# On the Zeros of Laplace Transforms 

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#### Abstract

Suppose that $f$ is a positive, nondecreasing, and integrable function in the interval $(0,1)$. Then, by Pólya's theorem, all the zeros of the Laplace transform $$
F(z)=\int_{0}^{1} e^{z t} f(t) d t
$$ lie in the left-hand half-plane $\operatorname{Re} z \leq 0$. In this paper, we assume that the additional condition of logarithmic convexity of $f$ in a left-hand neighborhood of the point 1 is satisfied. We obtain the form of the left curvilinear half-plane and also, under the condition $f(+0)>0$, the form of the curvilinear strip containing all the zeros of $F(z)$.

KEY words: Laplace transform, distribution of zeros of the Laplace transform, logarithmically convex function, regularly varying function.


## 1. INTRODUCTION

We consider the Laplace transforms of integrable functions concentrated in a finite interval, which, without loss of generality, we take as the interval $(0,1)$. Thus, let

$$
\begin{equation*}
F(z)=\int_{0}^{1} e^{z t} f(t) d t, \quad f \in L^{1}(0,1) . \tag{1}
\end{equation*}
$$

Such (obviously, entire) functions often occur in different areas of calculus, for example, in spectral theory, in the theory of difference-differential equations, in the study of nonharmonic Fourier series, etc. The question of the distribution of the zeros of the functions (1) is of great interest; it was studied in the papers of Hardy [1], Pólya [2], Titchmarsh [3], Cartwright [4, 5], etc. The following theorem is the starting point for us.

Theorem A [2]. Suppose that $f$ is an integrable, positive, and nondecreasing function in the interval $(0,1)$. Then all the zeros of the function $F(z)$ lie in the left-hand half-plane $\operatorname{Re} z \leq 0$; if, moreover, $f$ is not a step-function of the form

$$
f(t)=c_{i}, \quad t_{i}<t<t_{i+1}, \quad i=0, \ldots, n ; \quad t_{0}=0, \quad t_{n+1}=1, \quad 0<c_{i}<c_{i+1}
$$

where the numbers $t_{i}$ are rational, then all the zeros of the function $F(z)$ lie in the half-plane $\operatorname{Re} z<0$.

From the class of functions $f$ satisfying the assumptions of Theorem A, we extract a sufficiently wide subclass and prove more detailed assertions on the distribution of the zeros of the functions (1) for it; namely, assertions that all the zeros of the functions (1) belong to well-defined proper subsets of the set $\operatorname{Re} z<0$. The additional condition defining the subclass in question is that the function $f$ is logarithmically convex in some left-hand neighborhood of the point 1. For this
subclass, we obtain the form of the left curvilinear half-plane and, under the condition $f(+0)>0$, also the form of the curvilinear strip containing all the zeros of the function (1). In both cases, the description of the set containing zeros is, in a certain sense, exact. Moreover, we also prove that if $f$ is logarithmically convex on the whole interval $(0,1)$, then, regardless of whether $f$ is monotone or not, all the zeros of the function $F(z)$ lie in the union of the horizontal strips $(2 n-1) \pi<|\operatorname{Im} z|<2 \pi n, n \in \mathbb{N}$.

## 2. LEMMAS

A positive function $f$ in the interval $(a, b)$ is said to be logarithmically convex in this interval if the function $\log f$ is convex in $(a, b)$. For a twice differentiable positive function, its logarithmic convexity in $(a, b)$ is equivalent to the condition $\left(f^{\prime}\right)^{2} \leq f f^{\prime \prime}$ everywhere in $(a, b)$ [6]. In the general case, the test for logarithmic convexity consists in the following.

