The Goldbach conjecture

A solution concept enabled by the alternative Zeta function theory

Baseline
The RH solution based on the Hilbert transformed fractional part function provides a connection to the Goldbach conjecture as it links the distribution of zeros of the Riemann Zeta function to the circle method. It basically goes along with a replacement of the log(x)-function by its periodical counterpart log(sin(x) (which is a L(2) function). The latter one plays also a key role in Ramanujan's theory of divergent series. In combination with the proposed fractional Hilbert space framework this leads to a "Goldbach conjecture" analysis on the Schnirelman constant (resp. his concept of (positive) density) under the Riemann Hypothesis. The considered series are related to appropriately defined generalized (Dirichlet) functions. This then is about a distributional asymptotic analysis (VlV).

Solution concept in a nutshell

The following is supported by corresponding "opportunity notes" in (BrK).
The topic of additive number theory is about calculating the probability of certain representations of all integers “n” as a sum of a given series of integers, e.g. the series of primes or the series of prime powers. The Goldbach conjecture is about the existence of such a representation (even integers represented a a sum of two primes). The corresponding probability (number of such representations of n divided by n) leads to the concept of “positive (Snirelmann) density” of a given series of integers. The number of primes has a “zero-density” as a consequence of the PNT. However, there is a distributional density representation based on the Dirac function, which is an element of the H(-1/2) Hilbert space (ViJ). The distribution property of the series E(n)/n in the unit interval (E(n) denotes the Euler function) is analyzed in (ScI).

The Goldbach conjecture would be proven if the related (Snirelmann) density (which is concerned with Fourier coefficients of continuous, periodic bounded variation functions) is positive and greater or equal than ½.

The uniform distribution of numbers mod 1 has been analyzed in ((ScI), (WeH)). We claim that a Snirelman density of order 1/2 with respect to its related bounded variation distribution function corresponds to a Snirelman density of order one with respect to its related uniform (H(-1/2) distributional) distribution "functions".

The circle method deals with complex numbers of the open unit disk, while the (number theoretical) probability calculation requires corresponding “densities on the unit circle”. Instead of walking along the x-axis to calculate existing relevant representations we propose to run around the unit circle. As there is an isomorphism between both domains just the mapping would not add any kind of additional value. We propose to measure the winding numbers while walking through the circles not per the zeros of the exp(ix) function, but per the Zeros of appropriate hypergeometric confluent functions. At the same time the “Hadamard gap” challenge of trigonometric gap series (“lacunary series”) is addressed by an appropriately chosen Hilbert space H(1/2) on the circle. This Hilbert space also appears in harmonic analysis in the context of boundary values of real harmonic functions of finite Dirichlet energy in the unit disk. There is also a “natural” relationship to periodical Hilbert transform, conformal mapping and the “Dirichlet space” of harmonic functions (NaS).

The “Hadamard gap” challenge of trigonometric gap series is about the convergence of certain trigonometric series; the proposed generalized Fourier coefficients in the context of (distributional) fractional Hilbert scales addresses the divergence problem of (purely) L(2)-based defined Fourier coefficients, whereby the Hilbert transform plays a key role enabling a distributional trigonometric series representation of the cot(x)-“density” function.

The cardinal series theory (e.g. (BrK)) applies Fourier-Stieltjes series and integrals to Littlewood's converse of Abel's theorem. Two specific cardinal series representations based on the Claussen integral function are proposed to be applied to multiplicative and additive number theoretical functions.

Current situation

A proof of the Goldbach conjecture is basically a proof of appropriate asymptotic behavior of corresponding infinite series. The analysis tool is the Hardy-Littlewood circle method. It faces the conceptual challenge that it operates on the open unit disk (to ensure convergence of appropriate infinite series), while the (Goldbach) problem adequate Dirichlet series domain is the unit circle itself. Therefore today's solution concepts require a two kinds of appropriate asymptotic estimates:

(1) framework adequate (Weyl sums) asymptotic estimates between the circle method approximated Goldbach asymptotic (on the open unit disk, r=exp(-(1/n))) and the truly Goldbach conjecture asymptotic. Those estimates are Goldbach problem independent i.e. especially independent if one analyzes the binary or tertiary Goldbach conjecture
(2) problem adequate Goldbach conjectures asymptotic analysis on the unit circle itself (e.g. to prove the Hardy-Littlewood asymptotic formula conjecture for the number of representations of an even integer as sum of two (odd) primes) resp. on certain subsets of it (e.g. major arcs domains).

Based on the Ramanujan sum Vinogradov introduced a specific singular series S(n). For odd integers n he was able to derive adequate asymptotic estimates to prove the tertiary Goldbach conjecture for all n greater a large number n(0). The sophisticated estimates concerning S(n) are a combination of (1) (ViI) and (2). As a consequence, as S(n)=0 for even integer n, his method cannot be applied for the binary Goldbach conjecture.

Solution concept details

We propose alternatively the "fractional part function" related Hilbert space framework (enabling a modified circle method on the unit circle) to formulate the Goldbach problem (as most prominent example of an additive number theory problem) as "*Tauberian problem for generalized functions".* As tool set we propose distributional Dirichlet series (VlV) in combination with supporting and opportunity lemmata in (BrK), e.g.

- define an appropropriate distributional Hilbert scale framework (with problem adequate orthogonal basis functions (differently to "standard" exp(inx) basis)and its relationship to Besov spaces)

- define appropriate *distributional Dirichlet series* representations as 1st derivatives of "standard" Dirichlet series with same asymtotics as the *2nd derivative of the log-Zeta function* on each "positive-delta-distance-line" parallel to the right hand side of the critical line

- approximate number theoretical functions by identical function term but alternative discrete domain, mapping integers 2n isomorph to the zeros of an appropriate confluent hypergeometric function

- apply Schnirelmann's density concept related to the second derivative of the log-Zeta function, while revisiting (GeL), (KaL), (LaE), (PrK) VI, in the light of the (circle method compatible) alternative (cot(x)-function based) li(x)-function from (BrK), particularly the opportunity notes O24-O30 in (BrK)

- change from Hilbert space framework 1 with domain 1 (which is the open unit disk) to a (periodical distributional function) Hilbert space framework 2 with domain 2 (which is the unit circle). Both Hilbert space frameworks are elements of same Hilbert scale environment
- change from the (cosnx,sinnx)=exp(-inx) orthogonal system to the Fresnel integrals (C(square(x),S(square(x)) with its relationship to the Kummer function in scope of (BrK).
Note: the tool set is about *distributions* (probability theory, differentials, Stieljes integrals) in combination with *distributions* (functional analysis & fractional Hilbert scales)

Snirelmann's "positive density" concept

- The set of primes build a finite basis of the set of all integers.
- The set of all sums of two primes unified with the numbers "0" and "1" has positive Schnirelman density.
- A subset A of the set of all integers unified with "0" and finite Schnirelman density > 0 has a basis of finite order.
- The Schnirelmann-Goldbach theorem states that every integer greater than 1 can be represented as a sum of a finite number of primes (NaM). The Schnirelman number is the number of primes which one needs maximal to build this representation. In other words, the set of primes builds a basis of finite order h of the set of integer numbers. The tertiary Goldbach conjecture is about a Schnirelman number 3. The theorem from Ramaré gives a proof for a Schnirelman number 7
- if the Snirelmann density of the concerned series can be proven greater or equal than 1/2 then the Goldbach conjecture would be confirmed.

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