NEW QUESTIONS RELATED TO THE TOPOLOGICAL DEGREE

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To I. M. Gelfand with admiration

1. Topological degree and VMO.

Degree theory for continuous maps has a long history and has been extensively studied, both from the point of view of Analysis and Topology. If $f \in C^0(S^n, S^n), \deg f$ is a well-defined element of $\mathbb{Z}$, which is stable under continuous deformation. Starting in the early 80's, the need to define a degree for some classes of discontinuous maps, emerged from the study of some nonlinear PDE's (related to problems in liquid crystals and superconductors). These examples involved Sobolev maps in the limiting case of the Sobolev imbedding, see Sections 2 and 3 below (topological questions for Sobolev maps strictly below the limiting exponent have been investigated in [14] and [13]). In these cases the Sobolev imbedding asserts only that such maps belong to the space VMO (see below) and *need not be continuous.*

In connection with degree for $H^{1/2}(S^1, S^1)$, L. Boutet de Monvel and O. Gabber suggested a concept of degree for maps in VMO(S^1, S^1)(see [2] and Section 3 below). In our joint work with L. Nirenberg [15] we followed-up on their suggestion and established on firm grounds a degree theory for maps in VMO(S^n, S^n). Here is a brief summary of our contribution.

First recall the definition of BMO (bounded mean oscillation), a concept originally introduced by F. John and L. Nirenberg in 1961. Let Ω be a smooth bounded open domain in $\mathbb{R}^n$, or a smooth, compact, n-dimensional Riemannian manifold (with or without boundary). An integrable function $f : \Omega \to \mathbb{R}$ belongs to BMO if

$$|f|_{\text{BMO}} = \operatorname*{Sup}_{B\subset\Omega} \fint_B \fint_B |f(x) - f(y)| dx\ dy < \infty,$$

where the Sup is taken over all (geodesic) balls in Ω. It is easy to see that an equivalent semi-norm is given by

$$\operatorname*{Sup}_{B\subset\Omega} \fint_B \left| f(x) - \fint_B f(y) dy \right| dx.$$

A very important subspace of BMO, introduced by L. Sarason, consists of VMO (vanishing mean oscillation) functions, in the sense that

$$\lim_{|B|\to 0} \fint_B \fint_B |f(x) - f(y)| dx dy = 0.$$

It is easy to see that

$$\text{VMO } (\Omega, \mathbb{R}) = \overline{C^0(\overline{\Omega}, \mathbb{R})}^{\text{BMO}}.$$

The space VMO is equipped with the BMO semi-norm $|f|_{\text{ BMO}}$. Clearly $L^\infty \subset$ BMO. It is well-know that BMO is strictly bigger than L^∞ (a standard example is $f(x) = |\log|x||$); however, as a consequence of the classical John-Nirenberg inequality,

$$\text{BMO } \subset \bigcap_{p<\infty} L^p.$$

Thus, BMO is "squeezed" between L^∞ and $\bigcap_{p<\infty} L^p$, and for many purposes serves as an interesting "substitute" for L^∞.

Concerning VMO, it is easy to see that $L^\infty \not\subset$ VMO, but of course $C^0 \subset$ VMO. A useful example, showing that the inclusion is strict, is the function

$$f(x) = |\log|x||^\alpha,$$

which belongs to VMO for every $\alpha < 1$. In some sense, VMO serves as a "substitute" for C^0. The Sobolev space $W^{1,n}$ provides an important class of VMO functions. Recall that for every $1 \le p < \infty$,

$$W^{1,p}(\Omega, \mathbb{R}) = \{f \in L^p(\Omega); \nabla f \in L^p(\Omega)\}.$$

Poincaré's inequality asserts that

$$\int_B \left| f - \fint_B f \right| \le C|B|^{1/n} \int_B |\nabla f|,$$

from which we deduce, using Hölder, that

$$\fint_B \left| f - \fint_B f \right| \le C \left[\int_B |\nabla f|^n \right]^{1/n}$$

and thus $W^{1,n} \subset$ VMO.

