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Pseudodifferential Operators on $SU(2)$

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ABSTRACT. *We construct an algebra of left-invariant pseudodifferential operators on* $SU(2)$. *We require only that the symbols be homogeneous and* C^2. *For Fourier-bandlimited symbols, we derive the expected formulae for composition and commutators and construct an orthonormal basis of common approximate eigenvectors that could be used to study spectral theory. Some remarks on applications to matrices of operators are made.*

Introduction

The idea of time-frequency localization is ubiquitous in mathematics. Analysts using tools such as wavelets and windowed Fourier transforms look for the "time-frequency content" of functions (e.g., [M]). Pseudodifferential operator theorists discuss microlocal regularity and wavefront sets (e.g., [T]). Representation theorists seek to unify their subject, which is after all the study of functions on groups, through the orbit method (see [Ki]), which is based on the idea that different representations "live" in different coadjoint orbits that correspond to them in an almost functorial way.

The Fourier transform of a function $f(x)$ is defined by

$$\hat{f}(\xi) = \int f(x) e^{-2\pi i x \cdot \xi}\, dx$$

and may be thought of as extracting the frequency content of f. Indeed, $\hat{f}(\xi)$ is the inner product of the function f with a "pure wave" of frequency ξ. As the reader undoubtedly knows, this transform is an indispensable tool for the mathematical analyst, and it would be unreasonable for us to try to list the applications. The Fourier transform is equally important for applied analysts, being the main tool for processing stationary signals.

As pointed out in [M], time-frequency methods were introduced by both pure and applied analysts in order to study more general objects. With a desire to generalize results obtained by Fourier transform methods, mathematicians introduced various notions of time-scale analysis (e.g., atomic decompositions and Stein's generalization of Littlewood–Paley theory to $\mathbb{R}^n$). Applied analysts need time-frequency methods to process "signals whose spectral characteristics are varying with time" [B, p. 418].

Time-scale and time-frequency methods amount to studying functions and operators in terms of their interaction with "atomic" functions that roughly speaking, have particular time-frequency support. (As will be explained, one cannot construct "atoms" that have precise time-frequency support.) So, for example, the continuous wavelet transform is defined, for appropriate ψ, by

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$$W_f(t,s) = \int f(x)\psi\left(\frac{x-t}{s}\right) dx$$

and is seen to be the result of decomposing f into its inner products with a certain function "shifted" in time and scale. Similarly, the windowed Fourier transform

$$S_w(f)(t,\xi) = \int f(x)w(x-t)e^{ix\cdot\xi}\, dx$$

is given by the values of the function f paired with a window function shifted in time and frequency.

The Fourier inversion theorem states roughly that

$$f(x) = \int \hat{f}(\xi)e^{2\pi i x\cdot\xi}\, d\xi.$$

Hence the functions $\{e^{2\pi i x\cdot\xi}\}_{\xi\in\mathbb{R}^n}$ behave like an orthonormal basis, and we may say that the values of $\hat{f}$ are independent of one another. At the risk of being pedantic, we may also write that roughly

$$f(x) = \int f(y)\delta(x-y)\, dy,$$

which is a formula that is its own inverse. This last expression is the statement that the Dirac delta functions $\{\delta(x-y)\}_{y\in\mathbb{R}^n}$ are as much like an orthonormal basis as the Fourier "basis."

Hand in hand with these observations is the observation that the operator $T_S : f(x) \mapsto \chi_S(x)f(x)$ is a projection that annihilates a nontrivial set of functions for any set S (provided that the complement of S has positive measure). Similarly, one has the corresponding frequency projections B_F that map f to the function whose Fourier transform is given by $\chi_F(\xi)\hat{f}(\xi)$ for any set of frequencies F. These too are projections that annihilate a nontrivial set of functions when the complement of F has positive measure.

One might wish to say that a function f has "time-frequency support" in the set $S \times F$ if f is supported in S and $\hat{f}$ is supported in F. However, one has Heisenberg uncertainty type theorems, the weakest of which states that if S and F are bounded sets, then no function has "time-frequency support" in $S \times F$. If one tries to force things by first projecting f via T_S, so that the result will be supported in S, and then projecting via B_F, so that the result will be frequency supported in F, one finds that the intermediate property of being supported in S has vanished. The operators T_S and B_F do not commute.

One may attempt to rectify matters by replacing the notion of support by one of almost support, saying that a function is supported in a set S if most of its energy (L^2-norm) is contained in that set. The traditional Heisenberg uncertainty theorem states that the area of $S \times F$ must be at least one for a function to be almost time-frequency supported on $S \times F$. By using concepts such as almost time-frequency support, analysts are able to perform useful phase-space analyses. The reader is urged again to consult [M] for an excellent survey of these matters and to see [B] for further discussion of applications of time-frequency methods to signal analysis.

