# The three Millennium problem solutions, RH, NSE, YME, and a Hilbert scale based quantum geometrodynamics

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**„looking back, part (A)"**

on a 10 year journey to …

## Riemann Hypothesis solutions

A modified Zeta function theory is proposed to overcome current challenges

(a) to verify several **Riemann Hypothesis** (RH) criteria

(b) to prove the binary **Goldbach conjecture**

The current two baseline functions to define the Zeta functions, the Gaussian function and the (periodical) fractional part function (resp. their corresponding Mellin transforms) are replaced by their corresponding Hilbert transforms, which are the Dawson function (which is a specific Kummer function) and the Fourier series representation of the $\log(\sin x)$-function. The convergence analysis is based on corresponding Hilbert space frameworks, supporting especially Cardon's *„convolution operator representation"* and Bagchi's "Nyman-Beurling" RH criteria applied to the modified entire Zeta function. Thereby, the convolution operator representation goes along with convergent (Mellin transform) integrals, overcoming the corresponding challenge of an only formally valid self-adjoint invariant operator representation of the standard entire Zeta function ((EdH) 10.3).

Basically, the Bagchi RH criterion and the Cardon RH criterion are two sides of the same coin, which is about the construction of appropriately defined operators, i.e. a defined mapping rule in combination with a defined domain.

The Bagchi-"*Nyman-Beurling*" RH criteria comes along with the $\log(\sin x)$ $L_2^\#(0,1)$-function (the Hilbert transform of the fractional part function $\rho(x)$) in the form

$$\rho_H(x) = \sum_1^\infty \frac{cos\, 2\pi\nu x}{\pi\nu} = -\frac{1}{\pi} log\, 2\, sin(\pi x) \in L_2^\#(0,1)$$

defined in the (periodical) Hilbert scale framework $H_0^\#(0,1)$. It enables the definition of new arithmetical functions going along with a Hilbert space based "circle method" defined on the „boundary of the unit circle", alternatively to the "open unit disk" domain of the Hardy-Littlewood circle method. The counterpart of the zeros of the $e^{ix}$-function are the zeros of the Kummer function ${}_1F_1(\frac{1}{2};\frac{3}{2},-x) = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}\frac{x^n}{n!}$ enabling the definition of a pair of two different arithmetical functions to analyse $(p,q)$-binary number theoretical problems. The concept also supports the verification of the Snirelmann density criterion to prove the Goldbach conjecture.

The proposed modified Zeta function theory supports the proof of several RH criteria, which might be grouped into the following three classes (A1)-(A3), basically defined by the applied underlying function space frameworks:

(A1) this class is about RH criteria which can be re-formulated in terms of distributional Hilbert scale functions $H_\alpha\,(-\infty,\infty)$

(A2) this class is about RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions $H_\alpha^\#(0,1)$

(A3) this class is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "*a distributional way to prove the Prime Number Theorem*" (ViJ).

The common key challenge is in the context of the "Polya theorem", where the not-vanishing constant Fourier term provides the key handicap. From the related "Polya theorem" pdf (see pdf in the "literature" page) we quote:

"*Attempts have been made to apply Polya's general theorem about the zeros of the Fourier transform of a real function to the Zeta function (PoG). After a change of variable* $t \rightarrow logx$ *and approximating the integral over the half-line (positive x-axis) by integrals over finite intervals (which are the essential restriction/handicap of Polya's theorem to be applied to the Zeta function), one can obtain a theorem about zeros of Mellin transforms. For the Zeta function, the Müntz formula has been used, but the Polya theorem only applies in the interval (0,1), and no information is obtained about the zeros of the Zeta function in the critical open stripe of validity of Müntz's formula*".

(OlJ): „*The theory of Dirichlet series offers a bridge between number theory and analysis. Perhaps the most appealing example of the power of this connection is given by the tauberian approach tot the classical prime number theorem*." … *The asymptotic behavior in terms* of the related *Chebyshev-type inequality* can be connected to *local function theoretic properties of* (distributional) (Bagchi-type) *Hilbert spaces* $H_\omega$.

The Hilbert-Polya conjecture (which is about the existence of a proper self-adjoint integral operator going along with the concept of convolution operators, (CaD)) needs to overcome the mathematical problem of the not vanishing constant Fourier term of the Jacobian theta function.

Every Hilbert transformed $L_2$ function has a vanishing constant Fourier term. The Paley-Wiener functions form a Hilbert subspace of $PW$ of $L_2(R)$, which is a reproducing kernel Hilbert space (HiJ). Thus, the inner product of any $f \in PW$ with $w := w(t,x) = \frac{\sin(\pi(t-x))}{\pi(t-x)}$, i.e. $f(t) = (f,w)_0 = \int_{-\infty}^{\infty} f(x)w(t-x)dt$. $PW$ and $PW^\perp$ are invariant subspaces for the Hilbert transforms on $L_2(R)$. Calderón's reproducing formula provided the baseline for the continuous wavelet transformation theory coming along with the concept of „windowed Fourier transforms" (LoA). The $PW_\pi$ reproducing kernel provided the baseline for the cardinal series theory, addressing e.g. the question „*how power series coefficients* $a_n$ *of a function* $f(z) = \sum a_n z^n$ *determine its singularities*", (HiJ). Corresponding $\pi$ –cardinal series are convergent under the condition $\sum \frac{|a_n|}{n} < \infty$. Cardinal series can be obtained formally by considering the Lagrange interpolation formula, which is proposed alternatively to the Euler-Mclaurin /Newton-Gauss summation formulas.

The canonical resolution of the „$E_\mu$ *spectrum of the dynamical Hamiltonian mechanics system*" (consisting of a discontinuous and a continuous part) is concerned with certain self-adjoint and unitary operators in a Hilbert space. The conception that such a spectrum reveals the mechanical properties of the system in its own structure lead to the concepts of an inner product $(E_\mu f, g)$ on the Hilbert space, $PW$-like formulas (KoB1), which is the well known spectral theory in Hilbert spaces for unitary operators (e.g. (HiF)). The physical motivations for related ergodic theorems are based on the Maxwell-Boltzmann gas theory and the Gibbs statisical mechanics (HoE).

With respect to part B, the common mathematical tools to analyze the proposed complementary kinematical energy space $H_1^\perp$ and the non-linear Landau damping phenomenon are

- the wavelets, where a vanishing constant Fourier term of a $L_2$ function is a sufficient condition to be a wavelet function (HoM), (MeY); the underlying „theory of frames", especially the Nonharmonic, but exact Fourier frames are closely related to the Riesz basis of a separable Hilbert space, (YoR) p. 157, (NaA); the latter one plays a key role in the context of the Kadec 1/4-theorem (related to the Paley-Wiener space $PW$ (DuR), resp. indefinite metrics, (AzT), (BoJ), as applied in (A2), (A3) below; general references to frames are provided in (YoR) p. 190.
- ergodic theory on a Hilbert space (HaP), which is about measure-preserving transformation, e.g. in relationship to turbulence problems in hydrodynamics (HoE), (LiP), in combination with non-harmonic Fourier series theory (YoR), based on complex exponentials forming a Riesz basis for $L_2^\#$ and sequences of real or complex numbers with uniform density one (DuR).

The parts (A1-A3) below are concerned with Hilbert scales in the form $H_\alpha$ , $l_2^\alpha$, $\alpha \in R$:

The Dirichlet series theory is an extension of the concept of power series replacing $\sum_1^\infty a_n e^{-xn} \to \sum_1^\infty a_n e^{-x\log n}$. The relationship between the Dirichlet series

$$f(s) := \sum_1^\infty a_n e^{-s\log n} \quad g(s) := \sum_1^\infty b_n e^{-s\log n}$$

and the Hilbert space $H_{-1/2}^{\#} \cong l_2^{-1/2}$on the critical line is given by ([LaE] §227, Satz 40):

$$((f,g))_{-1/2} := \lim_{\omega\to\infty} \frac{1}{2\omega}\int_{-\omega}^{\omega} f(\tfrac{1}{2}+it)g(1/2-it)dt = \sum_1^\infty \frac{1}{n}a_n b_n.$$

The cardinal series theory is an extension of the Dirichlet series theory. In our case this is about absolute convergent series $C(x) = \frac{\sin(\pi(x-\lambda))}{\pi}\sum_{n=1}^\infty (-1)^n \frac{C(n+\lambda)}{x-\lambda-n}$ with cardinal series $C(x) = \frac{w}{\pi}\sum_{-\infty}^\infty a_n \frac{\sin\frac{w}{\pi}(x-a-nw)}{x-a-nw}$ for the points $(a+nw, a_n)$, whereby $\sum_2^\infty (|a_n|+|a_{-n}|)\frac{\log n}{n} < \infty$ to ensure that the cardinal series $\frac{\sin\pi x}{\pi}\left[\frac{a_0}{x} + \sum_{n=1}^\infty (-1)^n (\frac{a_n}{x-n} + \frac{a_{-n}}{x+n})\right]$ is absolutely convergent (WhJ1).