Similarly, the fractional Sobolev space $W^{s,p}(\Omega)$ is contained in VMO for all $0 < s < 1$ and all $1 < p < \infty$ with $sp = n$ (the limiting case of the Sobolev imbedding). Indeed, in the Gagliardo characterization, we have

$$W^{s,p}(\Omega) = \{f \in L^p(\Omega); \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx\ dy < \infty\}. \tag{1.1}$$

Clearly

$$\int_B \int_B |f(x) - f(y)| dx\ dy = \int_B \int_B \frac{|f(x) - f(y)|}{|x - y|^{(n/p)+s}} |x - y|^{(n/p)+s} dx\ dy$$

$$\leq C|B|^{(1/p)+(s/n)} \int_B \int_B \frac{|f(x) - f(y)|}{|x - y|^{(n/p)+s}} dx\ dy.$$

Using Hölder we deduce that

$$\int_B \int_B |f(x) - f(y)| dx\ dy \leq C|B|^{(1/p)+(s/n)+2-(2/p)} \left[\int_B \int_B \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx\ dy\right]^{1/p}.$$

and thus, when $sp = n$,

$$⨍_B ⨍_B |f(x) - f(y)| dx\ dy \leq C \left[\int_B \int_B \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx\ dy\right]^{1/p},$$

which implies that $W^{s,p} \subset$ VMO.

One of the basic results in [15] is the following

Theorem 1. (H. Brezis - L. Nirenberg [15]). *Every map $f \in VMO(S^n, S^n)$ has a well-defined degree. Moreover:*

(a) this degree coincides with the standard degree when f is continuous,

(b) the map $f \mapsto \deg f$ is continuous on $VMO(S^n, S^n)$ under BMO-convergence.

It is quite easy to define the VMO-degree. For any given measurable map $f : S^n \to S^n$ and $0 < \varepsilon < 1$, set

$$\bar{f}_\varepsilon(x) = ⨍_{B_\varepsilon(x)} f(y) dy.$$

Next, an elementary lemma which is extremely useful

Lemma 1. *If $f \in VMO(S^n, S^n)$, then*

$$|\bar{f}_\varepsilon(x)| \to 1 \text{ as } \varepsilon \to 0, \text{ uniformly in } x \in S^n.$$

Proof. Set

$$\rho_\varepsilon(x) = \fint_{B_\varepsilon(x)} \fint_{B_\varepsilon(x)} |f(y) - f(z)| dy\, dz,$$

so that $\rho_\varepsilon(x) \to 0$ as $\varepsilon \to 0,$ uniformly in $x \in S^n$, since $f \in$ VMO. Then, observe that

$$1 - \rho_\varepsilon(x) \leq |\bar{f}_\varepsilon(x)| \leq 1.$$

If $f \in$ VMO(S^n, S^n) we may now set

$$f_\varepsilon(x) = \frac{\bar{f}_\varepsilon(x)}{|\bar{f}_\varepsilon(x)|}, x \in S^n, 0 < \varepsilon < \varepsilon_0(f).$$

Using ε as a homotopy parameter we see that $\deg f_\varepsilon$ is well-defined and *independent of* ε for $\varepsilon > 0$ sufficiently small. This integer is, by definition, VMO- $\deg f$. The proof of (a) in Theorem 1 is straightforward. For the proof of (b) we refer to [15].

The space VMO(S^n, S^n) is larger than $C^0(S^n, S^n)$. However its structure, from the point of view of connected (or equivalently path-connected) components, is similar to $C^0(S^n, S^n)$. More precisely, there is a VMO version of the celebrated Hopf result:

Theorem 2. *The homotopy classes (i.e. the path-connected components) of VMO(S^n, S^n) are characterized by their VMO-degree.*

Remark 1. By contrast, it is *not* possible to define a degree for maps in $L^\infty(S^n, S^n)$. In fact, the space $L^\infty(S^n, S^n)$ is path-connected (see [15], Section I. 5).

2. Degree for $H^1(S^2, S^2)$ and beyond.

In my earlier paper with J. M. Coron [11](see also [8],[9]) we were led to a concept of degree for maps in $H^1(S^2, S^2)$. Our original motivation came from solving a nonlinear elliptic system, proposed in [16],which amounts to finding critical points of the Dirichlet integral

$$E(u) = \int_\Omega |\nabla u|^2$$

subject to the constraint

$$u \in H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega; S^2); u = \varphi \text{ on } \partial\Omega\},$$

where Ω denotes the unit disc in $\mathbb{R}^2$ and $\varphi : \partial\Omega \to S^2$ is given (smooth). In the process of finding critical points it is natural to the study the connected components of $H^1_\varphi(\Omega, S^2)$, a question which is closely related to the study of the components of $H^1(S^2, S^2)$. The way we defined a degree for $H^1(S^2, S^2)$ was with the help of an *integral formula.* Recall that if $f \in C^1(S^n, S^n)$, Kronecker's formula asserts that

$$\deg f = \fint_{S^n} \det(\nabla f) \tag{2.1}$$

where $\det(\nabla f)$ denotes the $n \times n$ Jacobian determinant of f. When $n = 2$, the right-hand side of (2.1) still makes sense when f is not C^1, but merely in $H^1(S^2, S^2)$ because $\det(\nabla f) \in L^1$. We were able to prove (via a density argument) that the RHS in (2.1) belongs to $\mathbb{Z}$ and we took it as a definition of the H^1-degree of f. Similarly, one may use (2.1) to define a degree for every map $f \in W^{1,n}$. In view of the discussion in Section 1, we know that $W^{1,n} \subset$ VMO and thus any $f \in W^{1,n}(S^n, S^n)$ admits a VMO-degree in the sense of Section 1. Fortunately, the two definitions coincide. In fact, we have