Lemma 1. Suppose that $f$ is a positive function in the interval $(a, b)$. Then for it to be logarithmically convex in this interval, it is necessary and sufficient that the function

$$
f_{x}(t):=e^{x t} f(t)
$$

be convex in $(a, b)$ for all $x \in \mathbb{R}$.
Proof. In both the necessary and sufficient parts of the lemma, we deal with continuous functions $f$ and $\log f$. We shall use the fact that for a function $\varphi \in C(a, b)$ the convexity $\varphi$ in $(a, b)$ is equivalent to the condition

$$
\begin{equation*}
\varphi\left(\frac{t_{1}+t_{2}}{2}\right) \leq \frac{\varphi\left(t_{1}\right)+\varphi\left(t_{2}\right)}{2} \tag{2}
\end{equation*}
$$

for all points $t_{1}, t_{2} \in(a, b)[6]$. Then the logarithmic convexity of the function $f$ in $(a, b)$ is equivalent to the condition

$$
\begin{equation*}
f\left(\frac{t_{1}+t_{2}}{2}\right) \leq \sqrt{f\left(t_{1}\right) f\left(t_{2}\right)} \tag{3}
\end{equation*}
$$

for all $t_{1}, t_{2} \in(a, b)$.
Suppose that the function $f$ is logarithmically convex in $(a, b)$. Then inequality (3) holds. Multiplying (3) term-by-term by $\exp \left(x\left(t_{1}+t_{2}\right) / 2\right)$, we obtain

$$
f_{x}\left(\frac{t_{1}+t_{2}}{2}\right) \leq \sqrt{f_{x}\left(t_{1}\right) f_{x}\left(t_{2}\right)}
$$

Next, for $\varphi=f_{x}$, the application of Cauchy's inequality to the right-hand side yields property (2), which indicates that the function $f_{x}$ is convex.

Suppose that the function $f_{x}$ is convex in $(a, b)$. Then, for $\varphi=f_{x}$, inequality (2) is valid; it can be written as

$$
\begin{equation*}
f\left(\frac{t_{1}+t_{2}}{2}\right) \leq \frac{1}{2}\left(e^{x\left(t_{1}-t_{2}\right) / 2} f\left(t_{1}\right)+e^{x\left(t_{2}-t_{1}\right) / 2} f\left(t_{2}\right)\right) \tag{4}
\end{equation*}
$$

By assumption, for fixed $t_{1}$ and $t_{2}$, this inequality holds for all $x \in \mathbb{R}$, and, in particular, also for the value of $x$ which yields the minimum of the right-hand side. By differentiation, we find that for this value of $x$ the summands on the right-hand side of (4) coincide. But, in this case, their half-sum is equal to their geometric mean, yielding property (3), which indicates the logarithmic convexity of $f$. The lemma is proved.

Lemma 2 [7]. If $\varphi$ is an integrable and convex function in the interval ( $0,2 \pi$ ) (respectively, in the interval $(\pi / 2,5 \pi / 2))$, then

$$
\int_{0}^{2 \pi} \varphi(t) \cos t d t \geq 0 \quad\left(\text { respectively }, \quad \int_{\pi / 2}^{5 \pi / 2} \varphi(t) \sin t d t \geq 0\right)
$$

In what follows, $x=\operatorname{Re} z, \quad y=\operatorname{Im} z, r=|z|, \quad \theta=\arg z$. We denote

$$
\begin{equation*}
\Phi_{u}(z)=\int_{u}^{1} e^{-z t} g(t) d t \tag{5}
\end{equation*}
$$

Lemma 3. Suppose that $g$ is a function of bounded variation on $\left[a_{0}, 1\right], 0 \leq a_{0}<1$, with $g(1-0) \neq 0$. Then, for any fixed $b_{0} \in\left(a_{0}, 1\right)$ and $u \in\left[a_{0}, b_{0}\right]$, the following estimate is valid:

$$
\begin{equation*}
\left|\Phi_{u}(z)\right| \geq C \frac{e^{-x}}{r}(1+o(1)), \quad x \rightarrow-\infty \tag{6}
\end{equation*}
$$

where $C>0$ is independent of $u \in\left[a_{0}, b_{0}\right]$.
Proof. We assume that $g(1)=g(1-0)$; by assumption, $g(1) \neq 0$. Using the left-continuity of $g$ at the point 1 , we choose $\delta \in\left(0,1-b_{0}\right)$ so small that

$$
\begin{equation*}
\operatorname{var}(g(t): 1-\delta \leq t \leq 1) \leq \frac{|g(1)|}{2} \tag{7}
\end{equation*}
$$