Remark: The inverse mapping of $\sigma(x) := \tilde{T}(\log x) := \sum_{n\le x}\frac{1}{n}\log(\frac{x}{n})$, $x \ge 1$, is given by ((ScW), lemma 3.3, p.36) $\sigma^{-1}(x) = \tilde{T}^{-1}(\log x) = \sum_{n\le x}\frac{\mu(n)}{n}\log(\frac{x}{n})$. The link to the Riemann, von Mangoldt and Landau density functions $J(x) := \frac{1}{2}\left[\sum_{p^n<x}\frac{1}{n} + \sum_{p^n\le x}\frac{1}{n}\right]$, $\psi(x) := \sum_{n\le x}\Lambda(x)$, $\vartheta(x) = \sum_{n=1}^\infty \Lambda(n)\log(\frac{x}{n}) = \psi(x)\log x$ (fulfilling $d[x\vartheta'] = d\psi = \log x\, dJ$) is given by $x\sigma'(x) = \sum_{n\le x}\frac{1}{n}$. The considered sequences below enable modified Riemann density functions $J^*(x), \psi^*(x), \vartheta^*(x)$ with $dJ = dJ^*, d\psi^* = d\psi, d\vartheta = d\vartheta^*$ (see also lemma 5 v), vii), viii) below).

From (ApT) p. 66 we recall that the PNT can be derived from the formula $\sum_{n=1}^\infty \frac{\mu(n)}{n} = 0$. What cannot derived from the PNT is the convergence of the series, (LaE) §160, $\sum_{n=1}^\infty \frac{\mu(n)}{n}\log(\frac{1}{n}) = -\sum_{n=2}^\infty \mu(n)\frac{\log n}{n} = 1$.

Putting $a_n := \sqrt{n}\mu(n)\log(\frac{1}{n})$ ($a_n^* := \mu(n)\log(\frac{1}{n})$) and $b_n := \frac{1}{\sqrt{n}}$ ($b_n^* := 1$) one gets

$$((f,g))_{-1/2} = ((f^*,g^*))_{-1/2} = 1 \text{ and } ((g,g))_{-1/2} = ((g^*,g^*))_{-1} = \sum_1^\infty \frac{1}{n}b_n^2 = \sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6},$$

i.e. $f(\frac{1}{2}+it), g(\frac{1}{2}+it) \in H_{-1/2}$ resp. $g^*(\frac{1}{2}+it) \in H_{-1}$ and therefore $f^*(\frac{1}{2}+it) \in H_0$.

We note that the RH is equivalent to an order of magnitude $O(x^{\frac{1}{2}+\varepsilon})$ ($\varepsilon > 0$) of $M(x) := \sum_{n\le x}\mu(n)$, ((ApT) p. 301), i.e., that $M \in H_{\frac{1}{2}+\varepsilon}$. From (LaE6), (MiM), (MiM1), we recall the equivalent Farey series based criterion of $M(x) := \sum_{n\le x}\mu(n) = \sum_{k=1}^{A(x)}\cos(2\pi r_k)$ with $A(x) := \sum_{n\le x}\varphi(n)$. From the Sobolev embedding theorem we recall, that $H_{\frac{1}{2}+\varepsilon}$ is a sub-space of $C^0$ (more precisely the space of Hölder continuous functions with exponent $1/2$) and that the Dirac function $\delta$ is an element of the dual space $\delta \in H_{-\frac{1}{2}-\varepsilon}$. The link to *„a quick distributional way to the Prime Number Theorem (PNT)"*, (ViJ) is given by the distributions $\delta, H[\delta] \in H_{-1/2-\varepsilon}$, where $\pi H[\delta(x)] = \frac{1}{x} = \log'(\frac{x}{n})$ and $\psi'(x) = \sum_{n\le x}\Lambda(n)\delta(x-n)$. We further note, that $H_{\alpha+\varepsilon}$ is compactly embedded into $H_\alpha$.

We mention the Riemann error function

$$\int_x^\infty \frac{dt}{t(t^2-1)\log t} = \frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\Gamma(1+\frac{s}{2})}{s}\right]x^s ds \cong -\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\Gamma(1+\frac{s}{2})\,x^s\frac{ds}{s},$$

derived from the $\Gamma(1+\frac{s}{2})$ term of the entire Zeta function ((EdH) 1.16).

The parts (A1-A3) below are also concerned with the Kummer functions are ${}_1F_1\left(1,\frac{3}{2};-x\right)$ and ${}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)$, which are connected by the formula ${}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)=e^x\,{}_1F_1\left(1,\frac{3}{2};-x\right)$, (AbM) (7.1.21), (GrI) 9.212.

The challenging part to verify the RH criterion $\pi(x)-li(x)=O(\sqrt{x}\log x)=O(x^{\frac{1}{2}+\varepsilon})$, (which could be even $O(x^{\frac{1}{2}})$ in a variational representation avoiding the $L_\infty-$ norm) is the asymptotical behavior of the exponential (integral) function ((EdH) 1.14 ff., (BeB) IV)

$$Ei(x):=-\int_x^\infty e^{-y}d\log y=-\int_x^\infty e^{-y}\frac{dy}{y}=\int_{-\infty}^x\frac{e^t}{t}dt\approx\frac{e^{-x^2}}{x^2}\sum_{k=0}^\infty(-1)^{k+1}\frac{k!}{x^{2k}}\ .$$

In the following we summarize some properties of the concerned Kummer functions and their relationships to the baseline functions of the Zeta function theory:

The Kummer function based representations of the $li(x)$ function is given by

$$li(x)=\int_{..}^x\frac{dt}{logt}dt=-x\ {}_1F_1(1;1,-\log x)=Ei(logx)=-\int_{-logx}^\infty\frac{e^{-t}}{t}dt=\int_{-\infty}^{logx}\frac{e^t}{t}dt\sim\frac{1}{logx}$$

with

$$Ei(x)=logx+\gamma+\sum_{k=1}^\infty\frac{1}{k}\frac{x^k}{k!}\ .$$

The series representations and asymptotics of the error function

$$Erf(z):=\int_0^z e^{-t^2}dt\ \text{resp.}\ Erfc(z):=\int_z^\infty e^{-t^2}dt$$

are given by (AbM) 7.1.1, 7.1.5, 7.1.23, (OlF) 3 §1,

$$Erf(z):=z\sum_{k=0}^\infty\frac{(-1)^k}{2k+1}\frac{z^{2k}}{k!}\quad,\quad Erfc(z)\approx\frac{1}{z}e^{-z^2}\left[1+\sum_{k=1}^\infty(-1)^k\frac{1\cdot3\cdot\ldots\cdot(2k-1)}{x^{2k}}\right].$$

The relationships to the concerned Kummer functions and the related functions $\Psi(a,c,z^2)$ and $\frac{e^x-1}{x}$ are given by (LeN) 9.13,

$$li(z)=-z\ {}_1F_1(1,1,-\log z)\quad,\quad Ei(z)=-e^z\Psi(1,1,-z)\ ,\quad\frac{e^x-1}{x}=\ {}_1F_1(1,2;x)$$

$$Erf(z)=z\ {}_1F_1(\tfrac{1}{2},\tfrac{3}{2},-z^2)=ze^{-z^2}\ {}_1F_1(1,\tfrac{3}{2},z^2)$$

$$Erf(z)=\frac{1}{2}e^{-z^2}\Psi(\tfrac{1}{2},\tfrac{1}{2},z^2)\ ,\ Erfc(z)=\frac{1}{2}ze^{-z^2}\Psi(1,\tfrac{3}{2},z^2)\ .$$