Lemma 2. *For every $f \in W^{1,n}(S^n, S^n)$,*

$$W^{1,n}\text{-}\deg f = VMO\text{-}\deg f.$$

Moreover the components of $W^{1,n}(S^n, S^n)$ are characterized by their degree.

Using this concept of degree we managed to prove in [11] that if φ is not a constant, then E achieves its minimum on two distinct components of $H^1_\varphi(\Omega, S^2)$. A very interesting question remains open:

Open problem 1. Does E admit a critical point in each component of $H^1_\varphi(\Omega, S^2)$ when φ is not a constant?

Even the special case

$$\varphi(x, y) = (Rx, Ry, \sqrt{1 - R^2}), \ \ 0 < R < 1, \ \ x^2 + y^2 = 1,$$

is open.

It is also interesting to study the homotopy structure of $W^{1,p}(S^n, S^n)$ for values of $p \neq n$. This was done in my joint paper with Y. Li [13]:

Theorem 3. *When $p > n$, the standard (C^0) degree of maps in $W^{1,p}$ is well-defined and the components of $W^{1,p}$ are characterized by their degree. When $1 \leq p < n, W^{1,p}$ is path-connected.*

Following the earlier paper[14] we started to investigate with Y.Li[13] the homotopy structure of $W^{1,p}(M, N)$ when M and N are general Riemannian manifolds

(M possibly with boundary, while $\partial N = \phi$). When $p \geq \dim M$ the homotopy structure of $W^{1,p}(M,N)$ is identical to the one of $C^0(M,N)$. When $\dim M > 1$ and $1 \leq p < 2$, we proved in [13] that $W^{1,p}(M,N)$ is *always* path-connected. When p decreases from $\dim M$ to 2, the set $W^{1,p}(M,N)$ becomes larger and larger while various surprising phenomena may occur:

(a) some homotopy classes persist below the Sobolev threshold $p = \dim M$, where maps need not belong to VMO.

(b) as p decreases, the set $W^{1,p}(M,N)$ increases and in this process some of the homotopy classes "coalesce" as p crosses distinguished *integer* values - and usually there is a cascade of such levels where the homotopy structure undergoes "dramatic" jumps.

(c) as p decreases new homotopy classes may "suddenly" appear, at some (integral) levels; every map in these new classes must have "robust" singularities: they cannot be erased via homotopy.

We refer the interested reader to [13] and to the subsequent remarkable paper by F.B. Hang - F.H. Lin [17].

3. Degree for $H^{1/2}(S^1,S^1)$. Can one hear the degree of continuous maps?.

Another important example which motivated my work with L. Nirenberg [15] was the concept of degree for maps in $H^{1/2}(S^1,S^1)$ due to L. Boutet de Monvel - O. Gabber (it is presented in the Appendix of [2]). The motivation in [2] came from a Ginzburg-Landau model arising in superconductivity. This $H^{1/2}$-degree also plays an important role in our study of the Ginzburg-Landau Vortices with F. Bethuel - F. Hélein (see [1]). For example it is at the heart of the proof of

Lemma 3. *Let Ω be the unit disc in $\mathbb{R}^2$ and let φ be a smooth map from $\partial\Omega = S^1$ into S^1. then*

$$[H^1_\varphi(\Omega, S^1) \neq \phi] \Leftrightarrow [\deg \varphi = 0].$$

The way Boutet de Monvel and Gabber originally defined a degree for $H^{1/2}(S^1,S^1)$ went as follows. First, observe that if $f \in C^1(S^1, \mathbb{C}\setminus\{0\})$, then the Cauchy formula asserts that

$$\deg f = \frac{1}{2i\pi}\int_{S^1} \frac{\dot f}{f}. \tag{3.1}$$

In particular, if $f \in C^1(S^1,S^1)$ we may write (3.1) as

$$\deg f = \frac{1}{2i\pi}\int_{S^1} \bar f \dot f = \frac{1}{2\pi}\int_{S^1} \det(f,\dot f) \tag{3.2}$$