To put the notion of time-frequency support in the context of an earlier paragraph, as well as the rest of the paper, one can say that we are replacing the commuting of T_S and B_F by the almost commuting of the good classes of operators we shall introduce. The eigenfunctions of these operators also must have time-frequency support, which is as large as a Heisenberg box. The question we shall be addressing is the location of the box. A pseudodifferential operator is, roughly speaking, an operator written in the form

$$Lf(x) = \int\int a(x,\xi)e^{i(x-y)\cdot\xi} f(y)\, dy\, d\xi. \qquad (0.1)$$

Another way of writing this, in the spirit of the spectral theorem, is

$$L = \int a(x, \xi)\, dT_x\, dB_\xi. \tag{0.1*}$$

The function $a(x, \xi)$ is called the symbol of the operator. Usually, there is a regularity requirement on $a(x, \xi)$ forcing a to be smooth in ξ and controlling the growth in ξ of a and its derivatives in x. The action of converting a symbol to an operator is called a quantization. Sometimes, instead of $a(x, \xi)$ in the integral one writes $a(y, \xi)$ or even $a(\frac{x+y}{2}, \xi)$. These are, respectively, the right and Weyl quantizations. The quantization (0.1) is called the Kohn–Nirenberg quantization. Intuitively, one should think of a pseudodifferential operator as acting as a "multiplier," i.e., that it multiplies the functions it acts on, represented as functions in time and frequency, by its symbol. Pseudodifferential operator theory is concerned with making approximate sense of this idea. One does so by introducing the notion of the wavefront set of a function. This is, roughly speaking, the set of points (x, ξ) so that all localizations of the function f near x have the property that their Fourier transforms do not decay rapidly in the direction of ξ. The wavefront set is a sort of time-frequency singular set. The action of pseudodifferential operators does not increase wavefront sets. This is one example of the time-frequency locality of pseudodifferential operators. Many things can be shown about the behavior of the wavefront sets of solutions to PDEs in terms of the highest order terms of the PDE. This sort of result is commonly referred to as microlocal regularity.

Given a Lie group G, its coadjoint orbits are the orbits in the cotangent space at the identity T_e^*G of the natural group action there. Because of the group action, the cotangent bundle of the group $T^G = T_e^*G \times G$ is trivial. The representations of the group may all be viewed as spaces of functions on the group in the usual way. Then one can ask where in the cotangent bundle of the group does the time-frequency support of a given representation lie. A good answer is on the product of a coadjoint orbit with the group. One way to see this is to notice that the coadjoint orbits are the simultaneous level sets of the symbols of all of the biinvariant operators on the group—operators for which the representations are eigenspaces. Coadjoint orbits are also an attractive answer because they are symplectic manifolds sporting a natural Poisson bracket related to the commutation relations of the Lie algebra of G. As we shall mention a little later, this suggests as a goal for geometry, the association to each symplectic manifold of a space of functions for which the symplectic manifold should be viewed as the time frequency space and a quantization procedure for making operators out of the functions on the manifold.

In this paper, we seek to unify some of these ideas. We construct an algebra of left-invariant pseudodifferential operators on $SU(2)$. This project began out of a desire to understand better the effect that the symbol of an operator has on its spectrum. The classical Weyl's lemma for the Laplacian states that the number of eigenvalues of the Laplacian on a compact manifold M that are less than λ can be asymptotically related to the volume of the set in the cotangent bundle of M where the symbol of the Laplacian is less than λ. Various Szëgo limit theorems hint that the time-frequency support of small eigenvalues for other operators should be essential where their symbols are small. See [LS] for a complete description. We wanted a more direct approach to these ideas, in particular to study the spectral asymptotics of matrices of differential operators with identically vanishing principal symbol determinant. Here, each eigenvalue of the symbol matrix in the appropriate sense of eigenvalue should relate the asymptotic behavior of infinitely many eigenvalues. The goal was to see that each eigensymbol is essentially equally important in any region of phase space.

Left-invariant operators on semisimple compact groups seemed a good model case in which to test these ideas since such operators have many common finite-dimensional invariant subspaces and are nonetheless noncommutative enough to be interesting. We found that to prove some of the results we wanted for matrices of operators, it was necessary to be able to construct rather well-behaved pseudodifferential operators. We find that the operators we construct have an orthonormal basis of common approximate eigenvectors. This basis is essentially a local Fourier basis.

All of our results are stated for operators with bandlimited symbols. This restriction is not essential, however, since all C^2 functions are in an appropriate sense "almost bandlimited," i.e., their Fourier series decay like $\frac{1}{n^2}$. Thus, any of our operators may be decomposed into the sum of one with bandlimited symbol and one with arbitrarily small norm.

The operators we discuss are not new. A much more general class was studied in [CW] where boundedness between various L^p spaces was obtained. What is new, however, is our symbol calculus for these operators; and the basis of approximate eigenvectors for those whose symbols are in C^2. We prove the following result.

Theorem 5.2.

Let f be a bandlimited symbol. Then $x_1^n, \ldots, x_{n+1}^n$ is an orthonormal basis of approximate eigenvectors for $q_n^{(d)}(f)$ with error bounded by $C(n+1)^{d-\frac{1}{4}}$ where C is a constant depending only on the C^2 norm of f and its bandwidth.

Here $q_n^{(d)}$ is our quantization of homogeneous symbols of order d on the nth representation space of $SU(2)$.

Many generalizations are possible. The restriction to $SU(2)$ does not seem to be essential. We are presently working on generalizing our construction to other semisimple compact groups, the geometry of whose coadjoint orbits we understand less well. Also, some of our spectral theoretic results even in the $SU(2)$ case are not completely satisfactory.

A more general context in which to view our constructions is that of quantization of symplectic manifolds. Our algebra constitutes an asymptotic quantization of the sphere (see [KM]). Modulo some technical difficulties, a similar construction on toric manifolds (for definition see [KT]) is indicated. In [KM], rather general constructions of asymptotic quantizations are given; however, they operate on a sheaf of wave packets which is a rather unwieldy object. Our operators, however, will act on the usual spaces given by geometric quantization [AK, Ko, GS, KT] that are usually viewed as the spaces of global holomorphic sections of a line bundle. We hope that our work will be a step forward toward the aspiration of [KM] for "A Fourier transformation theory in which nonlinear symplectic manifolds would serve as phase spaces, i.e., the spaces where the wavefront sets lie, [which] could become a definitive mathematical theory of quantization of these manifolds. This kind of theory would combine both of the aforementioned approaches to quantization." We hope our paper inspires future work.

