonal to $\underset{\sim}{\Omega}$. We observe that the integral over ϕ of $\underset{\sim}{\Omega}\cdot\nabla_{\underset{\sim}{\Omega}}$ also vanishes and that

$$\nabla_{\underset{\sim}{\Omega}}\nabla_{\underset{\sim}{\Omega}} = \underset{\sim}{e}_\theta\underset{\sim}{e}_\theta\partial_\theta^2+\underset{\sim}{e}_\phi\underset{\sim}{e}_\phi\left(\frac{1}{\sin^2\theta}\,\partial_\phi^2+\frac{\cos\,\theta}{\sin\,\theta}\,\partial_\theta\right) + \text{non-diagonal terms}$$

so we can write

$$I_{\perp}:\nabla_{\underset{\sim}{\Omega}}\nabla_{\underset{\sim}{\Omega}} = \Delta_{\underset{\sim}{\Omega}} = \partial_\mu(1-\mu^2)\partial_\mu+\frac{1}{1-\mu^2}\,\partial_\phi^2\;. \tag{A-3}$$

Thus

$$S\psi = \sigma_s^s\psi + H_s\psi$$

where

$$\sigma_s^s(\underset{\sim}{r}) = \int\sigma_s^s(\underset{\sim}{r},\underset{\sim}{\Omega}\cdot\underset{\sim}{\Omega}')d\underset{\sim}{\Omega}' \;, \tag{A-4}$$

$$H_s = \beta\sigma\Delta_{\underset{\sim}{\Omega}}\;. \tag{A-5}$$

The Fokker-Planck coefficient $\beta\sigma$ is given by

$$\beta(\underset{\sim}{r})\sigma(\underset{\sim}{r}) = (\pi/2)\int_{1-\varepsilon}^{1}(1-\mu^2)\sigma_s^s(\underset{\sim}{r},\mu)\,d\mu \approx \pi\int_{1-\varepsilon}^{1}(1-\mu)\sigma_s^s(\underset{\sim}{r},\mu)\,d\mu. \quad \text{(A-6)}$$

This coefficient equals the coefficient T" obtained in the general energy-dependent derivation by Ligou[1] in the limit of the one-speed approximation, i.e. as the ratio of the scatterer mass to particle mass tends to infinity.

With this approximate result for the quasi-singular scattering kernel, the final form for the BFP equation can be written as

$$(v^{-1}\partial_t + \underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla} + \sigma_r)\psi = (H_r + H_s)\psi + Q$$

where $\sigma_r = \sigma - \sigma_s^s$, H_r is the regular scattering operator, and Q denotes any external source.

APPENDIX B

SCATTERING COEFFICIENTS FOR THE BFP EQUATION

In this appendix we give a numerical example to illustrate the way to decompose the usual scattering kernel into the regular and quasi-singular parts discussed in Appendix A. We assume the scattering kernel can be expanded in (K+1) Legendre polynomials as

$$\sigma_s/\sigma = (4\pi)^{-1}\sum_{n=0}^{K}\omega_n^B P_n(\mu) , \quad \text{(B-1a)}$$

and attempt to approximate this kernel over the interval $-1\leq\mu\leq(1-\varepsilon)$ with the regular part

$$\sigma_s^r/\sigma = (4\pi)^{-1} \sum_{n=0}^{N} \omega_n P_n(\mu), \quad N < K . \tag{B-1b}$$

In both equations, we use $\omega_n = (2n+1) f_n$. We determine the ω_n so that σ_s^r is the best least-square approximation to σ_s in [-1, 1-ε] by minimizing the error

$$E = (4\pi)^{-2} \int_{-1}^{1-\varepsilon} \left[\sum_{n=0}^{K} (\omega_n^B - \omega_n) P_n(\mu) \right]^2 d\mu ,$$

for $\omega_n = 0$, $n > N$. That is, the ω_n obtained from the set of equations

$$\sum_{n=0}^{K} (\omega_n^B - \omega_n)\ A_{nk} = 0 , \quad k = 0 \text{ to } N , \tag{B-2}$$

where

$$A_{nk} = A_{kn} = 2\delta_{nk}/(2n+1) - \tilde{A}_{nk},$$

and

$$\tilde{A}_{nk} = \int_{1-\varepsilon}^{1} P_n(\mu) P_k(\mu) d\mu \approx \varepsilon - (\varepsilon^2/4)[n(n+1) + k(k+1)].$$