We integrate by parts

$$
\begin{equation*}
\Phi_{u}(z)=-\frac{1}{z}\left(e^{-z} g(1)-e^{-z u} g(u)-\int_{u}^{1} e^{-z t} d g(t)\right) \tag{8}
\end{equation*}
$$

Next, we split the integral into the summands $K_{1}$ and $K_{2}$ corresponding, respectively, to the closed intervals $[u, 1-\delta]$ and $[1-\delta, 1]$. For $x \leq 0$, in view of (7), we obtain the estimates

$$
\left|K_{1}\right| \leq V e^{-(1-\delta) x}, \quad\left|K_{2}\right| \leq \frac{|g(1)|}{2} e^{-x}
$$

which, together with the triangle inequality applied to (8), yield the inequality

$$
\left|\Phi_{u}(z)\right| \geq \frac{|g(1)|}{2} \frac{e^{-x}}{r}\left(1-M e^{\left(1-b_{0}\right) x}-V e^{\delta x}\right), \quad x \leq 0
$$

which means that Lemma 3 is valid.

## 3. THE RIGHT BOUNDARY FOR THE ZEROS

Theorem 1. Suppose that the assumptions of Theorem A are satisfied and, moreover, the function $f$ is logarithmically convex on some interval $(1-a, 1), 0<a \leq 1$. Then all the zeros of the function (1) with a sufficiently large modulus of their imaginary part lie in the set

$$
x \leq-\log \left(y^{2} \cdot \int_{0}^{\pi /(2|y|)} t f(1-t) d t\right)+C, \quad|y|>y_{0}
$$

where $C$ is a constant.
Note that Theorem 1 is meaningful (as compared to Theorem A) if, for $f(1-0)=+\infty$, the following inequality holds:

$$
\begin{equation*}
y^{2} \int_{0}^{\pi /(2 y)} t f(1-t) d t>\frac{\pi^{2}}{8} f\left(1-\frac{\pi}{2 y}\right) \rightarrow+\infty, \quad y \rightarrow+\infty \tag{9}
\end{equation*}
$$

Proof of Theorem 1. Since $F(\bar{z})=\bar{F}(z)$, it suffices to consider $y>0$. We have

$$
\begin{gather*}
e^{-z} F(z)=\int_{0}^{1} e^{-z t} f(1-t) d t=: G(z) \\
V(z):=-\operatorname{Im} G(z)=\int_{0}^{1} e^{-x t} f(1-t) \sin y t d t \tag{10}
\end{gather*}
$$

Let us find a lower bound for $|V(z)|$ for sufficiently large $y$ and for $x<0$ (recall that, by Theorem A, $F(z)$ has no zeros for $x \geq 0)$. We put

$$
N=N(y):=\max \left(n \in \mathbb{N}: \frac{\pi}{2 y}+\frac{2 \pi n}{y} \leq a\right), \quad a_{N}:=\frac{\pi}{2 y}+\frac{2 \pi N}{y}
$$

and note that $a_{N} \leq a$. Moreover, since $\pi /(2 y)+2 \pi(N+1) / y>a$, we have $a_{N}>a-2 \pi / y$ and, therefore,

$$
\begin{equation*}
\frac{a}{2}<a_{N} \leq a \tag{11}
\end{equation*}
$$

for sufficiently large $y$. Suppose that

$$
V(z)=\left(\int_{0}^{\pi /(2 y)}+\int_{\pi /(2 y)}^{a_{N}}+\int_{a_{N}}^{1}\right) e^{-x t} f(1-t) \sin y t d t=: V_{1}+V_{2}+V_{3}
$$

Since $x<0$, we have

$$
V_{1}>\frac{2}{\pi} y \int_{0}^{\pi /(2 y)} t f(1-t) d t
$$

Further, by assumption, the function $f(1-t)$ is logarithmically convex on $(0, a)$. By Lemma 1, the function $e^{-x t} f(1-t)$ is convex on $(0, a)$, and hence, by Lemma 2 , we have $V_{2} \geq 0$. Since

$$
\begin{aligned}
V_{3} & =-\operatorname{Im} \int_{a_{N}}^{1} e^{-z t} f(1-t) d t=\operatorname{Im}\left(\frac{1}{z} \int_{a_{N}}^{1} f(1-t) d e^{-z t}\right) \\
& =\operatorname{Im}\left(\frac{1}{z}\left(f(+0)-f\left(1-a_{N}\right) e^{-z a_{N}}-\int_{a_{N}}^{1} e^{-z t} d f(1-t)\right)\right),
\end{aligned}
$$

in view of (11), for $x<0$ we have

$$
\left|V_{3}\right| \leq A \frac{e^{-x}}{r} \leq A \frac{e^{-x}}{y}, \quad A=3 f\left(1-\frac{a}{2}\right)
$$