The asymptotics of the Kummer functions are given by, (LeN) 9,

$${}_1F_1(a,c;z)=\frac{\Gamma(c)}{\Gamma(c-a)}e^{\pm i\pi a}z^{-a}\left[\sum_{k=0}^n\frac{(-1)^k}{k!}(a)_k(1+a-c)_kz^{-k}+O(|z|^{-n-1}\right]+$$

$$+\frac{\Gamma(c)}{\Gamma(a)}e^{-(c-a)}z^{-(c-a)}\left[\sum_{k=0}^n\frac{(-1)^k}{k!}(1-a)_k(c-a)_kz^{-k}+O(|z|^{-n-1}\right],$$

It especially holds $z\ {}_1F_1(\frac{1}{2},\frac{3}{2},z)=\frac{1}{2e}\left[\sum_{k=0}^n(-1)^k(\frac{1}{2})_kz^{-k}+O(|z|^{-n-1}\right]$ , i.e.

$$Ei(x)\approx-2e\ {}_1F_1(\tfrac{1}{2},\tfrac{3}{2},-x)\ \approx\frac{1}{x}\left[\sum_{k=0}^\infty(\tfrac{1}{2})_k(\tfrac{1}{x})^k\right].$$

We further note that the function represented by the series $\sum_{k=0}^\infty\frac{(-1)^k}{(2k+1)\cdot x^{2k+1}}$, $|x|>1$, has the value $\pi/2$ as $x\to0$, $x>0$, (BeB) IV, (10.2).

The Hilbert transform of the Gaussian function is the Dawson function given by

$$F(x):=e^{-x^2}\int_0^x e^{t^2}dt=\int_0^\infty e^{-t^2}\sin(2xt)dt=x\ {}_1F_1\left(1,\tfrac{3}{2};-x^2\right)=xe^{-x^2}\ {}_1F_1\left(\tfrac{1}{2},\tfrac{3}{2};x^2\right).$$

and it holds (GaW)

$$H\left[e^{-x^2}\right]=2\sqrt{\pi}F(x)$$

resp.

$$H^+\left[\frac{e^{-x}}{\sqrt{\pi x}}\right]=2\frac{F(\sqrt{x})}{\sqrt{x}}\approx\frac{1}{x}\left[1+\sum_{k=1}^\infty\frac{1\cdot3\cdot\ldots(2k-1)}{(2x)^k}\right].$$

The link of the Hermite polynomials is given by (AbM), 7.1.15, resp. (RyG)

$$\frac{1}{\sqrt{\pi}}H\left[e^{-x^2}\right]=F(x)=\frac{1}{\sqrt{\pi}}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{H_k^{(n)}}{x-x_k^{(n)}}=\frac{1}{\sqrt{\pi}}\lim_{h\to 0}\sum_{n\ odd}\frac{e^{-(x-nh)^2}}{n}\approx\frac{1}{x}.$$

where $x_k^{(n)}$ and $H_k^{(n)}$ are the zeros and weight factors of the Hermite polynomials.

Let $f(x):=\int_0^\infty\frac{e^{-t^2}}{t+x}dt$, then, $f(x)=\sqrt{\pi}F(x)-\frac{e^{-x^2}}{2}Ei(x^2)=-logx-\frac{\gamma}{2}+o(1),\ x\to 0$ , (OlF) 2 §4.

Remark: The functions ${}_1F_1\left(\frac{1}{2},\frac{3}{2};-z\right)$ and ${}_1F_1\left(1,\frac{3}{2};z\right)$ possesses the same zeros (SeA), as the related Whittaker function $z^{-3/4}M_{\frac{1}{4},\frac{1}{4}}(z)$, which is the solution of the corresponding self-adjoint (ODE) Whittaker operator, (BuH), §17.1.

Lemma:

i) $${}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)=\sum_{k=0}^{\infty}\frac{1}{k+\frac{1}{2}}\frac{x^k}{k!}=\sum_{k=0}^{\infty}\frac{2}{2k+1}\frac{x^k}{k!}=\frac{1}{2}\int_0^1\frac{e^{xt}}{\sqrt{t}}dt\ ,\quad {}_1F_1\left(\frac{1}{2},\frac{3}{2};0\right)=1\ \text{(LeN) (9.11.1)}$$

ii) $$xe^{-x^2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right)=x\ {}_1F_1\left(1,\frac{3}{2};-x^2\right)=\int_0^\infty e^{-t^2}\sin(2xt)\,dt$$

iii) $${}_1F_1'\left(\frac{1}{2},\frac{3}{2};x\right)=\frac{1}{3}\ {}_1F_1\left(\frac{3}{2},\frac{5}{2};x\right)$$

iv) $${}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)+x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2};x\right)=e^x$$

v) $${}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right)=\frac{2}{x}\int_0^x e^{t^2}dt\quad \text{resp.}\quad {}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)=\frac{1}{\sqrt{x}}\int_0^x e^t\frac{dt}{\sqrt{y}}\ \text{(AbM) 7.1.5, (GaW)}$$

vi) $$\int_0^\infty x^{s/2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}=\frac{\Gamma(\frac{s}{2})}{1-s}\quad\text{resp.}\ \int_0^\infty x^{s/2}\left[-x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2};-x\right)\right]\frac{dx}{x}=\frac{s}{1-s}\Gamma(\tfrac{s}{2})\quad \text{(GrI) 7.612}$$

resp. $$\Gamma\left(1+\frac{s}{2}\right)=s(1-s)\int_0^\infty x^s\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x^2\right)\frac{dx}{x}\ \text{in the critical stripe}$$

Putting

$$g(x):=\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)+x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2};-x\right)=\sum_{k=0}^{\infty}\frac{k+1}{k+1/2}\frac{(-x)^k}{k!}$$

one gets

$$-xg(x)=\sum_{k=0}^{\infty}\frac{k+1}{k+\frac{1}{2}}\frac{(-x)^{k+1}}{k!}=-x\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)-x^2\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2};-x\right)\approx\left[\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)_k\left(\frac{1}{x}\right)^k\right]+\left[\sum_{k=0}^{\infty}\left(\frac{1}{2}\right)_k(k+1)(\tfrac{1}{x})^k\right]$$

$$-\int_0^\infty x^{1-s}g(x)\frac{dx}{x}=-\int_0^\infty x^{1-s}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};-x\right)\frac{dx}{x}+\int_0^\infty x^{1-s}\left[-x\ {}_1F_1'\left(\frac{1}{2},\frac{3}{2};-x\right)\right]\frac{dx}{x}=\frac{s\Gamma(1-s)}{1-2s}.$$

Putting $T(x):=\int_0^x g(t)dt$ resp. $\tilde{T}(x):=-\int_0^x tg(t)dt$ it follows

i) $$T(x)=\int_0^x\left[\sum_{k=0}^{\infty}(-1)^k\frac{k+1}{k+\frac{1}{2}}\frac{t^k}{k!}\right]dt=x\cdot\left[\sum_{k=0}^{\infty}(-1)^k\frac{1}{k+\frac{1}{2}}\frac{x^k}{k!}\right]\ \text{with}\ dT(x)=g(x)dx$$

ii) $$\lim_{x\to\infty}\frac{1}{x}\int_0^x g(t)dt=\lim_{x\to\infty}\frac{T(x)}{x}=1\ \text{ i.e. } T(x)\sim x,\ \text{as}^{(*)}\quad sf(s)=s\int_0^\infty x^{-s}g(x)\frac{dx}{x}=-\frac{s+1}{2s+1}\Gamma(1-s)$$

iii) $$g(x)=\sum_{k=0}^{\infty}\frac{k+1}{k+1/2}\frac{(-x)^k}{k!}=\sum_{k=0}^{\infty}\frac{1}{k+1/2}\frac{(-x)^k}{k!}+\sum_{k=0}^{\infty}\frac{k}{k+1/2}\frac{(-x)^k}{k!}=\frac{T(x)}{x}+\sum_{k=0}^{\infty}\frac{k}{k+1/2}\frac{(-x)^k}{k!}$$

iv) $$\int_0^\infty x^{-s}\frac{d\tilde{T}(x)}{x}=-\int_0^\infty x^{1-s}g(x)\frac{dx}{x}=\frac{s\Gamma(1-s)}{1-2s}\ \text{ i.e. }\lim_{x\to\infty}\frac{\tilde{T}(x)}{x}=0\ .$$

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(*) Lemma: (OsH) Vol1, 89): Let $T(x)$ monotone increasing, and

$$f(s)=\int_0^\infty e^{-sx}dT(x)=\int_0^\infty x^{-s}dT(logx)=\int_0^\infty x^{-s}\frac{dT(x)}{x}\quad ,\ s=\sigma+it,\ \sigma>0$$

convergent. Then, if $\lim_{s\to 0^+}sf(s)$ exists, this holds also for $\lim_{x\to\infty}\frac{T(x)}{x}$, and both limits are identical, i.e. $\lim_{s\to 0^+}sf(s)=\lim_{x\to\infty}\frac{T(x)}{x}$ .