1. The Setup

The group $\mathrm{G} = SU(2)$ may be thought of as the unit quaternions. In other words, as a set,

$$\mathrm{G} = \{w + ix + jy + kz \in \mathbb{H} \mid w^2 + x^2 + y^2 + z^2 = 1\}.$$

All of the tangent spaces may be identified by the group action to the one at the identity, namely, the hyperplane $w = 1$, which we will denote by $\mathfrak{g}$. Note that in this coordinate system, the bracket operation for the Lie algebra $\mathfrak{g}$ is given by

$$[e_1, e_2] = 2e_3, \qquad [e_2, e_3] = 2e_1, \qquad [e_3, e_1] = 2e_2,$$

where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, and $e_3 = (0, 0, 1)$.

The tangent and cotangent spaces, $\mathfrak{g}$ and $\mathfrak{g}^*$, are identified by the natural metric

$$g = dx^2 + dy^2 + dz^2.$$

We would like to have a procedure for converting a function on the cotangent space into a left-invariant operator on functions on G that agrees with pseudodifferential and representation theoretic intuition.

Recall that the complexification of $\mathfrak{g}$ is isomorphic as a Lie algebra to $sl(2,\mathbb{C})$, which is the Lie algebra generated by H, X and Y with commutation relations $[H,X]=2X$, $[H,Y]=-2Y$, and $[X,Y]=H$. The map from $sl(2,\mathbb{C})$ to the complexification of $\mathfrak{g}$ is given by $H=i(0,0,1)$, $X=i(1/2,i/2,0)$, and $Y=i(1/2,-i/2,0)$. The reader is invited to check that this is a Lie algebra isomorphism.

The spheres $x^2+y^2+z^2=n^2$ are the integer coadjoint orbits that are usually associated with the representations of highest weight n and dimension $n+1$. Alas, we shall find it more convenient to associate the sphere of radius $n+1$ to the representation of highest weight n. We shall choose a Cartan subalgebra, which is the same as choosing a direction in $\mathfrak{g}$, and is also the same as choosing a direction in $\mathfrak{g}^*$. We choose the direction $(0,0,1)$.

Notice that in the representation of highest weight n the matrices for H, X, and Y are given by

$$\pi_n(X)=\begin{pmatrix} 0 & \sqrt{n} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{2(n-1)} & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & 0 & \cdots & 0 & \sqrt{n} \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \pi_n(Y)=\begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ \sqrt{n} & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{2(n-1)} & & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & & \sqrt{n} & 0 \end{pmatrix},$$

and

$$\pi_n(H)=\begin{pmatrix} n & 0 & \cdots & 0 & 0 \\ 0 & n-2 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & -n+2 & 0 \\ 0 & 0 & \cdots & 0 & -n \end{pmatrix}.$$

One can see that the values of the matrix correspond exactly to the Fourier coefficients of the functions $(x+iy)/2$, $(x-iy)/2$ and z on the sphere of radius $n+1$, along horizontal circles at integer height. Since the symbols of these operators are, respectively, $i(x+iy)/2$, $i(x-iy)/2$, and iz, a quantization is suggested that associates to functions, matrices built out of their Fourier coefficients.

We will let (z,θ) be coordinates on S^2, the unit sphere, where θ is the angle of longitude on the sphere, about the z-axis. We will write functions on the two-sphere as $f(z,\theta)$ and expand them in Fourier series in θ as

$$f(z,\theta)=\sum f_n(z)e^{in\theta}.$$

Before introducing the quantization procedure, we will state and prove a few lemmas about regularity of the functions $f_n(z)e^{in\theta}$.

2. Regularity of Spherically Directed Fourier Components

Lemma 2.1.
If $f\in C^k(S^2)$, then $f_n(z)e^{in\theta}\in C^k(S^2)$.

Proof. Then nth component of f,

$$f_n(z)e^{in\theta}=\frac{1}{2\pi}\int_0^{2\pi} S_\theta f e^{-in\theta}\,d\theta,$$

where S_θ denotes the action of the circle on functions on the sphere. Thus, $f_n(z)e^{in\theta}$ is an integral of C^k functions on the sphere and hence is C^k itself. □

Lemma 2.2.

If $f(z,\theta) \in H^k(S^2)$, the kth L^2 *Sobolev space on* S^2, *then* $f_n(z)e^{in\theta} \in H^k(S^2)$ *and* $\|f_n(z)e^{in\theta}\|_{2,k} \le \|f\|_{2,k}$.

Proof. The vector field $\frac{\partial}{\partial\theta} = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$ preserves the degree of polynomials. The Hilbert space $H^k(S^2)$ decomposes into an orthogonal direct sum of the spaces of spherical harmonics of fixed degree. The nth component of f is just the projection of f into the in eigenspace of $\frac{\partial}{\partial\theta}$. Since $\frac{\partial}{\partial\theta}$ preserves the spaces of spherical harmonics, the growth of the spherical harmonic coefficients of $f_n(z)e^{in\theta}$ can be no greater than the growth of the coefficients of f. Hence if f is in $H^k(S^2)$, then so is $f_n(z)e^{in\theta}$. $\square$

Unfortunately, in order to obtain the needed estimates for our quantization procedure, we must understand the regularity of $f_n(z)e^{in\theta}$ not just as a function on the sphere but as a function of z. This is very different. For instance, $x + iy = \sqrt{1-z^2}e^{i\theta}$ is C^∞ on the sphere, but $\sqrt{1-z^2}$ is not even C^1 as a function of z. This is because $\frac{\partial}{\partial z}$ is not a continuous vector field on the sphere. Its length is $\frac{1}{\sqrt{1-z^2}}$.