The function $G(z)$, and hence the function $F(z)$, has no zeros wherever $V_{1}+V_{2}>\left|V_{3}\right|$. Therefore, it follows from the estimates for $V_{i}, i=1,2,3$, that $F(z)$ has no zeros wherever

$$
\frac{2}{\pi} y \int_{0}^{\pi /(2 y)} t f(1-t) d t>\frac{A}{y} e^{-x}, \quad y>y_{0}>0
$$

Taking the logarithm, we obtain the assertion of Theorem 1.
In view of Theorem A, Theorem 1 and inequality (9) yield the following assertion.

Corollary 1. Suppose the assumptions of Theorem 1 are satisfied. Then all the zeros of the function $F(z)$ lie in the union of the set

$$
x \leq \min \left(0,-\log f\left(1-\frac{\pi}{2|y|}\right)+C\right), \quad|y|>\frac{\pi}{2}
$$

where $C$ is a constant, with the half-strip $x<0,|y| \leq \pi / 2$.
We denote by

$$
\left(z_{n}\right)_{1}^{\infty} \quad\left(\text { where } \quad \forall n \quad\left|z_{n+1}\right| \geq\left|z_{n}\right|\right)
$$

the sequence of all zeros of the function $F(z)$ (it is well known [8] that the function (1) has an infinite set of zeros). Consider conditions on $f$ for which

$$
\begin{equation*}
\operatorname{Re} z_{n} \rightarrow-\infty, \quad n \rightarrow \infty . \tag{12}
\end{equation*}
$$

In [9], it was proved that, under the assumptions of Theorem A, all the zeros of the function $F(z)$ lie in some right half-plane $\operatorname{Re} z \geq h>-\infty$ (i.e., in the vertical strip $h \leq \operatorname{Re} z \leq 0$ ) if and only if $0<f(+0) \leq f(1-0)<+\infty$. Thus, for relation (12) to hold, it is necessary that at least one of the conditions $f(+0)=0$ or $f(1-0)=+\infty$ be satisfied. Corollary 1 yields the following result.

Corollary 2. Suppose that the function $f$ satisfies the assumptions of Theorem 1. In that case, if $f(1-0)=+\infty$, then relation (12) holds for the zeros of the function $F(z)$.

## 4. THE LEFT BOUNDARY FOR THE ZEROS

This boundary depends on the behavior of the function $f$ in the neighborhood of the point 0 . We assume that $f(+0)>0$.
Theorem 2. Suppose that the assumptions of Theorem 1 are satisfied, and also $f(+0)>0$. Then there exist constants $C \in \mathbb{R}$, and $H, y_{0}>0$ such that all the zeros of the function $F(z)$ lie in the union of the set

$$
x \geq-\log \left(|y| \int_{0}^{\pi /(2|y|)} f(1-t) d t\right)+C, \quad|y|>y_{0}
$$

with the set $x \geq-H,|y| \leq y_{0}$.
Proof. Suppose that $G(z)$ and $a_{N}$ are the same as in the proof of Theorem 1. It suffices to consider $y \geq 0$. First, let us show that in any sector $\cos \theta \leq-\delta<0$ there are at most a finite number of zeros of the function $G(z)$. To do this, we denote by $J_{1}, J_{2}$ the summands in the integral (10) corresponding to the intervals $(0,1 / 2)$ and $(1 / 2,1)$, respectively. Then

$$
\left|J_{1}\right| \leq e^{-x / 2}\|f\|_{1}, \quad x \leq 0,
$$

and since $J_{2}=\Phi_{1 / 2}$ in the notation (5), by Lemma 3, (with $g(t)=f(1-t)$ ) for $J_{2}$ we have estimate (6). Hence if $H$ is sufficiently large, then, by the triangle inequality, for $x \leq-H<0$ we obtain the estimate

$$
|G(z)| \geq C_{1} \frac{e^{-x}}{r}-C_{2} e^{-x / 2} \geq \frac{C_{1}}{r} e^{-x}\left(1-C_{3} r e^{x / 2}\right)
$$

The last bracket is positive if $\cos \theta \leq-\delta<0$ and $r$ is sufficiently large; hence the intermediate assertion dealing with the sector is proved.