In the context with *„the connection between $\zeta(s)$ and primes*" we refer to (EdH) 1.11.

In the context with Polya's theorem (PoG) we recall from (EdH) 12.5, „Transforms with zeros on the line"*:*

*„Polya's theorem is that a real self-adjoint* $^{(*)}$ *operator of the form* $f(x) \to \int_{1/a}^{a} f(ux)F(u)du$ *(where* $F(u)$ *is real and satisfies* $u^{-1}F(u^{-1}) = F(u)$*) which has the property that* $u^{\frac{1}{2}}F(u)$ $^{(**)}$ *is nondecreasing on the interval* $[1,a]$ *has the property that the zeros of its transform all lie on the line* $Re(s) = 1/2$*.*

$^{(*)}$ *The notion „self-adjoint" is different from the definition in functional analysis, but very much close to it, when considering the operator equation in a Hilbert space framework for functions with domain on the critical line. In this case it means that the Polya condition has the same effect than replacing s by (1-s).*

$^{(**)}$ *The unnatural-seeeming factor* $u^{-1}$ *can be eliminated by renormalizing so that* $\int_0^\infty x^{-s}F(x)dx$ *is written* $\int_0^\infty x^{\frac{1}{2}-s}\left[x^{\frac{1}{2}}\tilde{F}(x)\right]dlog(x) = \int_0^\infty x^{-s}\tilde{F}(x)dlog(x)$*. Then the self-adjoint condition is simply* $\tilde{F}(u^{-1}) = \tilde{F}(u)$*. Polya's condition is that* $\tilde{F}$ *be non-decreasing on* $[1,a]$*, and the conclusion of the theorem is that the zeros lie on* $Im(s) = 0$*.*

In the context with *„the connection between $\zeta(s)$ and Tauberian theorems resp. Abel or Cesaro average*" we refer (EdH) 12.7.

In the context with some relevance of the considered Kummer functions to plasma physics we refer to (KoV), (PaY) regarding

- the linear response of magnetized Bose plasmas at T=0 for large and small values of its parameter; the large parameter expansion plays a determining role in the behaviour of these Bose systems in the limit that the external magnetic field B approaches zero. This particular expansion is generalized for the Hurwitz zeta function, (KoV)
-
- the linearized collision operator in the Boltzmann equation with repulsive intermolecular (inverse-power) potentials $V(r) = a \cdot r^{-\alpha}$ for $\alpha > 2$; the collision operator has a purely discrete spectrum and its eigenfunctions are infinitely differentiable $L_2$ -functions which are complete in $L_2$ . The proof relies on the formalism of pseudo-differential operators; the special case $\alpha = 2$ is about the Maxwell's molecules, (PaY).

In the context with the building of distributional Hilbert scales based on a linear operator with discrete spectrum and eigenfunctions, which are complete in $L_2$ , its underlying approximation theory, and an „exponential decay" inner product resp. norm with parameter $t > 0$, given by

$$(x,y)_{\alpha.(t)} = \sum_k \sigma_k^\alpha e^{-\sqrt{\sigma_k}t}(x,\varphi_k)(y,\varphi_k) \quad , \quad \|x\|_{\alpha.(t)}^2 := (x,x)_{\alpha.(t)}$$

govering all „polynomial decay" Hilbert scale norms we refer to (NiJ), (NiJ1).

In the context with some relevance of the considered Kummer functions to the Navier-Stokes equation we refer to (PR1) regarding an integral representation of the Navier-Stokes equations for an incompresslble viscous fluid. „*Making use of standard integral transform methods and considering the longitudinal components of the velocity field, thereby eliminating the pressure field, the Navier-Stokes equations are cast in integral form. The intrinsically non linear character of the equations has proved to be an unsurmountable difficulty that has severely restricted their practical use. The limited understanding of the turbulent motion of fluids and the lack of a comprehensive theory of turbulence is a consequence of this mathematical complication. … The final result is a non linear integral equation for the velocity fieId alone, involving a single convolution over the space and time variables*."

The convolution kernel of the integral representation of the Navier-Stokes equations is build on the functions

$$I_0(\vec{r},t) := \frac{1}{(4\pi\nu t)^{3/2}} e^{-\frac{\vec{r}^2}{4\nu t}} \quad , \quad I_1(\vec{r},t) := (\frac{\nu t}{\vec{r}^2}) \frac{1}{(4\pi\nu t)^{3/2}} \; {}_1F_1\left(\frac{1}{2},\frac{3}{2};-\frac{\vec{r}^2}{4\nu t}\right) .$$

The following part (A1) is related to RH criteria which can be re-formulated in terms of distributional Hilbert scale functions $H_\alpha$ $(-\infty,\infty)$.

Part (A2) is related to RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions $H_\alpha^\#(0,1)$.

Part (A3) is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "*a distributional way to prove the Prime Number Theorem*" (ViJ).

The parts (A2) and (A3) are concerned with appropriate sequences of real numbers and sequences of vectors; the latter ones are called Riesz sequences, it the sequence is a Riesz basis in the closure of the space spanned by those vectors.

With respect to the Bagchi criterion below we emphasis that there is only a positive answer to the following question in a certain „weak" sense (SeK): *„can every frame of complex exponentials* $\{e^{i\lambda_n x}\}$ *in* $L_2(-\pi,\pi)$ *be made into a Riesz basis by removing from* $\{e^{i\lambda_n x}\}$ *a suitable collection of the functions* $e^{i\lambda_n x}$ *resp. can every Riesz sequence* $\{e^{i\lambda_n x}\}$ *in* $L_2(-\pi,\pi)$ *be made into a Riesz basis by adjoining to* $\{e^{i\lambda_n x}\}$ *a suitable collection of exponentials* $e^{i\lambda x}$ *not elements of* $\{e^{i\lambda_n x}\}$ *?"* For the critical sequence in our cases (part (A2) and (A3)) this is about $\lambda := \beta_1 := \frac{\omega_1+\omega_2}{2} - \frac{1}{4}$ to ensure a Snirelmann density of ½ of the concerned sequence $\{\beta_n\}_{n\in N}$ (see also (ReR)).

Following part (A3), there is a section about a new tool set to prove the binary Goldbach conjecture, taking advantage of (A2-A3), e.g. the sequences based on the imaginary parts of the zeros of the considered Kummer functions with underlying domains with Snirelmann density of ½. The final tool is about a *„2-semi-(truly)-circle (even/odd integer) method*" with an underlying „major/minor arcs I/II" concept alternatively to the Hardy-Littlewood circle method.
(A1) This class is about RH criteria which can be re-formulated in terms of distributional Hilbert scale functions $H_\alpha$ $(-\infty,\infty)$.

(A1) The Euler product formula connects the Riemann function $\zeta(s)$ (the analytic continuation of the power function representation beyond the halfplane $Re(s) > 1$) with primes. Taking the log of both sides of the Euler product formula results into a Stieltjes integral representation of the $log\zeta(s)$ function with a (prime) density $dJ(x)$ in the form ((EdH) 1.11)

$$log\,\zeta(s) = s\int_0^\infty x^{-s-1}J(x)dx = \int_0^\infty x^{-s}dJ(x)$$

The entire Zeta function

$$\xi(s) := \frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = \xi(1-s)$$

is basically a product of the three functions $\zeta(s)$, $(s-1)$ and $\Gamma(1+\frac{s}{2})$, whereby $\Gamma(s)$ denotes the Gamma function ((EdH) 1.13).