Lemma 2.3.

Let $f_n(z)e^{in\theta}$ *be a* C^2 *function on* S^2. *Then near either pole,*

$$f_n(z) = C\sqrt{1-z^2} + R(z),$$

where $R(z)$ *is* C^1 *in* z *up to the pole and* C *is a constant that vanishes when* $|n| \neq 1$. *Also,* $R(z)$ *is* $o(\sqrt{1-z^2})$ *when* $n \neq 0$.

Proof. The function

$$f_n(z)e^{in\theta} = g(x,y) + \int_0^{\sqrt{1-z^2}} h(r,\theta)\,dr,$$

where $g(x,y)$ is a polynomial of degree 2 in x and y (the ≤ 2 part of the Taylor series of $f_n(z)e^{in\theta}$) and h is a function that is C^2 in θ and C^1 in r with h and $\frac{\partial h}{\partial r}$ vanishing at 0. The function $g(x,y)$ can be written as

$$\begin{aligned} g(x,y) &= A_1 + A_2(x+iy) + A_3(x-iy) + A_4(x+iy)^2 + A_5(x^2+y^2) + A_6(x-iy)^2 \\ &= A_1 + \sqrt{1-z^2}(A_2e^{i\theta} + A_3e^{-i\theta}) + (1-z^2)(A_4e^{2i\theta} + A_5 + A_6e^{-2i\theta}), \end{aligned}$$

where the A's are constants, most of which are zero. The functions 1 and $1-z^2$ are C^1 in z. Now, $A_2 = 0$ unless $n = 1$ and $A_3 = 0$ unless $n = -1$, so we need only to show that the lemma holds for $f_n(z)e^{in\theta} = \int_0^{\sqrt{1-z^2}} h(r,\theta)\,dr$. Now $r = \sqrt{1-z^2}$. So combining all our formulae yields

$$f_n(z) = \int_0^{\sqrt{1-z^2}} \int_0^{2\pi} h(r,\theta)e^{-in\theta}\,d\theta\,dr.$$

Hence,

$$f_n'(z) = \frac{-z}{\sqrt{1-z^2}} \int_0^{2\pi} h(\sqrt{1-z^2},\theta)e^{-in\theta}\,d\theta = (-z)\int_0^{2\pi} \frac{h(r,\theta)}{r}e^{-in\theta}\,d\theta.$$

By assumption on h, the function $\frac{h(r,\theta)}{r}$ is continuous in r. Hence, f_n' is continuous.

Finally, $R(z) = A_1 + R_1(z)$, with $R_1(z)$ being $o(\sqrt{1-z^2})$ and $A_1 = 0$ unless $n = 0$. $\square$

3. Quantization Procedure

Given $f(z,\theta) = \sum f_n(z)e^{in\theta}$, a C^2 function on S^2, the two-sphere of radius 1, define

$$q_n^{(d)}(f)_{j,k} = (-i(n+1))^d f_{k-j}\left(\frac{n+2-j-k}{n+1}\right), \qquad 1 \le j,k \le n+1 \text{ and } d \in \mathbb{R}.$$

Remark. As with classical pseudodifferential calculi, our symbols are really defined as functions on the cotangent bundle of the manifold in question. Since the symbols correspond to left-invariant operators on $SU(2)$, they are determined by their values at the fiber over the identity ($\mathfrak{g}^*$). Since we require our symbols to be homogeneous, they are determined by their values on the unit sphere and the use of $n+1$ and d above are just the expressions for extending f to be homogeneous of degree d, given its values at radius 1, and then evaluating on the sphere of radius $n+1$.

Now, as with the operators H, X, and Y, in order to define an operator on $SU(2)$, one needs only to say how the operator acts in the representation of highest weight n for each n. Such an operator is a sequence of square matrices, M_n, of size $(n+1)\times(n+1)$. Hence, finally, the quantization of f is given simply as such a sequence of matrices. □

For example, let $f(z,\theta) = \frac{i}{2}(x+iy) = \frac{i}{2}\sqrt{1-z^2}e^{i\theta}$. Then

$$q_n^{(1)}(f)_{j,k} = \delta_{j(k-1)}\frac{n+1}{2}\sqrt{\left(1-\frac{n+1-2j}{n+1}\right)\left(1+\frac{n+1-2j}{n+1}\right)} = \delta_{j(k-1)}\sqrt{j(n-j+1)}.$$

But

$$\pi_n(X)_{j,k} = \delta_{j(k-1)}\sqrt{j(n-j+1)},$$

so the quantization is correct for the operator X, and similarly for Y.

Moreover, for $f(z,\theta) = z$,

$$q_n^{(1)}(f)_{j,k} = \delta_{jk}(n+1)\frac{n+2-2j}{n+1} = \delta_{jk}(n+2-2j).$$

Hence, it is also correct for H. In the next section, we shall make use of the exact correspondence of q_n with π_n, when restricted to $\mathfrak{g}$, to show that the symbol of a commutator of two operators is approximately the Poisson bracket of the symbols.

4. Symbol Classes and Composition Rules

Let $f(z,\theta)$ be a C^2 function on the unit sphere. We will say that f is a *bandlimited symbol* of bandwidth N if $f_n(z) = 0$ for $|n| > N$.

Lemma 4.1.