(The approximate result for $\tilde{A}_{nk}$ is obtained by using the orthogonality relation for Legendre polynomials and expanding $P_n(\mu)$ in the interval $(1-\varepsilon) \le \mu \le 1$ in a Taylor series expansion.) The use of Eqs. (B-1) and (B-2) allows the least-square error also to be written as

$$E = (4\pi)^{-2} \sum_{k=N+1}^{K} \omega_k^B \sum_{n=0}^{K} (\omega_n^B - \omega_n) A_{nk} . \tag{B-3}$$

Once the ω_n have been calculated, the value of β follows from Eqs. (A-1), (A-6), and (B-1) as

$$\beta = 8^{-1} \sum_{n=0}^{K} \int_{1-\varepsilon}^{1} (1-\mu^2)(\omega_n^B - \omega_n) P_n(\mu)\, d\mu$$

$$= (12)^{-1} \sum_{n=0}^{K} (\omega_n^B - \omega_n)(\tilde{A}_{n0} - \tilde{A}_{n2}) \; . \qquad \text{(B-4)}$$

An approximate result for β is

$$\beta \approx (\varepsilon^2/8) \sum_{n=0}^{K} (\omega_n^B - \omega_n)$$

and, provided that $N \geq 2$, then Eqs. (B-2) and (B-4) yield

$$\beta = 6^{-1} \, [(\omega_0^B - \omega_0) - 5^{-1} (\omega_2^B - \omega_2)] \; .$$

As an example, we consider the scattering law for 0.6329 μm light incident on a 1 μm diameter sphere of carbon in air with an index of refraction $1.280 + 0.93i$. The coefficients ω_n^B, $n = 0$ to 15, for the usual Legendre polynomial expansion of this kernel are given in Table B-1 and our results for ω_n and β are given in Table B-2 for

TABLE B-1

Coefficients for Boltzmann Equation for 0.6329 μm Light Incident on a 1 μm Diameter Sphere of Carbon in Air

n	ω_n^B			
0 to 3	.1741	.4219	.6126	.7250
4 to 7	.7664	.7433	.6655	.5413
8 to 11	.3863	.2449	.1304	.5371-1
12 to 15	.1671-1[a]	.3946-2	.7333-3	.1104-3

[a]Read as $.1671 \times 10^{-1}$

TABLE B-2

Coefficients for Boltzmann-Fokker-Planck Equation for Scattering Law of Table B-1

ε	N	ω_0	ω_1	ω_2	ω_3	ω_4	ω_5	β	σ/σ_r	E
0.1	2	.5715-1[a]	.8301-1	.8601-1	---	---	---	.1939-2	1.132	.5-4
	5	.7669-1	.1391	.1716	.1670	.1418	.1053	.1534-2	1.108	.2-4
	8	.1226	.2721	.3775	.4246	.4255	.3883	.7386-3	1.054	.5-5
0.05	2	.9505-1	.1897	.2417	---	---	---	.8113-3	1.086	.5-2
	5	.1218	.2681	.3663	.4005	.3824	.3213	.5084-3	1.055	.2-3
	8	.1548	.3651	.5213	.6041	.6223	.5834	.1726-3	1.020	.1-4
0.01	2	.1540	.3618	.5134	---	---	---	.4846-4	1.086	.2-2
	5	.1637	.3910	.5615	.6545	.6775	.6372	.2451-4	1.010	.6-3
	8	.1715	.4142	.6000	.7074	.7442	.7168	.5927-5	1.003	.4-4
0.001	2	.1720	.4156	.6020	---	---	---	.5296-6	1.002	.3-2
	5	.1731	.4189	.6075	.7178	.7572	.7321	.2544-6	1.001	.8-3
	8	.1739	.4213	.6116	.7235	.7645	.7409	.5662-7	1.000	.5-4