The method for deriving the formula for $J(x)$ is basically about the calculation of the Fourier inverses of $log\xi(s)$, $-\log(s-1)$ and $-log\Gamma(1+\frac{s}{2})$. The Fourier inverse of the principle term $-\log(s-1)$ becomes the logarithmic integral represention of the $li(x)$ function. The Fourier inverse of the term $-log\Gamma(1+\frac{s}{2})$ leads to the famous Riemann approximation error function between the prime density function J(x) and the $li(x)$ function, ((EdH) 1.16),

$$\int_x^\infty \frac{dt}{t(t^2-1)logt} = \int_x^\infty \frac{1}{tlogt}(\sum t^{-2n})dt = \sum_{n=1}^\infty \int_x^\infty t^{-2n}\frac{dt}{tlogt} = \frac{1}{2\pi i}\frac{1}{logx}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{log\Gamma\left(1+\frac{s}{2}\right)}{s}\right]x^s ds$$

An appropriate convergence behavior of the Riemann error function is one of several RH criteria. Let $f(x)$ denote the Gaussian function. Then the entire Riemann Zeta function is given by

$$\xi(s) = \frac{s}{2}\Gamma(\frac{s}{2})(s-1)\pi^{-s/2}\zeta(s) = (1-s)\cdot\zeta(s)M[-xf'(x)](s) = \xi(1-s).$$

The Hilbert transform of the Gaussian function is the Dawson function given by

$$F(x) := e^{-x^2}\int_0^x e^{t^2}dt = \int_0^\infty e^{-t^2}\sin(2xt)dt = x\ {}_1F_1\left(1,\frac{3}{2};-x^2\right) = xe^{-x^2}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};x^2\right).$$

Replacing the Gaussian function by its Hilbert transform leads to an alternative entire Zeta function definition $\xi^*(s)$ with same zeros in the critical stripe as $\xi(s)$ resulting into a modified Riemann error function with improved approximation property. It is given by

$$\xi^*(s) := \frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\cdot\zeta(s) = \zeta(s)\cdot M\left[\frac{d}{dx}[-x\cdot f_H(x)]\right](s)$$

resp.

$$\zeta^*(s) := \frac{\cot(\frac{\pi}{2}s)}{1-s}\cdot\zeta(s) = \frac{\tan(\frac{\pi}{(2}(1-s))}{1-s}\cdot\zeta(s)$$

with same zeros as $\xi(s)$, as it holds $s(1-s)\xi^*(s)\xi^*(1-s) = \pi\xi(s)\xi(1-s)$. It is basically about a replacement of the term $\pi\frac{s}{2}$ of the term $\pi\Gamma\left(1+\frac{s}{2}\right) = \frac{\pi}{2}s\Gamma\left(\frac{s}{2}\right)$ by the term $tan(\frac{\pi}{2}s) = \cot(\frac{\pi}{2}(1-s)$. The method for deriving the formula for $J(x)$ is then about the calculation of the Fourier inverses of $log\xi(s)$, $-\log(s-1)$, $-log\Gamma(\frac{s}{2})$ and $-\log(tan(\frac{\pi}{2}s)) = -\log(\cot(\frac{\pi}{2}(1-s))$.

With the series expansion of $tan(\frac{\pi}{2}s)$ resp. $\frac{\pi}{2}s\cdot cot(\frac{\pi}{2}s)$ ((GrI) 1.421) it follows

$$\xi^*(s) = \frac{\xi(s)}{\sqrt{\pi}}\sum_{k=1}^\infty \frac{1}{(k-\frac{1}{2})^2-(\frac{s}{2})^2} \qquad \text{resp.} \qquad \xi(s) = \frac{\xi^*(s)}{\sqrt{\pi}}\cdot\frac{2}{\pi}\left[1+\sum_{k=1}^\infty \frac{s^2}{s^2-(2k)^2}\right].$$

(A2) This class is about RH criteria which can be re-formulated in terms of periodical distributional Hilbert scale functions $H_{\alpha}^{\#}(0,1)$

Bagchi's "Hilbert space based reformulation of the Nyman-Beurling RH criterion" (BaB) is based on the (periodical) fractional part $L_2^{\#}$-function

$$\rho(x) = x - [x] = \frac{1}{2} - \sum_1^{\infty} \frac{\sin 2\pi \nu x}{\pi \nu} .$$

Its convergent Mellin transform in the critical stripe defines the zeta function $\zeta(s)$ given by ((TiE) II 2.1),

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x}{x^s} \frac{dx}{x} = -s \int_0^{\infty} x^{-s} \rho(x) \frac{dx}{x} \qquad 0 < Re(s) < 1.$$

The conceptual challenge becomes obvious condering the general Mellin transform property $M[xh'](s) = -sM[h](s)$ and the related (insufficient) Mellin transform zeta function $\zeta(s)$ representation in the form

$$\zeta(s) = s \int_0^{\infty} \frac{[x]-x+\frac{1}{2}}{x^s} \frac{dx}{x} = -s \int_0^{\infty} x^{-s} \left[\rho(x) - \frac{1}{2}\right] \frac{dx}{x} \qquad -1 < Re(s) < 0.$$

The considered Hilbert space in [BaB] is about of all sequences $a = \{a_n | n \in N\}$ of complex numbers such that

$$\sum_{n=1}^{\infty} \omega_n |a_n|^2 < \infty \quad \text{with} \quad \frac{c_1}{n^2} \leq \omega_n \leq \frac{c_2}{n^2}$$

which is isomorph to the Hilbert space $H_{-1} \cong l_2^{-1}$. For $\gamma := \{1,1,1,1,......\}$ it holds

$$\|\gamma\|_{-1}^2 = \sum_1^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

i.e. $\gamma \in l_2^{\perp}$ resp. $\gamma \in (l_2^{-1/2})^{\perp}$, with $l_2^{-1} = l_2 \otimes l_2^{\perp}$, $l_2^{-1} = l_2^{-1/2} \otimes (l_2^{-1/2})^{\perp}$. For the Zeta function on the critical line it also holds $\Xi \in H_{-1}(-\infty,\infty) = \mathrm{H}_{-1} = H_0 \otimes H_0^{\perp}$, and more specifically, $\Xi, \gamma \in H_{-1/2}^{\perp}$, i.e. the orthogonal projection of $\Xi, \gamma$ onto $H_{-1/2}$ is $0$.

Theorem (Bagchi-Nyman criterion, (BaB)): Let

$$\gamma_k := \left\{\rho(\tfrac{m}{k}) \middle| m = 1,2,3,....\right\} \quad \text{for} \quad k = 1,2,3,...$$

and $\Gamma_k$ be the closed linear span of $\gamma_k$. Then the Nyman criterion states that the following statements are equivalent:

i) The Riemann Hypothesis is true

ii) $\gamma \in \bar{\Gamma}_k$.

The verification of the Bagchi criterion is enabled by ergodic theory on a Hilbert space, specific properties of the Mellin and Hilbert transforms regarding the basis functions $e_{\nu}$ and the fractional function in combination with non-harmonic Fourier series theory (based on complex exponentials forming a Riesz basis for $L_2^{\#}$, ((DuR), (YoR)) and related approximation theory in Hilbert scales.

The concerned $\gamma := \{1,1,1,1,......\} \in l_2^{-1}$ can be represented in the form $\gamma = \left\{\frac{1}{n}\gamma_1\right\}_{n \in N} = e_n^{odd} + e_n^{even}$ with $e_{odd} := (1,0,1,0,1,0,1 \ldots)$ and $e_{even} := (1,0,1,0,1,0,1 \ldots) + (0,1,0,1,0,.....)$. The analogue split for $\gamma_k$ with $k \geq 2$ is given by

$$\tilde{\gamma}_{2k} := \left\{\frac{1}{n}\gamma_{2k}\right\}_{n \in N} = \left\{\frac{e_n^{odd}}{n}\gamma_{2k}\right\}_{n \in N} + \left\{\frac{e_n^{even}}{n}\gamma_{2k}\right\}_{n \in N}$$

$$\tilde{\gamma}_{2k+1} := \left\{\frac{1}{n}\gamma_{2k+1}\right\}_{n \in N} = \left\{\frac{e_n^{odd}}{n}\gamma_{2k+1}\right\}_{n \in N} + \left\{\frac{e_n^{even}}{n}\gamma_{2k+1}\right\}_{n \in N}.$$

Let $S_n^{odd} := span\{\tilde{\gamma}_{2k}\}_{k=1,..,n+1}$ and $S_n^{even} := span\{\tilde{\gamma}_{2k+1}\}_{k=1,..,n+1}$ denote the n-dimensional subspaces of $H_{-1/2} \cong l_2^{-1/2}$ and $\gamma_{odd}^{(n)} := P_n(e_n^{odd})$ and $\gamma_{even}^{(n)} := P_n(e_n^{even})$ be the orthogonal projections $P_n : H_{-1} \to S_n^{odd}, S_n^{even}$, of $\gamma$ onto$S_n^{odd}, S_n^{even}$. Then the Bagchi criterion is equivalent to

$$(\gamma - \gamma_{odd}^{(n)}, v)_{-1/2} = (\gamma - \gamma_{even}^{(n)}, v)_{-1/2} = 0 \ , \ \ \forall v \in l_2^{-1/2}.$$

Lemma ([GrI] 3.761): For $a > 0$ it holds

$$M[\sin(a\,\cdot)](s) = \int_0^\infty x^s \sin(ax)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\sin(\frac{\pi}{2}s) \qquad 0 < |Re(s)| < 1$$

$$M[\cos(a\,\cdot)](s) = \int_0^\infty x^s \cos(ax)\frac{dx}{x} = \frac{\Gamma(s)}{a^s}\cos(\frac{\pi}{2}s) \qquad 0 < Re(s < 1$$

and $H[sin](\omega x) = -\cos(\omega x)$ resp. $H[cos](\omega x) = sin(\omega x)$.