(which is the simplest form of Kronecker's formula (2.1)). Then Boutet de Monvel-Gabber observed that the right-hand side in (3.2) still makes when f is not C^1, but merely in $H^{1/2}$. To do so, they interpret the RHS in (3.2) as a scalar product in the duality $H^{1/2} - H^{-1/2} (\bar{f} \in H^{1/2}, \dot{f} \in H^{-1/2})$. Using a density argument they prove that the RHS in (3.2) belongs to $\mathbb{Z}$ and they take it as definition for the $H^{1/2}$-degree of f. On the other hand recall (see Section 1) that $H^{1/2}(S^1) \subset$VMO(S^1). Therefore any $f \in H^{1/2}(S^1, S^1)$ admits a VMO-degree in the sense of Section 1 and in fact we have

Lemma 4. *For every* $f \in H^{1/2}(S^1, S^1)$

$$H^{1/2}\text{-}\deg f = VMO\text{ - }\deg f.$$

Lemma 2 and Lemma 4 show the unifying character of the VMO-degree, putting various concepts of degree (for continuous maps, for $W^{1,n}(S^n, S^n)$ maps, for $H^{1/2}(S^1, S^1)$ maps, etc) under a common roof.

In 1996, I. M. Gelfand invited me to present at his seminar the VMO-degree theory we had just developed with Louis Nirenberg. He asked me to elaborate on the special case of the $H^{1/2}(S^1, S^1)$ - degree. I wrote down Gagliardo's characterization of $H^{1/2}$, which, in this special case, takes the form

$$H^{1/2}(S^1) = \{f \in L^2(S^1); \int_{S^1}\int_{S^1} \frac{|f(x) - f(y)|^2}{|x - y|^2} dx\ dy < \infty\}.$$

Since I. M. Gelfand was not fully satisfied with Gagliardo's formulation, I also wrote down the characterization of $H^{1/2}$ in terms of the Fourier coefficients (a_n) of f:

$$H^{1/2}(S^1) = \{f \in L^2(S^1); \sum_{n=-\infty}^{+\infty} |n||a_n|^2 < \infty\}$$

(see also Lemma 5 below). At that point I. M. Gelfand asked whether there is a connection between the degree and the Fourier coefficients. At first I was surprized by his question, but I realized shortly afterwards that if one inserts the Fourier expansion

$$f(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta}$$

into (3.2) one finds

$$\deg f = \sum_{n=-\infty}^{+\infty} n|a_n|^2. \tag{3.3}$$

Formula (3.3) is easily justified when $f \in C^1(S^1, S^1)$. The density of $C^1(S^1, S^1)$ into $H^{1/2}(S^1, S^1)$ and the stability of degree under VMO-convergence (and thus under $H^{1/2}$-convergence) yield

Theorem 4. *For every $f \in H^{1/2}(S^1, S^1)$*

$$VMO\text{-}\deg f = \sum_{n=-\infty}^{+\infty} n|a_n|^2. \tag{3.4}$$

Formula (3.4) raises some intriguing questions. But, first, a consequence of Theorem 4:

Corollary 1. *Let (a_n) be a sequence of complex numbers satisfying*

$$\sum_{n=-\infty}^{+\infty} |n|\ |a_n|^2 < \infty, \tag{3.5}$$

$$\sum_{n=-\infty}^{+\infty} |a_n|^2 = 1 \tag{3.6}$$

and

$$\sum_{n=-\infty}^{+\infty} a_n \bar{a}_{n+k} = 0 \quad \forall k \neq 0. \tag{3.7}$$

Then

$$\sum_{n=-\infty}^{+\infty} n|a_n|^2 \in \mathbb{Z}. \tag{3.8}$$

Proof. Set

$$f(\theta) = \sum_{n=-\infty}^{+\infty} a_n e^{in\theta},$$

so that $f \in H^{1/2}(S^1, \mathbb{C})$. Moreover we have

$$\int_{S^1} (|f(\theta)|^2 - 1)e^{ik\theta} d\theta = 0 \quad \forall k. \tag{3.9}$$

Indeed, for $k = 0$, (3.9) follows from (3.6) and for $k \neq 0$, (3.9) follows from (3.7). Thus we obtain

$$|f(\theta)| = 1 \ a.e. \tag{3.10}$$

Applying Theorem 4 we find (3.8).

Pedagogical question. Is there an elementary proof of Corollary 1 which does not rely on Theorem 4?