Thus, to find the left boundary for the zeros, we need only consider the set

$$
\begin{equation*}
x<-H, \quad y>\frac{r}{2} \tag{13}
\end{equation*}
$$

where $H$ is sufficiently large. We can assume that $a<1$. Let

$$
G(z)=\left(\int_{0}^{a_{N}}+\int_{a_{N}}^{1}\right) e^{-z t} f(1-t) d t=: G_{1}(z)+G_{2}(z)
$$

In view of (11), by Lemma 3, we have estimate (6) for $\Phi_{a_{N}}(z)=G_{2}(z)$. Hence, on the set (13), for sufficiently large $H$ the following estimate is valid:

$$
\begin{equation*}
\left|G_{2}(z)\right| \geq B \frac{e^{-x}}{y}, \quad B>0 \tag{14}
\end{equation*}
$$

Now, suppose that $G_{1}=U-i V$, where

$$
U=\int_{0}^{a_{N}} e^{-x t} f(1-t) \cos y t d t, \quad V=\int_{0}^{a_{N}} e^{-x t} f(1-t) \sin y t d t
$$

We write

$$
V=\int_{0}^{\pi / y}+\int_{\pi / y}^{3 \pi /(2 y)}+\int_{3 \pi /(2 y)}^{a_{N}-\pi / y}+\int_{a_{N}-\pi / y}^{a_{N}}=: V_{1}+V_{2}+V_{3}+V_{4}
$$

Then, since $\sin y t$ is negative, $V_{2}<0$, and the successive applications of Lemmas 1 and 2 yields the inequality $V_{3} \leq 0$. Taking into account the fact that the function $f(1-t)$ is nonincreasing and also the inequality $-x / y \leq M_{1}<+\infty$, on the set (13) we have

$$
0<V_{1}<2 e^{-\pi x / y} \int_{0}^{\pi /(2 y)} f(1-t) d t<C \int_{0}^{\pi /(2 y)} f(1-t) d t
$$

Further, taking (11) and the monotonicity of $f$ into account and denoting $I_{N}=\left(a_{N}-\pi / y, a_{N}\right)$ and $C_{1}=3 f(1-a / 2)$, for $y>y_{0}$ we obtain

$$
\begin{align*}
\left|V_{4}\right| & =\left|\operatorname{Im}\left(\frac{1}{z} \int_{I_{N}} f(1-t) d e^{-z t}\right)\right| \\
& \leq \frac{1}{r}\left|f\left(1-a_{N}\right) e^{-z a_{N}}-f\left(1-a_{N}+\frac{\pi}{y}\right) e^{-z\left(a_{N}-\pi / y\right)}-\int_{I_{N}} e^{-z t} d f(1-t)\right|<C_{1} \frac{e^{-a x}}{y} \tag{15}
\end{align*}
$$

But

$$
\begin{equation*}
V=\int_{0}^{\pi /(2 y)}+\int_{\pi /(2 y)}^{a_{N}}>0 \tag{16}
\end{equation*}
$$

since the integrand in the first summand is positive and, in view of Lemma 1, Lemma 2 can be applied to the second summand. Since $V_{2}, V_{3} \leq 0$, we have

$$
0<V=\left(V_{1}+V_{4}\right)+\left(V_{2}+V_{3}\right) \leq V_{1}+V_{4} \leq V_{1}+\left|V_{4}\right|
$$

i.e.,

$$
\begin{equation*}
|V| \leq C \int_{0}^{\pi /(2 y)} f(1-t) d t+C_{1} \frac{e^{-a x}}{y} \tag{17}
\end{equation*}
$$