Let g and h be bandlimited symbols. Let $d_1, d_2 \in \mathbb{R}$. Then

$$q_n^{(d_1)}(g)q_n^{(d_2)}(h) = q_n^{(d_1+d_2)}(gh) + O((n+1)^{d_1+d_2-1}).$$

Proof. By linearity of q^{d_j} and the distributive property of function and matrix multiplication it suffices to consider the case $g = g(z)e^{il_1\theta}$ and $h = h(z)e^{il_2\theta}$. Then we have by definition

$$q_n^{(d_1)}(g)_{jk} = (-i(n+1))^{d_1}\delta_{l_1(k-j)}g\left(\frac{n+2-j-k}{n+1}\right).$$

Similarly,

$$q_n^{(d_2)}(h)_{jk} = (-i(n+1))^{d_2}\delta_{l_2(k-j)}h\left(\frac{n+2-j-k}{n+1}\right).$$

Hence using summation convention,

$$q_n^{(d_1)}(g)_{jk} q_n^{(d_2)}(h)_{kr}$$
$$= (-i(n+1))^{d_1+d_2} \delta_{(l_1+l_2)\,(r-j)} g\left(\frac{n+2-j-r+l_2}{n+1}\right) h\left(\frac{n+2-r-j-l_1}{n+1}\right).$$

On the other hand,

$$q_n^{(d_1+d_2)}(gh)_{jr} = (-i(n+1))^{d_1+d_2} \delta_{(l_1+l_2)\,(r-j)} g\left(\frac{n+2-j-r}{n+1}\right) h\left(\frac{n+2-j-r}{n+1}\right).$$

For fixed r and j, let $g_0 = g(\frac{n+2-j-r}{n+1})$, $g_1 = g(\frac{n+2-j-r+l_2}{n+1})$, $h_0 = h(\frac{n+2-j-r}{n+1})$, and $h_1 = h(\frac{n+2-r-j-l_1}{n+1})$.

Then the difference $q_n^{(d_1)}(g) q_n^{(d_2)}(h) - q_n^{(d_1+d_2)}(gh)$ is given by

$$(-i(n+1))^{d_1+d_2} \delta_{(l_1+l_2)\,(r-j)} \left(g_0(h_1-h_0) + h_1(g_1-g_0)\right).$$

Now, if $|l_2| \neq 1$, then $h_1 - h_0 \leq \frac{l_2 C_h}{n+1}$, by the mean value theorem and the fact that $h(z)$ is C^1 in z. On the other hand, if $|l_2| = 1$, then $h_1 - h_0 \leq \frac{l_2 C_h}{\sqrt{1-z^2}(n+1)}$. Also, if $l_1 \neq 0$, then $g_0 \leq C\sqrt{1-z^2}$, where $z = \frac{n+2-j-r}{n+1}$; whereas if $l_1 = 0$, then g_0 is merely bounded.

Consider the term $g_0(h_1 - h_0)$ in the difference above. If $l_1 = 0$, then $h_1 - h_0 = 0$, so the term is identically zero. If $l_1 \neq 0$, then $g_0(h_1 - h_0) \leq \frac{l_2 C}{n+1}$ because if $|l_2| = 1$, then the difference is $\frac{1}{n+1}$ times something that grows only as fast as g_0 shrinks near the pole and if $|l_2| \neq 1$, then the difference is like $\frac{C}{n+1}$ and g_0 is bounded.

By symmetry, the term $h_1(g_1 - g_0) \leq \frac{l_1 C}{n+1}$, which proves the lemma. □

Lemma 4.2.

Let g and h be bandlimited symbols. Let $d_1, d_2 \in \mathbb{R}$. Then

$$[q_n^{(d_1)}(g), q_n^{(d_2)}(h)] = 2i q_n^{(d_1+d_2-1)}(\{g,h\}) + o((n+1)^{d_1+d_2-1}).$$

Proof. Again it suffices to consider the case $g = g(z)e^{il_1\theta}$ and $h = h(z)e^{il_2\theta}$. Observe that since g and h are C^1 functions on S^2, their Poisson bracket, given by[1]

$$\{g,h\} = (l_2 g'(z)h(z) - l_1 g(z)h'(z))e^{i(l_1+l_2)\theta},$$

is a continuous function on S^2. By the computations of the previous lemma,

$$[q_n^{(d_1)}(g), q_n^{(d_2)}(h)]_{jr} = (-i(n+1))^{d_1+d_2} \delta_{(l_1+l_2)\,(r-j)} (g^{j-l_2}h^{j+l_1} - g^{j+l_2}h^{j-l_1}).$$

Here we have defined $g^x = g(\frac{n+2-x-r}{n+1})$ and similarly $h^x = h(\frac{n+2-x-r}{n+1})$. But notice that

$$g^{j-l_2}h^{j+l_1} - g^{j+l_2}h^{j-l_1} = g^{j-l_2}(h^{j+l_1} - h^{j-l_1}) - h^{j-l_1}(g^{j+l_2} - g^{j-l_2}).$$

Now consider the term $T = g^{j-l_2}(h^{j+l_1} - h^{j-l_1})$. Let $z = \frac{n+2-j-r}{n+1}$. Then

$$T = g\left(z + \frac{l_2}{n+1}\right)\left(h\left(z - \frac{l_1}{n+1}\right) - h\left(z + \frac{l_1}{n+1}\right)\right).$$

Hence, by the mean value theorem, there is an ϵ with $|\epsilon| < \frac{l_1}{n+1}$, such that

$$T = \frac{2l_1}{n+1} g\left(z + \frac{l_2}{n+1}\right) h'(z+\epsilon).$$

[1]This is because the vector fields $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial z}$ on S^2 are mutually orthogonal and have reciprocal lengths, so the standard symplectic structure (volume element) on S^2 is given by $\frac{\partial}{\partial\theta} \wedge \frac{\partial}{\partial z}$ (Archimedes' principle).