Suppose now $f \in C^0(S^1, S^1)$ and $f \notin H^{1/2}$. Then the series

$$\sum_{n=-\infty}^{+\infty} |n||a_n|^2$$

is divergent. The LHS in (3.4) is well-defined, but the RHS is not. It is natural to ask whether $\deg f$ may still be computed as a "principal value" of the series $\sum_{n=-\infty}^{+\infty} n|a_n|^2$ (which is not absolutely convergent). In [10] we raised the question whether standard summation processes can be used to compute the degree of a general $f \in C^0(S^1, S^1)$. Let for example

$$\sigma_N = \sum_{n=-N}^{+N} n|a_n|^2$$

or

$$P_r = \sum_{n=-\infty}^{+\infty} n|a_n|^2 r^{|n|}, \ \ 0 < r < 1.$$

Is it true that, for any $f \in C^0(S^1, S^1)$,

$$\deg f = \lim_{N \to +\infty} \sigma_N \ \text{ or } \ \deg f = \lim_{r \downarrow 1} P_r?$$

J. Korevaar [19] has shown that the answer is negative. He has constructed interesting examples of maps $f \in C^0(S^1, S^1)$, of degree zero, such that σ_N (resp. P_r) need not have a limit as $N \to \infty$ (resp. $r \to 1$) or may converge to any given real number $\lambda \neq 0$, including $\pm\infty$. In view of this fact we now propose a more "modest" question: do the absolute values of the Fourier coefficients determine the degree? More precisely

Open problem 2. (Can one hear the degree of continuous maps?). Let $f, g \in C^0(S^1, S^1)$ and let $(a_n), (b_n)$ denote the Fourier coefficients of f and g respectively. Assume

$$|a_n| = |b_n| \quad \forall n \in \mathbb{Z}. \tag{3.11}$$

Can one conclude that

$$\deg f = \deg g \ \ ?$$

Same question if one assumes only that $f, g \in VMO(S^1, S^1)$.

Of course, the answer to Open problem 2 is positive if $f, g \in H^{1/2}(S^1, S^1)$. This is a consequence of Theorem 4. The answer is still positive in a class of functions strictly larger than $H^{1/2}$. The proof is based on

Theorem 5. *For every $f \in W^{1/3,3}(S^1,S^1)$ we have*

$$VMO\text{-}\deg f = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |a_n|^2 \frac{\sin^2 n\varepsilon}{n}. \tag{3.12}$$

Corollary 2. *Assume $f, g \in W^{1/3,3}(S^1,S^1)$ satisfy (3.11). Then*

$$VMO\text{-}\deg f = VMO\text{-}\deg g.$$

Corollary 3. (J. P. Kahane [18]). *Assume $f, g \in C^{0,\alpha}(S^1,S^1)$, with $\alpha > 1/3$, satisfy (3.11). Then*

$$\deg f = \deg g.$$

Note that $C^{0,\alpha} \subset W^{1/3,3}$ $\forall \alpha > 1/3$. (This is an obvious consequence of Gagliardo's characterization (1.1)). Thus Corollary 2 implies Corollary 3. Our proof of Theorem 5 is a straightforward adaptation of the ingenious argument of J. P. Kahane [18] for $C^{0,\alpha}, \alpha > 1/3$.

Remark 2. The conclusion of Theorem 5 holds if $f \in W^{1/p,p}(S^1,S^1)$ with $1 < p \leq 3$ (since $W^{1/p,p} \cap L^\infty \subset W^{1/3,3}$ $\forall p \leq 3$). [Note that when $1 < p \leq 2$ the conclusion of Theorem 5 is an immediate consequence of Theorem 4 since $\sum |n||a_n|^2 < \infty$. However in the range $2 < p \leq 3$ the conclusion is far from obvious since the series $|n||a_n|^2$ may be divergent]. It is interesting to point out that formula (3.12) *fails* if one assume only $f \in W^{1/p,p}(S^1,S^1)$ with $p > 3$. In fact, J. P. Kahane [18] has constructed an example of a function $f \in C^{0,1/3}(S^1,S^1)$ such that $\deg f = 0$ while

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2} \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |a_n|^2 \frac{\sin^2 n\varepsilon}{n} = \lambda,$$

where λ could be any real number $\lambda \neq 0$. The heart of the matter is the existence of a 2π-periodic function $\varphi \in C^{0,1/3}(\mathbb{R},\mathbb{R})$ such that

$$\int_0^{2\pi} (\varphi(\theta+h) - \varphi(\theta))^3 d\theta = \sin h \quad \forall h.$$

This still leaves open the question whether Corollary 2 holds when $W^{1/3,3}$ is replaced by $W^{1/p,p}, p > 3$.