Treating the function $U$ in a similar way, we can write

$$
U=\int_{0}^{\pi /(2 y)}+\int_{\pi /(2 y)}^{\pi / y}+\int_{\pi / y}^{a_{N}-3 \pi /(2 y)}+\int_{a_{N}-3 \pi /(2 y)}^{a_{N}}=: U_{1}+U_{2}+U_{3}+U_{4}
$$

and note that $U_{2}<0$, since $\cos y t$ is negative and $U_{3} \leq 0$ by Lemmas 1 and 2. Further, on the set (13) we have

$$
0<U_{1}<e^{-\pi x /(2 y)} \int_{0}^{\pi /(2 y)} f(1-t) d t<C \int_{0}^{\pi /(2 y)} f(1-t) d t
$$

and for $\left|U_{4}\right|$ an estimate of the type (15) is valid, i.e., $\left|U_{4}\right| \leq C_{1} e^{-a x} / y$. Similarly to (16), the following inequality holds:

$$
U=\int_{0}^{a_{N}-\pi /(2 y)}+\int_{a_{N}-\pi /(2 y)}^{a_{N}}>0
$$

and hence for $|U|$ we obtain estimate (17).
Now, recalling the relation $G=U-i V+G_{2}$ and using the estimates (14) and (17) (the last one is also valid for $|U|$ ), we obtain

$$
\begin{equation*}
|G(z)| \geq B \frac{e^{-x}}{y}-C_{2} \int_{0}^{\pi /(2 y)} f(1-t) d t-C_{3} \frac{e^{-a x}}{y} \tag{18}
\end{equation*}
$$

If $H$ is sufficiently large, then $(B / 2) e^{-x}>C_{3} e^{-a x}$ for $x<-H$, and it follows from (18) that $G(z)$, and hence also $F(z)$, has no zeros on the set

$$
\frac{B}{2} e^{-x}>C_{2} y \int_{0}^{\pi /(2 y)} f(1-t) d t, \quad y>y_{0}
$$

Theorem 2 is proved.
In view of Theorem A, Theorems 1 and 2 yield the following assertion.
Corollary 3. Suppose that the assumptions of Theorem 2 are satisfied. Then there exist constants $C_{1}, C_{2} \in \mathbb{R}$ and $H, y_{0}>0$ such that all the zeros of the function $F(z)$ lie in the curvilinear strip which is the union of the set

$$
\begin{equation*}
-\log \left(|y| \int_{0}^{\pi /(2|y|)} f(1-t) d t\right)+C_{1} \leq x \leq-\log \left(y^{2} \int_{0}^{\pi /(2|y|)} t f(1-t) d t\right)+C_{2}<0 \tag{19}
\end{equation*}
$$

where $|y|>y_{0}$, with set $-H \leq x<0,|y| \leq y_{0}$.
Remark 1. In view of Corollary 1, the set (19) in Corollary 3 can be replaced by the set

$$
-\log \left(|y| \int_{0}^{\pi /(2|y|)} f(1-t) d t\right)+C_{1} \leq x \leq-\log f\left(1-\frac{\pi}{2|y|}\right)+C_{2}<0
$$

## 5. THE CASE OF A REGULARLY CHANGING FUNCTION $f$

For a regularly varying function $f$ in the neighborhood of the point 1 , the asymptotics of the integrals appearing in (19) can be calculated, which allows us to simplify the form of the set (19).

The definitions and facts given below are taken from [10].
A positive and measurable function $g$ in the right-hand neighborhood of the point 0 , is called a regularly varying function of order $\alpha \in \mathbb{R}$ if, for all $\lambda>0$,

$$
\begin{equation*}
\lim _{t \rightarrow+0} \frac{g(\lambda t)}{g(t)}=\lambda^{\alpha} \tag{20}
\end{equation*}
$$

A regularly varying function of order $\alpha=0$ is called a slowly varying function. A regularly varying function of order $\alpha$ is of the form $g(t)=t^{\alpha} l(t)$, where $l(t)$ is a slowly varying function. For any slowly varying function $l(t)$, as $t \rightarrow+0$, there exists an equivalent continuously differentiable function $l_{0}(t)$ satisfying the condition

$$
\begin{equation*}
l_{0}^{\prime}(t)=o\left(\frac{l_{0}(t)}{t}\right), \quad t \rightarrow+0 \tag{21}
\end{equation*}
$$

