UPPER AND LOWER BOUNDS OF THE NORM OF SOLUTIONS OF DIFFERENTIAL SYSTEMS

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1. Let I denote the half-line $0 \leqq t < \infty$ and R^n the n-dimensional Euclidean space. We consider the differential system

$$x' = f(t, x); \qquad x(t_0) = x_0, \qquad (t_0 \geqq 0) \tag{1}$$

where x and f are n-dimensional vectors and the function $f(t, x)$ is continuous and defined on the product space $I \times R^n$. Let $|x|$ denote any convenient norm of x.

Let the function $V(t, x) \geqq 0$ be continuous and defined on $I \times R^n$. Suppose further that $V(t, x)$ satisfies a Lipschitz condition in x locally for each $t \in I$ and that $V(t, x) \to \infty$ as $|x| \to \infty$. Then we can prove the following results.

THEOREM 1. *Let the function $W(t, r)$ be continuous and defined for $t \in I$, $r \geqq 0$. Suppose that $r(t)$ is the maximal solution of the differential equation*

$$r' = W(t, r); \qquad r(t_0) = r_0, \tag{2}$$

existing for all t to the right of t_0. Assume that

$$V(t + \lambda^{-1}, x + \lambda^{-1} f(t, x)) \leqq V(t, x) + \lambda^{-1} W(t, V(t, x)) + o(\lambda^{-1}), \tag{3}$$

for each $t \in I$, $x \in R^n$ and for all sufficiently large $\lambda > 0$. Then, if $x(t)$ is any solution of (1) *such that $V(t_0, x_0) \leqq r_0$, $x(t)$ can be continued as far as $r(t)$ exists and*

$$V(t, x(t)) \leqq r(t), \qquad (t \geqq t_0). \tag{4}$$

If $V(t, x(t))$ is regarded as a measure of a solution $x(t)$ of (1), the following result gives a better control than (4).

THEOREM 2. *Suppose that the assumptions of Theorem* 1 *hold except that the condition* (3) *is replaced by*

$$\begin{aligned} V(t + \lambda^{-1}, x + \lambda^{-1} f(t, x)) \leqq {} & V(t, x)(1 + \lambda^{-1} L(t)) \\ & + \lambda^{-1} W(t, V(t, x) e^{\alpha(t)}) e^{-\alpha(t)} + o(\lambda^{-1}), \end{aligned} \tag{5}$$

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where $L(t)$ *is continuous for* $t\in I$ *and* $\alpha(t)=-\int_{t_0}^{t}L(s)ds$. *Then the inequality* (4) *is replaced by*

$$V(t, x(t))e^{\alpha(t)} \leqq r(t), \qquad (t \geqq t_0). \tag{6}$$

It is clear that Theorem 2 includes Theorem 1 and hence we prove Theorem 2.

PROOF OF THEOREM 2. Let $x(t)$ be any solution of (1) such that $V(t_0, x_0)\leqq r_0$. Define $m(t)=V(t, x(t))e^{\alpha(t)}$. Then $m(t_0)\leqq r_0$, since $\alpha(t_0)=0$. As $V(t, x)$ is assumed to satisfy a Lipschitz condition, we have, for small $h>0$,

$$m(t+h)-m(t) \leqq ke^{\alpha(t+h)}|\epsilon|h + e^{\alpha(t+h)}V(t+h, x(t) + hf(t, x(t))) - e^{\alpha(t)}V(t, x(t)),$$

where the vector ϵ tends to zero as h tends to zero and k is the Lipschitz constant. Now taking $h=\lambda^{-1}$ and using (5), one obtains

$$m(t+h)-m(t) \leqq ke^{\alpha(t+h)}|\epsilon|h + V(t, x(t))[e^{\alpha(t+h)}-e^{\alpha(t)}] + he^{\alpha(t+h)}[L(t)V(t, x(t)) + W(t, m(t))e^{-\alpha(t)} + o(h)],$$

which in its turn yields the inequality

$$\limsup_{h\to 0+}[m(t+h)-m(t)]/h \leqq W(t, m(t)).$$

This is sufficient to prove the result (6) as far as $x(t)$ exists, following the argument used in the lemma in [**6**].

Now suppose that $x(t)$ cannot be continued as far as $r(t)$ exists. Then there is a positive number t_1 such that $x(t)$ cannot be extended to the closed interval $t_0\leqq t\leqq t_1$. This implies that there cannot exist an increasing sequence $\{t_n\}$ tending to t_1 such that $|x(t_n)|$ is bounded, which means that $|x(t_n)|\to\infty$ as $t\to t_1-0$. Since $V(t, x)\to\infty$ as $|x|\to\infty$, we get from (6) that $r(t_1-0)\to\infty$. This contradiction proves that $x(t)$ exists as far as $r(t)$ exists in view of a result of Wintner [**10**].

REMARK. We observe that $W(t, r)$ need not be non-negative. Taking $V(t, x)=|x|$ and $W(t, r)=k(t)r$ or $k(t)g(r)$, where $k(t)$ is continuous and $g(r)>0$ for $r>0$, the upper bounds referred to in [**1**; **2**; **7**; **8**] can be obtained from Theorem 1 without demanding as much. In that case, condition (2) reduces to

$$|x+\lambda^{-1}f(t, x)| \leqq |x| + \lambda^{-1}k(t)g(|x|) + o(\lambda^{-1}),$$

which is weaker than the corresponding condition, viz;

$$|f(t, x)| \leqq k(t)g(|x|).$$

Obviously the latter condition demands $k(t)g(r)$ to be non-negative.

We also note that Theorems 1 and 2 contain the work of Conti [**3**].

The condition (2) is not strong enough to yield the lower bound referred to in [**7**; **8**]. We state the following result to that effect.