Taking $p \to 1$ in Remark 2 suggests that Theorem 5 holds for $f \in W^{1,1}$. This is indeed true and there is even a stronger statement:

Theorem 6. *For every* $f \in C^0(S^1, S^1) \cap BV(S^1, S^1)$ *we have*

$$\deg f = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \sum_{n=-\infty}^{+\infty} |a_n|^2 \sin n\varepsilon.$$

Consequently, we also have

Corollary 4. *Assume* $f, g \in C^0(S^1, S^1) \cap BV(S^1, S^1)$ *satisfy (3.11). Then*

$$\deg f = \deg g.$$

Remark 3. It was already observed by J. Korevaar in [19] that for every $f \in C^0 \cap BV$ one has

$$\deg f = \lim_{N \to \infty} \sum_{n=-N}^{+N} n|a_n|^2,$$

which also implies Corollary 4.

Proof of Theorem 5. We follow the argument of J. P. Kahane [18], except that we work in the fractional Sobolev space $W^{1/3,3}$ instead of the smaller Hölder space $C^{0,\alpha}, \alpha > 1/3$. Set

$$d = VMO - \deg f.$$

By Theorem 3 (and Remark 10) in [15] we may write

$$f(\theta) = e^{i(\varphi(\theta) + d\theta)}$$

for some $\varphi \in VMO(S^1, \mathbb{R})$. Applying Theorem 1 from [13] and the uniqueness of the lifting in VMO we know that $\varphi \in W^{1/3,3}$.

Write

$$\int_0^{2\pi} f(\theta + h)\bar{f}(\theta) d\theta = 2\pi \sum_{n=-\infty}^{+\infty} |a_n|^2 e^{inh} = \int_0^{2\pi} e^{idh} e^{i(\varphi(\theta+h) - \varphi(\theta))} d\theta, \tag{3.13}$$

$$e^{idh} = 1 + idh + O(|h|^2), \tag{3.14}$$

and

(3.15)
$$e^{i(\varphi(\theta+h) - \varphi(\theta))} = 1 + i(\varphi(\theta+h) - \varphi(\theta)) - \frac{1}{2}(\varphi(\theta+h) - \varphi(\theta))^2 + O(|\varphi(\theta+h) - \varphi(\theta)|^3).$$

Thus

(3.16)

$$\operatorname{Im}[e^{idh}e^{i(\varphi(\theta+h)-\varphi(\theta))}] = \operatorname{Im}[(1+idh)e^{i(\varphi(\theta+h)-\varphi(\theta))}] + O(|h|^2)$$
$$= (\varphi(\theta+h)-\varphi(\theta)) + dh + O|h|^2) + O(|h||\varphi(\theta+h)-\varphi(\theta)|^2) + O(|\varphi(\theta+h)-\varphi(\theta)|^3).$$

Integrating (3.16) with respect to θ yields

$$\left|\sum_{n=-\infty}^{+\infty} |a_n|^2 \sin nh - dh\right| \le C|h|^2 + C\int_0^{2\pi} |\varphi(\theta+h)-\varphi(\theta)|^3 d\theta. \tag{3.17}$$

Next, integrating (3.17) with respect to h on $(0, 2\varepsilon)$ gives

$$\left|\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |a_n|^2 \left(\frac{1-\cos 2n\varepsilon}{n}\right) - 2d\varepsilon^2\right| \le C\varepsilon^3 + C\int_0^{2\varepsilon} dh \int_0^{2\pi} |\varphi(\theta+h)-\varphi(\theta)|^3 d\theta$$

and therefore

$$\begin{aligned} &\left|\frac{1}{\varepsilon^2}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} |a_n|^2 \ \frac{\sin^2 n\varepsilon}{n} - d\right| \\ &\le C\varepsilon + \frac{C}{\varepsilon^2}\int_0^{2\varepsilon}\int_0^{2\pi} |\varphi(\theta+h)-\varphi(\theta)|^3 dh\ d\theta \\ &\le C\varepsilon + C\int_0^{2\varepsilon}\int_0^{2\pi} \frac{|\varphi(\theta+h)-\varphi(\theta)|^3}{|h|^2} dh\ d\theta, \end{aligned} \tag{3.18}$$

which implies (3.12) since $\varphi \in W^{1/3,3}$.

Proof of Theorem 6. Since $f \in C^0 \cap BV$, the corresponding φ satisfies $\varphi \in C^0 \cap BV$. We return to (3.17) with $h = \varepsilon$,

$$\left|\frac{1}{\varepsilon}\sum_{h=-\infty}^{+\infty} |a_n|^2 \sin n\varepsilon - d\right| \le C\varepsilon + \frac{C}{\varepsilon}\int_0^{2\pi} |\varphi(\theta+\varepsilon)-\varphi(\theta)|^3 d\theta. \tag{3.19}$$

Next we have

$$\int_0^{2\pi} |\varphi(\theta+\varepsilon)-\varphi(\theta)| d\theta \le \varepsilon\|\varphi\|_{BV}. \tag{3.20}$$

Inserting (3.20) in (3.19) gives

$$\left|\frac{1}{\varepsilon}\sum_{n=-\infty}^{+\infty} |a_n|^2 \sin n\varepsilon - d\right| \le C\varepsilon + C \operatorname*{Sup}_{\theta} \|\varphi(\theta+\varepsilon)-\varphi(\theta)\|^2_{L^\infty} \tag{3.21}$$

and the conclusion follows since $\varphi \in C^0$.