We denote

$$
g_{1}(\varepsilon)=\frac{1}{\varepsilon} \int_{0}^{\varepsilon} g(t) d t, \quad g_{2}(\varepsilon)=\frac{1}{\varepsilon^{2}} \int_{0}^{\varepsilon} t g(t) d t .
$$

Lemma 4. Suppose that the function $g$ is integrable on $(0,1)$ and is a regularly varying function of order $\alpha>-1$. Then, as $\varepsilon \rightarrow+0$,

$$
g_{1}(\varepsilon) \sim \frac{1}{1+\alpha} g(\varepsilon), \quad g_{2}(\varepsilon) \sim \frac{1}{2+\alpha} g(\varepsilon) .
$$

Proof. By assumption, $g(t)=t^{\alpha} l(t)$, where $l(t)$ is a slowly varying function Since the replacement of the function $g$ by an equivalent function as $t \rightarrow+0$ does not affect the the asymptotics of the functions $g_{1}$ and $g_{2}$, we can assume that $l(t)$ possesses property (21). By the L'Hospital rule, we have

$$
\begin{align*}
\lim _{\varepsilon \rightarrow+0} \frac{g_{1}(\varepsilon)}{g(\varepsilon)} & =\lim _{\varepsilon \rightarrow+0} \frac{1}{\varepsilon^{1+\alpha} l(\varepsilon)} \int_{0}^{\varepsilon} t^{\alpha} l(t) d t \\
& =\lim _{\varepsilon \rightarrow+0} \frac{\varepsilon^{\alpha} l(\varepsilon)}{(1+\alpha) \varepsilon^{\alpha} l(\varepsilon)+\varepsilon^{1+\alpha} l^{\prime}(\varepsilon)}=\lim _{\varepsilon \rightarrow+0} \frac{1}{1+\alpha+\left(\varepsilon l^{\prime}(\varepsilon) / l(\varepsilon)\right)} \tag{22}
\end{align*}
$$

which, in view of property (21), yields the required asymptotics for $g_{1}$. If $g_{1}$ is replaced by $g_{2}$ in (22), then $\alpha$ is replaced by $\alpha+1$, and everything is repeated. The lemma is proved.

Let us return to Corollary 3. The expressions under the signs of the logarithms in (19) are proportional to $g_{1}(\varepsilon)$ and $g_{2}(\varepsilon)$, respectively, where $\varepsilon=\pi /(2|y|)$, and $g(t)=f(1-t)$. Therefore, using Lemma 4 and relation (20), from Corollary 3 we obtain the following assertion.
Corollary 4. Suppose that the assumptions of Theorem 2 are satisfied and, moreover, the function $f(1-t)$ is a regularly varying function of order $\alpha \in(-1,0)$. Then all the zeros of the function $F(z)$ lie in the curvilinear strip which is the union of the set $-H \leq x<0 \quad|y| \leq y_{0}$ with the set

$$
\left|x+\log f\left(1-\frac{1}{|y|}\right)\right| \leq C, \quad|y|>y_{0}
$$

The example of the function $f(1-t)=1 /\left(t \log ^{2} t\right)$, with $g_{1}(\varepsilon)=1 /(\varepsilon|\log \varepsilon|)$, shows that for $\alpha=-1$ the functions $g_{1}$ and $g_{2}$ are no longer asymptotically proportional, and in this case we can only rely on Remark 1.