The ergodic theorem on a Hilbert space is about the convergent sequence $x_k := \frac{1}{k}(T^1x + T^2x + \cdots + T^kx)$ with respect to the norm topology. The underlying operator

$$T^* := \lim_{k\to\infty}\frac{1}{k}(T^1 + T^2 + \cdots + T^k)$$

is bounded and fulfills $T^k \circ T^* = T^*$, $T^* \circ T^k = T^*$, $T^* \circ T^* = T^*$. „*The mean ergodic theorem* ((HaP) p. 16) *is an amusing and simple piece of classical analysis in case the underlying Hilbert space is one-dimensional*" (HaP) p. 14.

Lemma ((HiF): Let $\mathrm{E}$ denote the eigen-space of the operator $\mathrm{T}^*$ with respect to the eigenvalue $1$, i.e. $\mathrm{E} := \{\mathrm{x} \in \mathrm{H}|\mathrm{T}^*\mathrm{y} = \mathrm{y}\}$ then it holds

*i)* $\mathrm{E} = (\mathrm{T}^* - \mathrm{Id})^{-1}(0)$ is closed

*ii)* the operator $\mathrm{T}^*$ is a projector onto the eigen-space $\mathrm{E}$ (because of $\mathrm{T}^*{}_{|\mathrm{E}} = \mathrm{Id}_{|\mathrm{E}}$).

The choosen operator $T^1 := H$ is the Hilbert operator (e.g. (LiI1) 11.1)

$$\underline{H}[u](t) := \frac{1}{i}\int_0^1 \cot\big(\pi(s-t)\big)\,u(s)ds \;,\; u \in \cap_{\alpha\in R} H_\alpha.$$

For each $\alpha \in R$ the extended operators $\underline{H}^{(\alpha)}: H_\alpha \to H_\alpha$ with domain $H_\alpha$ are bounded Fredholm operators of index zero with $\|\underline{H}^{(\alpha)}u\|_\alpha = \|u\|_\alpha$, with kernel $N(\underline{H}^{(\alpha)}) = span(1)$ (the set of all constants), and their range is $R(\underline{H}^{(\alpha)}) = \{v \in H_\alpha|(v,1) = 0\}$, ((LiI1) p.316); and for $e_\nu(x) := e^{2\pi i\nu x}, \nu \in Z$ it holds $\underline{H}e_\nu = sign(\nu)\cdot e_\nu$ (LiI1).

Putting $T^1 := \underline{H}^{(\alpha)}$ one gets with $T^k e_\nu = \begin{cases} sign(\nu)e_\nu & for\ k\ odd \\ e_\nu & for\ k\ even\end{cases}$ for $\nu \neq 0$

$$\sum_{i=1}^{2k-1} T^i e_\nu = \begin{cases} 2k-1 & for\ \nu>0 \\ -1 & for\ \nu<0\end{cases} \quad , \quad \sum_{i=1}^{2k} T^i e_\nu = \begin{cases} 2k & for\ \nu>0 \\ 0 & for\ \nu<0\end{cases} \quad .$$

Applying the Hilbert operator to the (periodical) fractional part $L_2^\#$-function ($\alpha = 0$)

$$\rho(x) = x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$$

leads to $^{(*)}$

$$\sum_{i=1}^{2k-1} T^i[\rho](x) = \begin{cases} \sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} & for\ \ 2k-1 = 1,5,9,13,\ldots \\ \sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu} & for\ \ 2k-1 = 3,7,11,\ldots\end{cases}$$

and

$$\sum_{i=1}^{2k} T^i[\rho](x) = \begin{cases} \sum_1^\infty \frac{\sin 2\pi\nu x + \cos 2\pi\nu x}{\pi\nu} & for\ \ 2k = 2,6,10\ldots \\ 0 & for\ \ 2k = 4,8,12,\ldots\end{cases}$$

whereby $\sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} = -\frac{1}{\pi}log|2sin(\pi x)| =: \rho_H(x) \in L_2^\#(0,1)$ and $\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu} = \frac{1}{2} - \rho(x)^{(**)}$.

We note that the first derivative of the Hilbert transform of $\rho(x)$ is given by the divergent (Cesàro summable) Fourier series representation of the $cot(\pi x)$ function, $cot(\pi x) = 2\sum_1^\infty sin(2\pi\nu x) = -\rho_H'(x)$ (see also (BeB); it is an element of the Hilbert space $H_{-1}^\#(0,1)$, i.e. the formulas above are valid in similar form for $\alpha = -1$ applied to the (in $H_{-1}^\#(0,1)$ convergent) Fourier series representation $\frac{1}{2}cot(\pi x) = \frac{a_0}{2} + \sum_1^\infty a_\nu \cos(2\pi\nu x) + b_\nu sin(2\pi\nu x)$ with $a_\nu = 0$ for $\nu \geq 0$ and $b_\nu = 1$ for $\nu > 0$ (i.e. $\gamma := \{b_1, b_2, b_3,, \ldots\ldots\}$).

(*) see also (CoG):

$T^1[\rho](x) = \sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu}$, $T^2[\rho](x) = \sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu}$, $T^3[\rho](x) = -\sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} = -T^1[\rho](x)$, $T^4[\rho](x) = -\sum_1^\infty \frac{\sin 2\pi\nu x}{\pi\nu} = -T^2[\rho](x)$, $T^5[\rho](x) = \sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} = T^1[\rho](x)$, $T^6[\rho](x) = \sum_1^\infty \frac{\sin\pi\nu x}{\pi\nu} = T^2[\rho](x)$, $T^7[\rho](x) = -\sum_1^\infty \frac{\cos 2\pi\nu x}{\pi\nu} = -T^1[\rho](x)$, $T^8[\rho](x) = -\sum_1^\infty \frac{\sin\pi\nu x}{\pi\nu} = -T^2[\rho](x)$ , .....

(* $[x] + \left[x + \frac{1}{2}\right] = [2x]$ resp. more general $\rho(nx) = \sum_{k=0}^{n-1}\rho\left(x + \frac{k}{n}\right)$, (ApT) p. 72

The sequence $\omega_n$ (the imaginary parts of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$) enables the definition of two sequences $\lambda_n^{(1)}=\omega_n$ and $\lambda_n^{(2)}=\frac{\omega_n+\omega_{n+1}-1}{2}$, leading to „nearly" harmonic even and odd winding numbers of the corresponding exponentials $\{e^{2\pi i\lambda_n x}\}_{n=0,\pm1,\pm2,\dots}$ occuring on the left and right part of the unit circle. From (YoR) p. 36, we recall

Kadec's $\frac{1}{4}$–Theorem (14): If $\sigma_\nu$ is a sequence of real numbers for which

$$|\sigma_\nu-\nu|\le L<\frac{1}{4}\quad,\ \nu=0,\pm1,\pm2,\dots$$

Then $\{e^{i\sigma_\nu x}\}_{\nu=0,\pm1,\pm2,\dots}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for $L_2^{\#}(-\pi,\pi)=H_0^{\#}(-\pi,\pi)$.

For the extension of Kadec's theorem to frames (being built by two real sequences $\{\theta_k^{(1)}\}_{k\in Z}$ $\{\theta_k^{(2)}\}_{k\in Z}$) we refer to (ChO) 9.8.