Now $g(z)$ is uniformly continuous, and if $|l_2| \neq 1$, then $h'(z)$ is uniformly continuous in $[-1, 1]$ and so

$$T - \frac{2}{n+1}(l_1 g(z)h'(z)) \to 0,$$

as $n \to \infty$, uniformly in z.

Again, the case $|l_2| = 1$ breaks down into subcases where $|l_1|$ is 0, 1 or bigger than 1. By Lemma 2.3, it is enough to consider the case $h(z) = \sqrt{1-z^2}$ since the z-smooth part of h can be dealt with as in the previous case.

Now, again $h'(z)$ exists, is continuous in $(-1, 1)$, but is not uniformly continuous, so the above argument will not work. However, as in the proof of Lemma 4.1, the important fact is that when $l_1 = 0$, $h^{j+l_1} - h^{j-l_1}$ and $l_1 g(z)h'(z)$ are zero, so their difference is identically zero.

When $|l_1| \geq 2$, then, as in Lemma 4.1, $g(z)$ dies at $|z| = 1$ as fast as the modulus of continuity of $h'(z)$ grows, so the result follows in this case also. Indeed, these facts imply that $g(z)(h'(z+\epsilon) - h'(z)) \to 0$ as $\epsilon \to 0$, uniformly in z.

When $|l_1| = 1$, as with h, we may as well assume that $g(z) = \sqrt{1-z^2}$, since the z-smooth part of g dies at the poles as fast as g does in the case when $|l_1| \geq 2$. But then, since X commutes with itself and $[X, Y] = H$, the result of the theorem is trivial.

In any case, the first terms of both types of brackets agree up to the correct order; the second terms also agree by a symmetric argument. Hence the lemma is proved. □

5. Spectral Asymptotics

We shall study the spectrum of selfadjoint operators arising from our calculus by constructing orthonormal bases of approximate eigenvectors for them. A vector v is an approximate eigenvector for a matrix A with error ϵ and approximate eigenvalue λ if

$$|Av - \lambda v| \leq \epsilon |v|.$$

Let v_λ^n denote the unit vector of weight λ in the representation of highest weight n. The "time-frequency support" of v_λ^n should be thought of as a horizontal strip at height λ with width 2 in the sphere of radius $n+1$. We will actually produce a family of approximate eigenvectors that are common to all homogeneous and sufficiently smooth left-invariant pseudodifferential operators. Most of our approximate eigenvectors will "live" in squares.

Define $w_{m,j,\theta}^n = \frac{1}{\sqrt{j}} \sum_{k=1}^{j} e^{i\theta(k-1)} v_{m-2+2k}^n$. With $m << n$ and $j << n$, we think of the time-frequency support of $w_{m,j,\theta}^n$ as being a rectangle on the $n+1$ sphere of height $2j$ centered at $(m+j, \theta)$. Clearly $w_{m,j,\theta}^n$ is a unit vector.

Define the shift operator S_μ by $S_\mu v_\lambda^n = v_{\lambda+\mu}^n$ where $v_\eta^n = 0$ if η is not one of the weights of the representation π_n. In our calculus, the symbol of S_μ is $e^{\frac{i\mu\theta}{2}}$. Observe that

$$|S_2 w_{m,j,\theta}^n - e^{i\theta} w_{m,j,\theta}^n| = \sqrt{\frac{2}{j}}.$$

Thus $w_{m,j,\theta}^n$ is an approximate eigenvector of S_2 with approximate eigenvalue $e^{i\theta}$ (the value of the symbol of S_2 at the "location" of $w_{m,j,\theta}^n$) with error $\sqrt{\frac{2}{j}}$. More generally, consider the following lemma.

Lemma 5.1.

Let $f(z, \theta)$ be a bandlimited symbol of bandwidth $N << j$ defined on the unit sphere. Let $r = \max\left(\frac{1}{\sqrt{1-(\frac{m}{n+1})^2}}, \frac{1}{\sqrt{1-(\frac{m+2j-2}{n+1})^2}}\right)$. Then there exist constants C_1 and C_2 depending on the *bandwidth and C^2 norm of f so that the vector $w^n_{m,j,\eta}$ is an approximate eigenvector for $q_n^{(d)}(f)$ with approximate eigenvalue, $(-i(n+1))^d f(\frac{m+j}{n+1}, \eta)$ with error bounded by*

$$\left(\frac{C_1}{\sqrt{j}} + \frac{rC_2 j}{n+1}\right)(n+1)^d.$$

Remark. The present lemma is simply the estimate needed to prove theorems such as that cited in the introduction and should not be taken too seriously as an end in itself. The reader may wish to consult the statement of the latter (Theorem 5.2), and Theorem 6.1, to see what results from adjusting the parameters of the present lemma. □