THEOREM 3. *Let the condition* (2) *of Theorem* 1 *be replaced by*

$$V(t + \lambda^{-1}, x + \lambda^{-1}f(t, x)) \geqq V(t, x) - \lambda^{-1}W(t, V(t, x)) + o(\lambda^{-1}).$$

Then, as long as $s(t) \geqq 0$ *and* $x(t)$ *exists*

$$V(t, x(t)) \geqq s(t),$$

where $s(t)$ *is the minimal solution of* $r' = -W(t, r)$, $s(t_0) \leqq V(t_0, x_0)$.

2. Consider the differential equation

$$x' = A(t)x + F(t, x) = f(t, x) \text{ say,} \tag{1'}$$

where $A(t)$ is a continuous $n \times n$ matrix. It is easy to show that by using our results one can generalize some known results pertaining to the above equation.

THEOREM 4. *Suppose that the assumptions of Theorem* 1 *are satisfied. Let*

$$b(|x|) \leqq V(t, x), \tag{7}$$

where $b(r)$ *is continuous, increasing in* r *and* $b(r) > 0$ *for* $r > 0$. *Then, if the scalar differential equation* (2) *is* (i) *stable*; (ii) *asymptotically stable, the system* (1′) *is* (i) *stable*; (ii) *asymptotically stable, respectively.*

PROOF. For any $\epsilon > 0$, if $|x| = \epsilon$, we have from (7), $b(\epsilon) \leqq V(t, x)$. If equation (2) is stable, given $b(\epsilon)$ and $t_0 \geqq 0$, there exists a $d = d(t_0, \epsilon)$ such that $r(t) < b(\epsilon)$ for $t \geqq t_0$, whenever $r_0 \leqq d$. Let $x(t)$ be any solution of (1′) satisfying $V(t_0, x_0) \leqq r_0 \leqq d$. Then we derive $|x_0| \leqq b^{-1}(d) = \gamma$ say. For such solutions, we have from Theorem 1

$$V(t, x(t)) \leqq r(t), \qquad (t \geqq t_0). \tag{8}$$

If possible, let $|x(t_1)| = \epsilon$ for some $t = t_1 > t_0$. Then one gets

$$b(\epsilon) \leqq V(t_1, x(t_1)) \leqq r(t_1) < b(\epsilon),$$

a contradiction which proves the stability of the system (1′).

To prove asymptotic stability, suppose if possible, $|x(t_n)| > \eta$, where $\{t_n\}$ is a divergent sequence and $\eta > 0$ is arbitrary. Then one obtains, as before,

$$b(\eta) \leqq V(t_n, x(t_n)) \leqq r(t_n).$$

Since equation (2) is asymptotically stable, $r(t_n) \to 0$ as $t_n \to \infty$. This implies a contradiction because $b(\eta)>0$. Hence $x(t)\to 0$ as $t\to\infty$ and the proof is complete.

THEOREM 5. *Let the assumptions of Theorem 2 be satisfied. Let the conditions corresponding to $b(r)$ and (7) also hold. Suppose that $\alpha(t)\to\infty$ as $t\to\infty$. Then, if the equation (2) is stable, the system (1′) is asymptotically stable.*

PROOF. Following the same argument as in Theorem 4 we have now to replace (8) by

$$V(t, x(t))e^{\alpha(t)} \leqq r(t), \qquad (t \geqq t_0).$$

Let $\{t_n\}$ be a divergent sequence and if possible, $|x(t_n)|>\eta$, where $\eta>0$ is arbitrary. Then one gets

$$e^{\alpha(t_n)}b(\eta) < b(\epsilon).$$

Since $\alpha(t_n)\to\infty$ as $t_n\to\infty$, this leads to a contradiction and proves the result.

The above theorems generalize some results of Halany [**5**] and Santoro [**3**].

REMARK. Since the previous considerations demand, as was pointed out, $W(t, r)$ to be non-negative, which implies that the solutions $r(t)$ of (2) are nondecreasing as t increases, one has limitations in assuming the properties that $r(t)$ should satisfy. For instance, $r(t)\to 0$ as $t\to\infty$ is impossible, as we have assumed in Theorem 4. The practical importance of our approach can be seen from the special case. Suppose that $V(t, x)=x\cdot x$. Then it is enough to take $L(t)=2\lambda(t)$, where $\lambda(t)$ is the largest eigenvalue of $\frac{1}{2}[A(t)+A^*(t)]$, $A^*(t)$ being the transpose of $A(t)$, and $x\cdot F(t, x)\leqq k(t)x\cdot x$. Such a choice works in both the theorems above, since $\lambda(t)$ and $k(t)$ need not be positive for all $t\geqq t_0$.

Theorem 1 can be used in a slightly different way so as to yield another type of information regarding the solutions of (1′).

Let $U(t)$ be the matrix solution of $U'(t)=A(t)U(t)$, $U(t_0)=$ unit matrix. Setting $x=U(t)y$ and using the method of variation of constants, it is easy to obtain the differential equation

$$y' = U^{-1}(t)F(t, U(t)y) = f(t, y) \text{ say.}$$

If the assumptions of Theorem 1 are satisfied for this $f(t, y)$, we have immediately

$$V(t, y(t)) \leqq r(t), \qquad (t \geqq t_0).$$

If further $V(t, x)=|x|$, one obtains from the definition of y that

$$|x(t)| \leqq r(t)\,|U(t)|, \qquad (t \geqq t_0),$$

where $x(t)$ is any solution of (1′) with $|x_0| \leqq r_0$. This implies that the behaviour of solutions of the perturbed system depends on that of the unperturbed system, if $r(t)$ is bounded. Such a result was obtained by Golomb [4] under stronger assumptions.

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