If the sequence $\sigma_\nu$ is symmetric, then the product $\prod(1-\frac{z^2}{\sigma_n^2})$ converges to an entire function $f(z)$ which belongs to the Paley-Wiener space, (YoR) p. 124.

The exponentials $e^{i\sigma_\nu x}$ can be transformed into the reproducing functions

$$K_n(z)=\frac{\sin(\pi(z-\sigma_n))}{\pi(z-\sigma_n)}\quad.$$

For $G(z):=z\prod(1-\frac{z^2}{\sigma_n^2})$ the related function $G_n(z):=\frac{G(z)}{G'(\sigma_n)(z-\sigma_n)}$ can be transformed into $g_n(t):=\int_{-\infty}^{\infty}G_n(x)e^{ixt}dt$. The solution of the moment problem $f(\sigma_\nu)=c_\nu\quad \nu=0,\pm1,\pm2,\dots.$, is given by (YoR) p.126,

$$(*)\qquad \mathrm{f(z)}=\sum_{\nu=-\infty}^{\infty}c_\nu\frac{G(z)}{G'\sigma_\nu)(z-\sigma_\nu)}\quad.$$

Remark (YoR) p.126: The formula (*) is a simple example of a „generalized" Lagrange interpolation formula for an entire function assuming the values $c_\nu$ at the points $\sigma_\nu$.

Remark: The product representation of the considered Kummer function is given by ((BuH) p.184)

$${}_1F_1(a,c,z)=\frac{1}{\Gamma(c)}e^{\frac{a}{c}\cdot z}\prod(1-\frac{z}{\alpha_n})e^{z/\alpha_n}.$$

The considered imaginary parts of its zeros fulfill the Kadec requirements with

$$\sigma_n=\alpha_n:=\omega_n+\frac{1}{4}.,\ \sigma_n=\beta_n:=\frac{\omega_n+\omega_{n+1}}{2}-\frac{1}{4}\ .$$

We mention that the approximation solution to $cot(\pi x)$, being testing against $H_0^{\#}(0,1)$ test functions, is an element of $H_{-1/2}^{\#}(0,1)$.

With respect to (A1) and the newly proposed entire Zeta function $\xi^*(s):=\frac{1}{2}(s-1)\pi^{\frac{1-s}{2}}\Gamma(\frac{s}{2})\,tan(\frac{\pi}{2}s)\cdot\zeta(s)$ we note the related (classical) series representations for the $\log(tan(\frac{\pi}{2}x))$ (whereby all summands of the first series are positive), and the (A2)-related series representations for the $log\,sin(\pi x)$ functions, given by (GrI) 1.518,

$$\log(tan(\frac{\pi}{2}x))=log\left(\frac{\pi}{2}x\right)+\sum_1^{\infty}(-1)^{k+1}\frac{1}{k}\frac{(2^{2k-1}-1)}{(2k)!}B_{2k}(\pi x)^{2k}\quad,\quad x^2<1$$

$$log\,sin(\pi x)=log(x\pi)+\sum_1^{\infty}(-1)^k\frac{1}{k}\frac{2^{2k-1}}{(2k)!}B_{2k}(\pi x)^{2k}\ ,\qquad x^2<1\ .$$

The term $-\log(tan(\frac{\pi}{2}s))$ is $L_2^{\#}$ -integrable and can be represented as a convergent series (ElL). It results into a correspondingly modified Riemann error function or a correspondingly modified definition of the $li(x)$ function. In both cases the approximation behavior between the $li(x)$ function and the prime density function gets improved.

(A3) This class is about RH criteria formulated by distributional arithmetical functions going beyond the distributional arithmetical functions applied for "*a distributional way to prove the Prime Number Theorem*" (ViJ).

In 1927 Polya published a very different sort of theorem as in (PoG) on the same general subject ("Über trigonometrische Integrale mit nur reellen Nullstellen") with quite weak conditions to the baseline function $F(u)$ (see also (EdH) 12.5). All attempts failed so far to build a proper (classical) baseline function $F(u)$ to prove the RH. The proposed distributional Hilbert space framework of this homepage provides an additional opportunity to build such a baseline function.

According to E. Landau the classical proofs of the Prime Number Theorem (PNT) are not being scalable for "*deeper going number theory problems*". Ikehara's theorem ((EdH) 12.7) is about a deduction from Wiener's general Tauberian theorem. In (ViJ) "*a distributional way to prove the PN*T" is based on the Mellin transform of e.g. the Dirac delta "function". The part B section is about a (distributional) quantum element/energy Hilbert space framework enabling a quantum gravity model. From a physical modelling perspective it goes along with a replacement of Dirac's model of the „density" of an „idealized point mass" resp. an „idealized point charge" (modelled by the Dirac or Delta „function") by Plemelj's concept of a „mass element $dm$" (PlJ) of the proposed „bosons" quantum state Hilbert space $H_{-1/2}$. „Tauberian Theorems for Generalized Functions" come along with the concept of an *„automodel „function*". It is claimed to be an appropriate framework to apply the Polya theorem to prove the RH, as the crucial condition of Polya's theorem is equivalent to one of the characterization criteria of an „automodel function", ((VlV) p. 57).

The proposed periodical, distributional Hilbert scale framework $H_\alpha^\#$ also enables also a truly unit circle method to tackle the binary Goldbach problem. It overcomes current handicaps of the famous Hardy-Littlewood circle method, which is based on the open unit circle disk.

Vinogradov applied the Hardy-Littlewood circle method to derive his famous (currently best known, but not sufficient) estimate regarding the tertiary Goldbach problem. Vinogradov's theorem states that any sufficiently large odd integer can be written as a sum of three prime numbers. The „any sufficiently large odd integer" condition is due to a not sufficiently "good" estimate: it is derived from two components based on a decomposition of the (Hardy-Littlewood) circle into two parts, the „major arcs" (also called „basic intervals") and the „minor arcs" (also called „supplementary intervals"). The sufficiently good estimate is based on „major arcs" estimate using also Goldbach problem relevant data; the not sufficiently good „minor arcs" estimate are purely Weyl sums estimates taking not any Goldbach problem relevant data into account. However, this estimate is optimal with respect to Weyl sums properties. In other words, the major/minor arcs decomposition is inappropriate to solve both Goldbach problems. The proposed periodical, distributional Hilbert scale framework $H_\alpha^\#$ with ist underlying domain (the boundary of the unit disk, the unit circle) is claimed to enable a truly unit circle based method to prove the binary Goldbach problem.

The proposed change also enables the definition of *two* appropriately different arithmetical functions to analyze binary number theoretical problems for any prime number pair $(p,q)$. We note that the winding numbers (i.e. the set of integers) of the unit circle are related to the zeros of the Weyl sum components, which are the basis functions $e^{2\pi inx}$. The sequence $\omega_n$ (the imaginary parts of the Kummer function ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ enables the definition of two sequences $\lambda_n^{(1)}=\omega_n$ and $\lambda_n^{(2)}=\frac{1}{2}(\omega_n+\omega_{n+1})$, leading to „nearly" harmonic even and odd winding numbers of the corresponding exponentials $\{e^{2\pi i\lambda_n x}\}_{n=0,\pm1,\pm2,\dots}$ occuring on the left and right semicircle of the unit circle.

Refering back to (A2) and the considered $\gamma:=\{1,1,1,1,\dots\dots\}$ with $\gamma\in(l_2^{-1/2})^\perp$ we mention that $\gamma$ can be intepreted as a „winding number list", while running periodically through the unit circle

The links of non-harmonic Fourier series theory to the cardinal series based interpolation theory in the Paley-Wiener space $PW$ are given by the following theorems (YoR) p. 170, p. 157):

for a complex exponentials system $\{e^{2\pi i\lambda_n x}\}_{n=0,\pm1,\pm2,\dots}$ to form a Riesz basis for $L_2^\#$ it is necessary and sufficient to the interpolation problem $f(\lambda_n)=c_n$, where $f\in PW$

a sequence of vectors $\varphi_n$ belonging to a separable Hilbert space $H$ is a Riesz basis if and only if it is an exact frame, i.e. $A\|f\|^2\le\sum_{n=1}^{\infty}|(f,\varphi_n)|^2\le B\|f\|^2$ (if $\varphi_n$ is a complete orthogonal sequence in H, then it holds $A=B=1$), or if and only if $\varphi_n$ is both, a frame and a Riesz sequence (SeK)

The link to a specific class of non-harmonic Fourier series is given by the considered sequences of real or complex numbers with uniform density one (DuR).