4. New estimates for the degree.

Going back to (3.3) we see that, for every $f \in C^1(S^1, S^1)$

$$|\deg f| \leq \sum |n| \; |a_n|^2. \tag{4.1}$$

Combining (4.1) with Gagliardo's characterization (1.1) of $H^{1/2}$ we find

$$|\deg f| \leq C \int_{S^1} \int_{S^1} \frac{|f(x) - f(y)|^2}{|x-y|^2} dx \; dy. \tag{4.2}$$

In fact, the sharp estimate

$$|\deg f| \leq \frac{1}{4\pi^2} \int_{S^1} \int_{S^1} \frac{|f(x) - f(y)|^2}{|x-y|^2} dx \; dy \tag{4.3}$$

is an immediate consequence of (4.1) and

Lemma 5. *For every $f \in H^{1/2}$ one has*

$$\int_{S^1} \int_{S^1} \frac{|f(x) - f(x)|^2}{|x-y|^2} dx \; dy = 4\pi^2 \sum_{n=-\infty}^{+\infty} |n| \; |a_n|^2 \tag{4.4}$$

Proof. Write

$$\begin{aligned}
\int_{S^1} \int_{S^1} \frac{|f(x) - f(y)|^2}{|x-y|^2} dx \; dy &= \int_0^{2\pi} \int_0^{2\pi} \frac{|\sum a_n e^{in\theta} - \sum a_n e^{in\psi}|^2}{|e^{i\theta} - e^{i\psi}|^2} d\theta \; d\psi \\
&= \int_0^{2\pi} \frac{d\gamma}{|e^{i\gamma} - 1|^2} \int_0^{2\pi} \left| \sum a_n (1 - e^{in\gamma}) e^{in\theta} \right|^2 d\theta \\
&= 2\pi \sum |a_n|^2 \int_0^{2\pi} \frac{|e^{in\gamma} - 1|^2}{|e^{i\gamma} - 1|^2} d\gamma.
\end{aligned}$$

But, for $|n| \geq 1$,

$$\frac{|e^{in\gamma} - 1|^2}{|e^{i\gamma} - 1|^2} = (e^{i(n-1)\gamma} + \ldots + 1)(e^{-i(n-1)\gamma} + \ldots + 1)$$

and thus

$$\int_0^{2\pi} \frac{|e^{in\gamma} - 1|^2}{|e^{i\gamma} - 1|^2} \; d\gamma = 2\pi |n|.$$

Inserting this into the previous equality yields (4.4).

Remark 4. Inequality (4.3) can be viewed as an estimate for the "least amount of $H^{1/2}$-energy" necessary to produce a map $f : S^1 - S^1$ with assigned degree. More precisely we have

$$\operatorname*{Inf}_{\substack{f:S^1\to S^1\\ \deg f=n}} \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^2}{|x-y|^2} dx\ dy = 4\pi^2|n| \tag{4.5}$$

and the Inf in (4.5) is achieved when $f(\theta) = e^{in\theta}$. The existence of a minimizer for similar problems where the standard $H^{1/2}$ norm is replaced by equivalent norms (e.g. the trace of an H^1 norm on the disc with variable coefficients) is a very delicate question because of "lack of compactness"; we refer to [20]

Remark 5. Estimate (4.2) serves as a building block in the study of the least $H^{1/2}$-energy of maps $u : S^2 \to S^1$ with prescribed singularities. Such a question has been investigated in [5]. More precisely, recall that

$$\|u\|^2_{H^{1/2}(S^2)} = \int_{S^2}\int_{S^2} \frac{|u(x)-u(y)|^2}{|x-y|^3} dx\ dy.$$

Given points $\Sigma = \{p_1, p_2, \dots, p_k\} \cup \{n_1, n_2, \dots, n_k\}$ consider the class of maps

$$A = \{u \in C^1(S^2 \setminus \Sigma, S^1); \deg(u, p_i) = +1 \text{ and } \deg(u, n_i) = -1, \quad \forall i\}.$$

Theorem 7. (Bourgain-Brezis-Mironescu [5]). *There exist absolute constants $C_1, C_2 > 0$ such that*

$$C_1 L(\Sigma) \le \operatorname*{Inf}_{u\in A} \|u\|^2_{H^{1/2}(S^2)} \le C_2 L(\Sigma) \tag{4.6}$$

where $L(\Sigma)$ is the length of a minimal connection connecting the points (p_i) to the points (n_i).