[a] Read as $.5715 \times 10^{-1}$

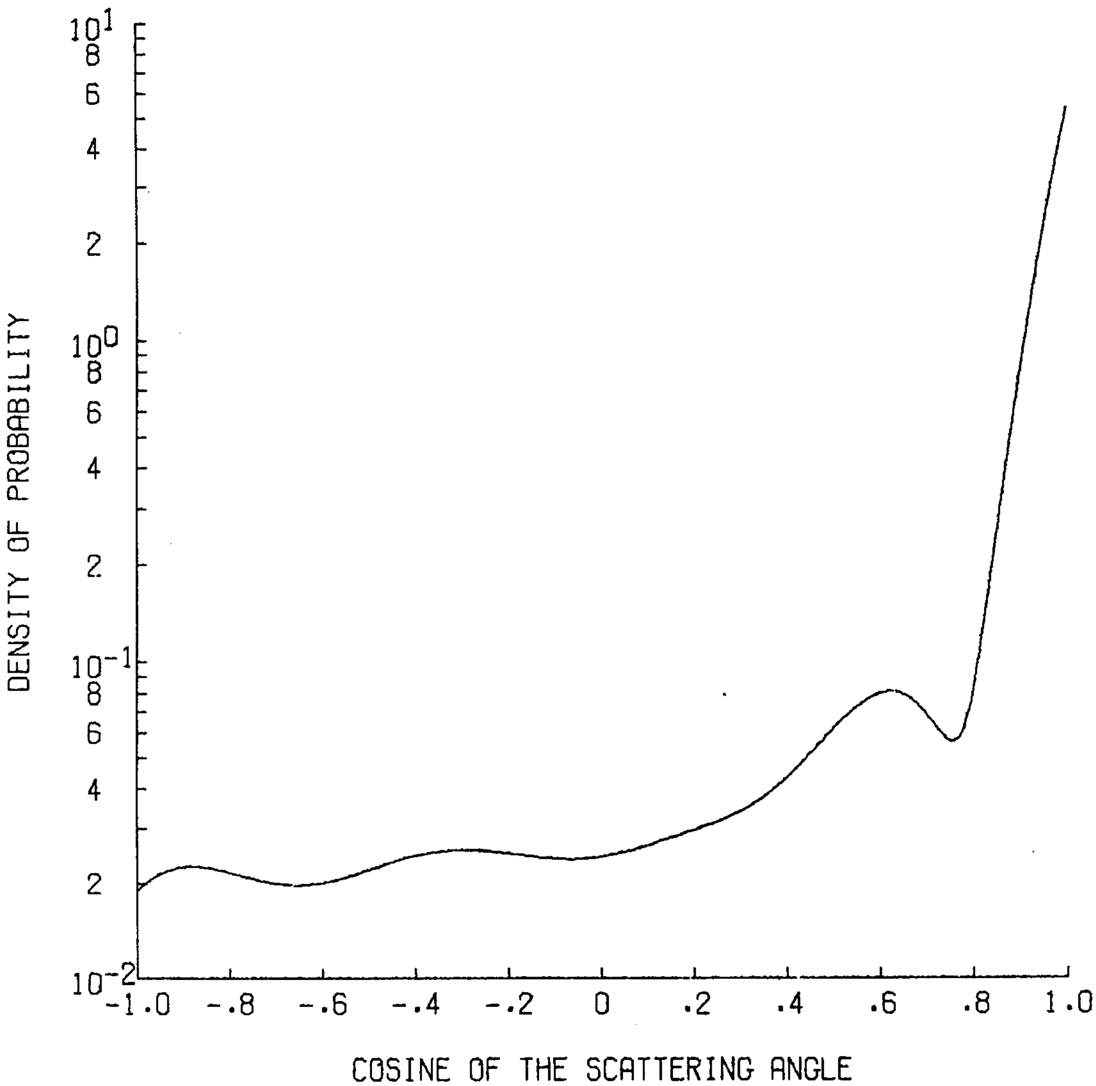

FIGURE B1. DENSITY OF PROBABILITY OF SCATTERING VERSUS THE COSINE OF THE SCATTERING ANGLE

different values of N and ε. We also give the least-square error E and the ratio

$$\sigma/\sigma_r = [1 - (\omega_0^B - \omega_0)]^{-1}$$

which appears as a factor multiplying the unknown coefficients in the steady-state form of the inverse equations.

The idea behind the use of the BFP kernel is to obtain a good fit (i.e., with small error E) to the kernel with as few expansion coefficients as possible for a small enough ε that $\psi(\underset{\sim}{r},\underset{\sim}{\Omega}',t)$ for $(1-\varepsilon) \leq \mu \leq 1$ can be expanded to only second order about $\underset{\sim}{\Omega}$. Thus when examining the data in Table B-2 we look for convergence of the ω_n with increasing values of N. For $\varepsilon = 0.1$, the error is small but the convergence is poor, thus suggesting that ε is too large. For $\varepsilon = 0.001$ the convergence of the ω_n with increasing N is quite rapid, but the error is approximately 10-60 times larger than for $\varepsilon = 0.1$. Since there is no obvious best value of ε and N for this scatterer, a reasonable compromise between these two cases would seem to be the $\varepsilon = 0.01$ and $N = 5$ model.