Proof. By linearity, it suffices to prove the lemma for $f(z,\theta) = f_l(z)e^{il\theta}$. We wish to approximate $q_n^{(d)}(f)$ by $(-i(n+1))^d f_l(\frac{m+j}{n+1})S_{2l}$. Let

$$A = q_n^{(d)}(f) - (-i(n+1))^d f_l\left(\frac{m+j}{n+1}\right) S_{2l}.$$

Then A is a matrix all of whose entries are l above the diagonal and of the form $(-i(n+1))^d(f_l(z) - f_l(\frac{m+j}{n+1}))$ where the z's are no further than $\frac{2j}{n+1}$ from $\frac{m+j}{n+1}$. The derivative of f_l is bounded by $\frac{rC_2}{2}$, where C_2 is a constant depending on the C^2 norm of f by the proof of Lemma 2.3. Thus,

$$||A|| \le (n+1)^d \frac{rC_2 j}{n}.$$

On the other hand, by the argument above, $w^n_{m,j,\eta}$ is an approximate eigenvector for $(-i(n+1))^d f_l(\frac{m+j}{n+1})S_{2l}$ with approximate eigenvalue $(-i(n+1))^d f_l(\frac{m+j}{n+1})e^{il\eta} = f(\frac{m+j}{n+1}, \eta)$ and with error $(n+1)^d\sqrt{\frac{2l}{j}}$. Setting $C_1 = N^2$ proves the lemma. □

Our approximate eigenvectors constitute a precise notion of microlocality. They give meaning to the values of the symbol. At the same time, the error terms give voice to the noncommutatvity inherent in the situation. They will not be swept under the rug. If j is increased, the first term becomes smaller, but the second is larger. If j is decreased, the reverse happens. It is a form of the uncertainty principle.

But our goal in this section is to study the spectrum of operators, and we must ask ourselves what can be said about the spectrum of an operator in relation to its approximate eigenvectors. For a general operator, we can say very little. The spectrum of the shift operators studied above is 0. However, for selfadjoint operators, the following theorem comes to our aid.

Theorem (Lidskii).

Let the $n \times n$ matrices A, B, and C be selfadjoint. Let the numbers $\alpha_1, \dots \alpha_n, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_n$ be their eigenvalues in order of size. Let $A = B + C$. Then the vector $(\alpha_1 - \beta_1, \dots, \alpha_n - \beta_n)$ is in the convex hull of all permutations of the vector $(\gamma_1, \dots, \gamma_n)$.

Corollary.

If $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex real-valued function, then $\sum_j \Phi(\alpha_j - \beta_j) \le \sum_j \Phi(\gamma_j)$.

The proofs may be found in [K, pp. 124–126].

How does this relate to our situation? Suppose that A is an $n \times n$ selfadjoint matrix. Suppose also that $x_1, \dots, x_n$ is an orthonormal family of approximate eigenvectors with error ϵ and approximate eigenvalues $(\beta_1, \dots, \beta_n)$. Then let B be the linear transformation satisfying $Bx_j = \beta_j x_j$. We would like to say that the spectrum of A is close to the spectrum of B. We know that $|(A-B)x_j| \le \epsilon$. Hence, $||A - B|| \le \sqrt{n}\epsilon$. Thus, if $(\alpha_1, \dots, \alpha_n)$ are the eigenvalues of A, Lidskii's theorem guarantees only that, for each j,

$$|\alpha_j - \beta_j| \le \sqrt{n}\epsilon.$$

Indeed, this is sharp. However, we also know that

$$||A - B||_{\mathrm{HS}} \leq \sqrt{n}\epsilon.$$

Here $||\ ||_{\mathrm{HS}}$ means Hilbert–Schmidt norm. Thus, by applying the corollary with $\Phi(y) = y^2$, we obtain that while any one α_j may be as far as $\sqrt{n}\epsilon$ from its corresponding β_j, the root mean square of the deviations is bounded by ϵ. In other words,

$$\sum_{j=1}^{n} (\alpha_j - \beta_j)^2 \leq n\epsilon^2.$$

However, to apply these observation to our situation, we must choose an orthonormal basis of approximate eigenvectors. Notice that our estimate for the error in Lemma 5.1 deteriorates near the north and south poles (i.e., for m or $m + 2j$ near $-n$ or n.) However, near the poles, the weight vectors themselves are approximate eigenvectors since the Fourier coefficients other than the 0th decay. Thus, we choose the following orthonormal basis for span$(v_n^n, v_{n-2}^n, \ldots, v_{-n}^n)$.

Let $M + 1$ be the smallest positive integer whose square is less than $n + 1$. Define

$$x_j^n = v_{n-2j+2}^n$$

for j running between 1 and M. Next we will define

$$x_{M+1+sM+t}^n = w_{n-(3+2s)M+2, M, \frac{2\pi t}{M}}^n,$$

where s and t run between 0 and $M - 1$. Finally we let

$$x_{M+M^2+j}^n = v_{n-2(\frac{1}{2}M+M^2)-2j+2}^n$$

for j running between 1 and $n + 1 - \frac{1}{2}M - M^2$. In other words, at the poles from height n to about $n - \sqrt{n}$ and from height at least $\sqrt{n} - n$ to height $-n$ on the sphere of radius $n + 1$, we use the weight vectors as elements of our basis. In the middle we use the local Fourier basis on windows of m elements. We now come to the main theorem of this paper.

Theorem 5.2.

Let f be a bandlimited symbol. Then $x_1^n, \ldots, x_{n+1}^n$ is an orthonormal basis of approximate eigenvectors for $q_n^{(d)}(f)$ with error bounded by $C(n+1)^{d-\frac{1}{4}}$ where C is a constant depending only on the C^2 norm of f and its bandwidth.