Theorem 7 is the $H^{1/2}$-version of an earlier result [12] concerning H^1 maps from S^3 into S^2 with singularities which had been motivated by questions arising in liquid crystals with point defects, while the analysis in [5] has its source in the Ginzburg-Landau model for superconductors. It is the LHS inequality in (4.6) which is related to (4.2). The RHS inequality in (4.6) comes from a "brute force" construction called the "dipole construction".

Remark 6. An immediate consequence of (4.3) is the estimate

$$|\deg f| \le \frac{1}{\pi^2}\int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^p}{|x-y|^2} \quad \forall f \in C^1(S^1, S^1), \forall p \in (1,2). \tag{4.7}$$

Estimate (4.7) deteriorates as $p \downarrow 1$ since the RHS in (4.6) tends to $+\infty$ unless f is constant (see [4]). It would be desirable to improve the constant $(1/\pi^2)$ and establish that

$$|\deg f| \leq C_p \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^p}{|x-y|^2} dx\ dy \quad \forall f \in C^1(S^1,S^1), \forall p \in (1,2). \tag{4.8}$$

with a constant $C_p \sim (p-1)$ as $p \downarrow 1$. In the limit as $p \downarrow 1$, one should be able to recover (in the spirit of [4]) the obvious inequality

$$|\deg f| \leq \frac{1}{2\pi}\int |\dot{f}|. \tag{4.9}$$

Inequality (4.8) is also valid for $p > 2$, but it cannot be deduced from (4.3) and its proof requires much work.

Theorem 8. (Bourgain-Brezis-Mironescu [6]). *For every $p > 1$, there is a constant C_p such that for any (smooth) $f : S^1 \to S^1$*

$$|\deg f| \leq C_p \int_{S^1}\int_{S^1} \frac{|f(x)-f(y)|^p}{|x-y|^2} = C_p \|f\|^p_{W^{1/p,p}}. \tag{4.10}$$

The proof of (4.10) we present in [6] makes use of the harmonic extension of f inside the disc (and the machinery of $W^{s,p}$-trace theory). It would be desirable to find a more direct proof of (4.10).

There is an estimate stronger than (4.10):

Theorem 9. (Bourgain-Brezis-Mironescu [7]). *For any $\delta > 0$ sufficiently small there is a constant C_δ such that, $\forall f \in C^0(S^1,S^1)$,*

$$|\deg f| \leq C_\delta \int_{S^1}\int_{S^1}\limits_{[|f(x)-f(y)|>\delta]} \frac{1}{|x-y|^2} dx\ dy. \tag{4.11}$$

The only proof we know for (4.11) is very involved and it is natural to raise

Open problem 3. Find a simpler proof for (4.11). Also, is there a more precise estimate of the form

$$|\deg f| \leq C\delta \int_{S^1}\int_{S^1}\limits_{[|f(x)-f(y)|>\delta]} \frac{1}{|x-y|^2} dx\ dy \tag{4.12}$$

with C independent of δ?

In the spirit of [4] one might then be able to recover (4.9) as $\delta \to 0$.

Higher dimensional analogs.

Theorem 8 can be extended to higher dimensions:

Theorem 8′. (Bourgain-Brezis-Mironescu [6]). *Let $n \geq 1$. For every $p > n$ there is a constant $C(p,n)$ such that for any (smooth) $f : S^n \to S^n$,*

$$|\deg f| \leq C(p,n) \int_{S^n}\int_{S^n} \frac{|f(x)-f(y)|^p}{|x-y|^{2n}} dx\ dy = C(p,n)\|f\|^p_{W^{n/p,p}}. \tag{4.13}$$

We have not been able to generalize Theorem 9 to higher dimensions. A natural analog would be

Open problem 4. Are there constants $\delta \in (0,1)$ and C such that, $\forall f \in C^0(S^1, S^n)$,

$$|\deg f| \leq C \underset{[|f(x)-f(y)|>\delta]}{\int_{S^n}\int_{S^n}} \frac{1}{|x-y|^{2n}} dx\ dy? \tag{4.14}$$

In a different direction, it might be interesting to estimate other topological invariants in terms of fractional Sobolev norms. One of the simplest examples could be

Open problem 5. Does one have

$$|\text{ Hopf-degree } f| \leq C_p \int_{S^3}\int_{S^3} \frac{|f(x)-f(y)|^p}{|x-y|^6} \quad \forall p > 3, \forall f \in C^1(S^3, S^2)?$$

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