Both Table B-2 and Fig. B-1 show that this scatterer is one that is not especially amenable to modelling with the BFP equation. The reason is the forward scattering peak is not particularly narrow or large so that relatively few (only 16) coefficients are needed to model the scattering kernel for the B equation.

This numerical example was selected to show the limitations of the BFP scattering model. For a scattering law that is much more sharply peaked in the forward direction, the usefulness of the BFP kernel would have been more clearly demonstrated.

APPENDIX C

INVERSE EQUATION FOR NONUNIFORM ILLUMINATION OBTAINED WITHOUT A SPATIAL FOURIER TRANSFORM

We begin again with the BFP operator of Eq. (2), but now Eq. (6) is replaced by

$$(\mu\partial_z + K_T)\psi(z,\underset{\sim}{\rho},\mu,\phi,s) = 0 \ ,$$

where ψ is the Laplace transform of the time-dependent angular flux and

$$K_T = sv^{-1} + \underset{\sim}{\Omega}\cdot\partial_{\underset{\sim}{\rho}} + K \ .$$

With an inner product of the form

$$(f,g) = \int_{z_-}^{z_+} dz \int_D dA \int_{4\pi} d\tilde{\Omega}\ fg\ , \tag{C-1}$$

where D is the entire x-y plane, the same procedure as before is used to eliminate derivatives of ψ in the direction normal to the surface. The new general form of the inverse equation is

$$\int_D dA \int_{4\pi} \mu\psi\ (K_T^{-1}\mu)^{\ell}(P\psi)\, d\tilde{\Omega}\Big|_{z_-}^{z_+} = 0\ , \tag{C-2}$$

where the particular solution P and the value of ℓ again have to be selected. In this case we choose for P the reflection operator $(Pf)(\tilde{\Omega}) = f(-\tilde{\Omega})$. Now, however, there is no general constraint that ℓ be odd, except that the value of $\ell = 0$ is a trivial solution valid for any ψ and hence should not be used. If we again select $\ell = -1$, then the final form for the inverse equation is

$$(\sigma_r + sv^{-1}) \int_D dA \int_{4\pi} \psi P\psi d\tilde{\Omega}\Big|_{z_-}^{z_+}$$

$$+ \int_D dA \int_{4\pi} \psi\tilde{\Omega}\cdot\partial_{\tilde{\rho}}(P\psi)\, d\tilde{\Omega}\Big|_{z_-}^{z_+} = \int_D dA \int_{4\pi} \psi(H_r + H_s)(P\psi)\, d\tilde{\Omega}\Big|_{z_-}^{z_+} . \tag{C-3}$$

The right-hand side of this equation is evaluated in a manner similar to that in Eqs. (18) and (19).

The principal difference between the Fourier transform approach of Eq. (16) and the final result in Eq. (C-3) is that a set of equations for different values of $\tilde{k}$ can be obtained in the former approach, and hence in principle fewer experiments would be required. This is because in Eq. (C-3) only the value of the Laplace transform variable can be changed to generate additional equations.

When $\beta = 0$, the final result reduces to a special case of that obtained by Larsen,[11] who studied the general energy-dependent problem in a body of arbitrary convex shape. His formulation is based on a direct angular flux, ψ, and an adjoint angular flux, ψ^*, that he has related to a directly-measurable ψ. For our one-speed problem, however, $\psi^*(\tilde{\Omega})$ can be selected to be simply $P\psi(\tilde{\Omega}) = \psi(-\tilde{\Omega})$, which gives our result.

Larsen's energy-dependent equation for our plane geometry case can be rederived with the approach developed here by redefining the inner product of Eq. (C-1) to include integration over all energies. Then the corresponding surface contribution becomes

$$\langle f,g \rangle = \int_D dA \int_0^\infty dE \int_{4\pi} d\tilde{\Omega}\ \tilde{\Omega}\cdot\hat{n}\ fg$$

and the inverse equation is

$$\langle L\psi^*, \psi \rangle = 0 ,$$

where

$$L \in A[(K_T^*)^{-1}\mu] .$$

Then the final form of the inverse equation for $\ell = -1$ is

$$\int_D dA \int_0^\infty dE \int_{4\pi} d\tilde{\Omega}\ \tilde{\Omega}\cdot\hat{n}\ \psi^* K_T \psi$$

where operator K_T is now generalized to include energy dependence. The quasi-singular portion of this operator, H_s, now must include a new term that is a derivative in the energy variable.[1]

Received: April 8, 1983
Revised: July 6, 1983