Proof. To prove the claim for x_j^n with j between $M+1$ and $M+M^2$, we need only to apply Lemma 5.1. Letting r be the same as in Lemma 5.1, we have that

$$r \leq \frac{2}{\sqrt{1 - \left(\frac{n+1-\sqrt{n+1}}{n+1}\right)^2}} = \frac{2}{\sqrt{\frac{2}{\sqrt{n+1}} - \frac{1}{n+1}}} \leq 2(n+1)^{\frac{1}{4}}.$$

The j of Lemma 4.1 is M and $M \sim \sqrt{n+1}$ since $M+1$ is the closest integer to $\sqrt{n+1}$. Thus, the total error is bounded by

$$\left(\frac{C_1}{\sqrt{j}} + \frac{rC_2 j}{n+1}\right)(n+1)^d \leq (2C_1 + 2C_2)(n+1)^{d-\frac{1}{4}}.$$

We need only to prove the error estimate for x_j with $j \leq M$ or $j \geq M^2 + M$. For these j, we have $x_j = v_{n-2j+2}^n$. Also observe that for these j, we know that

$$\sqrt{1 - \left(\frac{n-2j-2}{n+1}\right)^2} \leq 8(n+1)^{-\frac{1}{4}}. \qquad (\maltese)$$

Let $f_l(z)e^{il\theta}$ be the lth component of f with $l \neq 0$. Since f is C^2, and since the vector field $\frac{\partial}{\partial\theta}$ is continuous, we obtain that $|f_l(z)| \leq \frac{C}{l^2}\sqrt{1-z^2}$ by differentiating twice and observing that it commutes with projection into a band that is bounded in C^0. Thus,

$$\sum_{l\neq 0} |f_l(z)| \leq C\frac{\pi^2}{6}\sqrt{1-z^2}.$$

Combining this with (✠) implies that

$$|(q_n^{(d)}(f_0(z)) - q_n^{(d)}(f))x_j| \leq C(n+1)^{d-\frac{1}{4}}$$

for $j \leq M$ or $j \geq M + M^2 + 1$, which was the desired estimate. □

The above result may be thought of as being pitifully weak globally but containing a great deal of microlocal information. Simply applying Lidskii's theorem yields the result that the difference between $q_n^{(d)}(f)$ and a matrix diagonalized by the x^n's and having eigenvalues corresponding to the approximations for $q_n^{(d)}(f)$ has operator norm bounded by $C(n+1)^{d+\frac{1}{4}}$. This is clearly not sharp, since all operators involved are of order d. By more carefully examining the error, one can of course conclude the obvious bound of $C(n+1)^d$. Any such bound with an exponent smaller than d would, among other things, imply the strong Garding's inequality (see [F, §2.6]). We have not been able to improve the exponent. Thus as it is, by this method, we can only show that an operator with positive symbol has at most $o((n+1)^{\frac{1}{2}+\epsilon})$ negative eigenvalues. That is a lot, but not so much relative to $n+1$. What is more, we have seen that with the exception of $o((n+1)^{\frac{1}{2}+\epsilon})$ many eigenvalues, the values of the symbol (multiplied by $(-i(n+1))^d$, naturally) paint a fuzzy portrait of the spectrum of the operator. In the next section, we make some remarks about the relevance of eigensymbols to the spectrum of matrices of operators.

6. Matrices of Operators

Let (f_{jk}) be an $m \times m$ matrix of functions on the unit sphere satisfying $\bar{f}_{jk} = f_{kj}$. One might want to consider the spectrum of the selfadjoint matrix of operators, $A_{jk} = (q_n^{(d)}(f_{jk}))$. Let $p_1, \ldots, p_{n+1}$ be the points on the unit sphere at which any function attains the approximate eigenvalues of its quantization with respect to the vectors $x_1^n, \ldots, x_{n+1}^n$ (e.g., $p_{M+jM+k} = (\frac{n+1-(j+2)M}{n+1}, \frac{2\pi k}{M})$ for $0 \leq j, k \leq M-1$). Let X be any length m-column vector of numbers. Then one has, for any j,

$$A(Xx_j^n) = (-i(n+1)^d)((\sigma(A)(p_j)X)x_j^n + E),$$

where $\sigma(A)(p_j)$ is the matrix whose klth component is $f_{kl}(p_j)$ and $|E| \leq C(n+1)^{-\frac{1}{4}}$. Hence, the eigenvalues of $\sigma(A)$, which we refer to as eigensymbols, give a fuzzy portrait of the spectrum of A with error $(n+1)^{d-\frac{1}{4}}$.

In [Ka1], the second author studied matrices of operators A satisfying $\det\sigma(A) = 0$ identically. These are often referred to as degenerate determined systems and occur in geometry. At least one of the eigensymbols, as defined above, is zero. As was shown in [Ka1], saying that the eigensymbol is 0 does not properly reflect the local solvability properties of A. As we are about to remark, it does not properly reflect the spectral properties either. In [Ka1], a more general definition of eigensymbol is provided. A symbol λ is said to be an eigensymbol if there exists a column vector of operators X satisfying

$$\sigma(AX) = \lambda\sigma(X).$$

This is not equivalent to the previous definition because cancellation of highest order terms may occur in the composition of A and X. If A and X are operators in our calculus, then for each j, if λ is of order r, the vector Xx_j^n is an approximate eigenvector for A with eigenvalue with error

$C(n+1)^{r-\frac{1}{4}}$. Thus lower order eigensymbols give a finer portrait of the spectrum than higher order eigensymbols. With a little more effort, one obtains the following result.

Theorem 6.1.

Let A be an $m \times m$ matrix of operators with eigensymbols $\lambda_1, \ldots, \lambda_m$, having orders $r_1, \ldots, r_m$, respectively. Then there is an orthonormal basis of approximate eigenvectors $y^n_{j,k}$ for A with eigenvalues $(i(n+1))^{r_j}\lambda_j(p_k)$ and errors bounded by $C(n+1)^{r_j-\frac{1}{4}}$.

This answers at least partly in the affirmative a conjecture in [Ka2].

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