## 6. THE DISTRIBUTION OF ZEROS IN THE HORIZONTAL STRIPS

Theorem 3. Suppose that the function $f$ is integrable and logarithmically convex in the interval $(0,1)$. Then all the zeros of the function (1) lie in the union of the horizontal strips

$$
(2 n-1) \pi<|\operatorname{Im} z|<2 \pi n, \quad n \in \mathbb{N} .
$$

Proof. Suppose that

$$
G(z)=e^{-z} F(z)=\int_{0}^{1} e^{-z t} f(1-t) d t=: U(z)-i V(z)
$$

It suffices to show that the function $G(z)$ has no zeros in the strips

$$
2 \pi n \leq y \leq(2 n+1) \pi, \quad n \in \mathbb{Z}_{+}
$$

First, suppose that

$$
\begin{equation*}
\left(2 n+\frac{1}{2}\right) \pi \leq y \leq(2 n+1) \pi, \quad n \in \mathbb{Z}_{+} \tag{23}
\end{equation*}
$$

If $n=0$, then $\sin y t>0$ for $0<t<1$ and, since $f>0$, we have $V>0$. If $n \geq 1$, then, taking the integrand as $e^{-x t} f(1-t) \sin y t$, we can write

$$
V(z)=\int_{0}^{\pi /(2 y)}+\sum_{k=0}^{n-1} \int_{\pi(1 / 2+2 k) / y}^{\pi(5 / 2+2 k) / y}+\int_{\pi(1 / 2+2 n) / y}^{1}=: I_{1}+\sum_{k=1}^{n-1} V_{k}+I_{2}
$$

In both $I_{1}$, and $I_{2}$, we have $\sin y t>0$; this is obvious for $I_{1}$, but if we consider $I_{2}$, then $\pi(1 / 2+2 n) / y<t<1$, and hence $\pi / 2+2 \pi n<y t<y \leq(2 n+1) \pi$ (we have taken into account the right-hand inequality in (23)). Therefore, $I_{1}+I_{2}>0$. Successive applications of Lemmas 1 and 2 yield the inequality $V_{k} \geq 0$. Hence $V>0$ in the set (23), and on this set there are no zeros of $G(z)$.

It remains to consider the strips

$$
\begin{equation*}
2 \pi n \leq y<\frac{\pi}{2}+2 \pi n, \quad n \in \mathbb{Z}_{+} \tag{24}
\end{equation*}
$$

Now, we proceed with the function $U(z)$ in a similar way. If $n=0$, then $\cos y t>0$ for $0<t<1$ and, therefore, $U>0$. For $n \geq 1$, we can write

$$
U(z)=\sum_{k=1}^{n} \int_{2 \pi(k-1) / y}^{2 \pi k / y}+\int_{2 \pi n / y}^{1}=: \sum_{k=1}^{n} U_{k}+J
$$

where the integrand is $e^{-x t} f(1-t) \cos y t$. By Lemmas 1 and 2 , we again have $U_{k} \geq 0$, and $J>0$ because of the positivity of the integrand: indeed, if $2 \pi n / y<t<1$, then, in view of the right-hand inequality in (24), we have $2 \pi n<y t<y \leq \pi / 2+2 \pi n$. Hence $U>0$ on the set (24), and on this set $G(z)$ has no zeros, either. Theorem 3 is proved.

Theorems A and 3 yield the following assertion.

Corollary 5. Suppose that the function $f$ is integrable, nondecreasing, and logarithmically convex in the interval $(0,1)$. Then all the zeros of the function (1) lie in the union of the half-strips

$$
\operatorname{Re} z<0, \quad(2 n-1) \pi<|\operatorname{Im} z|<2 \pi n, \quad n \in \mathbb{N}
$$

As an example, consider the following function of Mittag-Leffler type of the order of unity:

$$
E_{1}(z ; \mu)=\frac{1}{\Gamma(\mu-1)} \int_{0}^{1} e^{z t}(1-t)^{\mu-2} d t, \quad \mu>1
$$

(see [11]). For $1<\mu<2$, the function $f(t)=(1-t)^{\mu-2}$ satisfies the assumptions of Theorem 3 . It is well known [12] that for $1<\mu<2$ all the zeros of the function $E_{1}(z ; \mu)$ lie in the half-plane $\operatorname{Re} z<\mu-2$. Combining this with Theorem 3, we obtain the following result.

Corollary 6. For $1<\mu<2$, all the zeros of the function $E_{1}(z ; \mu)$ lie in the union of the half-strips

$$
\operatorname{Re} z<\mu-2, \quad(2 n-1) \pi<|\operatorname{Im} z|<2 \pi n, \quad n \in \mathbb{N}
$$

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