## A tool set to prove the binary Goldbach conjecture

On the occasion of the 59th birthday
of my wife, Vibhuta
August 25, 2020

In the following we summaries the proposed tool set to tackle the binary Goldbach conjecture.

The following two lemmata are about the building of arithmetical functions based on a certain integrals:

Lemma 1 (Landau, (PoG1)): Let $q_n$ denote a divergent sequence of positive numbers $0 < q_1 \le q_2 \le q_3 \le \cdots$ $\lim_{n\to\infty} q_n = \infty$, $\tau(x)$ the corresponding counting function of the numbers of $q_n$ less than $\le x$ and $w(x)$ a positive, non-decreasing function with $\lim_{x\to\infty}\frac{w(2x)}{w(x)} = \lim_{x\to\infty}\frac{\tau(x)w(x)}{x} = 1$. Then

$$\lim_{x\to\infty}\frac{1}{\tau(x)}\sum_{q\le x}\rho(\frac{x}{q}) = 1-\gamma,$$

where $\rho(x)$ denotes the fractional part function.

In (PoG1) lemma 1 is generalized by

Lemma 2: Let $w(x)$ a positive, non-decreasing function with $\lim_{x\to\infty}\frac{w(\beta x)}{w(\alpha x)} = 1$ with $\alpha,\beta$ positive numbers. Then

$$\lim_{x\to\infty}\frac{w(x)}{x}\sum_{n\le x} f(\frac{x}{n}) = \int_0^1 f(t)dt.$$

Remark 1: Lemma 2 is valid for two kind of conditions about $f(t)$:

i) $f(t)$ is Riemann integrable in $[0,1]$

ii) $f(t)$ is integrable in every closed sub-interval of $[0,1]$, which does not contain $0$, and it exists an $\alpha \in (0,1)$ with $\lim_{t\to 0} t^{1-\alpha} f(t) = 0$.

Remark 2: The condition $\lim_{x\to\infty}\frac{w(\beta x)}{w(\alpha x)} = 1$ is related to the theory of quasi-asymptotics of generalized functions with its underlying concept of (slow) regular varying (automodel) functions $\vartheta(x)$, where $\lim_{x\to\infty}\frac{\vartheta(\beta x)}{\vartheta(x)}$ (resp. $\lim_{x\to\infty}\frac{x\vartheta'(x)}{\vartheta(x)}$) exists, (VlV) p. 56/57; (EsR). The latter condition, $\lim_{x\to\infty}\frac{x\vartheta'(x)}{\vartheta(x)} = c$, links back to the famous Polya criterion in (PoG) given by $\alpha \le -\frac{xf'(x)}{f(x)} \le \beta$ being valid for only (!) closed interval domains. We note that $-logx$ is slowly varying at $x = 0^+$, (SeE) p.47. We further note with respect to lemma 5 v) below, that for $\tilde{h}(x) := {}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)$ it holds $-\frac{x\tilde{h}'(x)}{\tilde{h}(x)} = 1 - \frac{e^x}{h\tilde{h}x)}$ .

In (LaE) §56, the following following lemma is provided. It has been applied in (LaE1) to derive the asymptotics integral formula for the Goldbach number counting function

$$H(2n) := \sum_{k=1}^{2n} G_{2k} = \sum_{p\le 2n}\pi(2n-p) \quad , p+q \le 2n,$$

in the form

$$H(2n) \sim 2n\int_2^n \frac{1}{\log(2n-u)}\frac{du}{logu} \sim 2n\int_2^n \frac{1}{\log(2n-\vartheta\frac{u}{n})}\frac{du}{logu} \sim \frac{1}{2}(\frac{2n}{\log(2n)})^2, \qquad 0 \le \vartheta \le 1$$

by putting $F(u,x) := \pi(2n-p)$.

We note that $\sum_{k\text{ odd}}^{2x} G_k \sim \frac{2x}{\log(2x)}$, while $\sum_{k=1}^{x} G_{2k} \sim \frac{1}{2}(\frac{2x}{\log(2x)})^2$, because the odd $k$ can be neglected as every odd number can be represented as sum of two primes, if $k-2$ is prime, otherwise not (LaE1).

We further note that $d\left[\sum_{k=1}^{x}\frac{1}{\varphi(k)}\right] \sim d[\log x] \sim \frac{dx}{x}$ (LaE1), and $d\vartheta = \left[\sum_{n\le x}\frac{\mu(n)}{n}log\left(\frac{x}{n}\right)\right] = \frac{1}{x}\left[\sum_{n\le x}\frac{\mu(n)}{n}\right]dx \sim \frac{dx}{x}$ (ApT) p.97.

Lemma 3: Let $F(u,x)$ be a function with real arguments with $2 \le u \le x$ fulfilling the following conditions

i) $F(u,x) \ge 0$

ii) For fixed $x > 2$ the function $\frac{F(u,x)}{logu}$ is never increasing for $u \in (2,x)$

iii) $F(2,x) = o(\int_2^x \frac{F(u,x)}{logu}du)$,

then it holds

$$\sum_{p\le x} F(p,x) \sim \int_2^x \frac{F(u,x)}{logu}du \quad .$$

The considered Stäckel approximation formula in (LaE1) is given by

$$\tilde{\mathrm{H}}(2n) := \sum_{k=1}^{n} \tilde{G}_{2k} = c_\lambda \cdot \sum_{\substack{k=2 \\ k\ even}}^{2n} \frac{1}{\varphi(k)}\left(\frac{k}{\log(k)}\right)^2 \quad \text{with } c_\lambda := \frac{\pi^4}{105\cdot\zeta(3)} \sim 0{,}772 \ldots.$$

The factor $c_\lambda$ was suggested to ensure that $\tilde{\mathrm{H}}(2n) < \mathrm{H}(2n)$.

Remark 3: We note that the above (appreciated) identical asymptotical behavior of both summation formulas does not allow a corresponding conclusion to the approximation behavior of the underlying Goldbach numbers $\mathrm{G}_{2k}$ resp. $\tilde{G}_{2k}$, as in the sum formula $\sum_{k=1}^{n}\mathrm{G}_{2k} - \tilde{G}_{2k}$ positive and negative terms cancel each other out.

Remark 4: With respect to (A2) above and the the term $\frac{1}{\varphi(n)}$ in the Stäckel approximation formula we note the inequality, (ApM) p. 71, $\frac{\sigma(k)}{k^2} \le \frac{1}{\varphi(k)} \le \frac{\pi^2}{6}\frac{\sigma(k)}{k^2} = \zeta(2)\frac{\sigma(k)}{k^2}$ , $k \ge 2$, where $\sigma(n) = \sigma_1(n)$ denotes the sum of the divisors of $n$ ((ApM) p. 38), and $\varphi(n)$ denotes the Euler totient function. It can be represented as discrete Fourier transform of the greatest common divisor in the form $\varphi(n) = \sum_{k=1}^{n}\gcd(k,n)\cdot\cos\left(2\pi\frac{k}{n}\right)$. We further mention that for even integers it holds $\varphi(2k) = 2\varphi(k) > \sqrt{k}$. At the same point in time, it holds $\frac{k}{log^2k} > \sqrt{k}$ for $\mathrm{k} \ge 2$.

Remark 5: It holds $\sum_{k\le x}\frac{1}{\varphi(k)} = O(logx)$, $\frac{1}{x}\sum_{k\le x}\mathrm{d(n)} = logx + (2\gamma - 1) + O(\sqrt{x})$, and $\sum_{k\le x}\frac{\mathrm{k}}{\varphi(k)} = O(x)$, $\sum_{k\le x}\frac{1}{\varphi(k)} = O(logx)$ (ApM) p. 71. The average order of $\mathrm{d(n)}$ is $logn$ i.e. $\frac{1}{x}\sum_{k\le x}\mathrm{d(k)} = logx + 2\gamma - 1 + O(\sqrt{x}) \sim logx$, (ApM) p. 52. For $\vartheta(x) := \sum_{n\le x}\frac{\mu(n)}{n}log\left(\frac{x}{n}\right)$ we recall from (ApT) p. 97, that the PNT theorem is equivalent to $d\vartheta(x) = \sum_{n\le x}\frac{\mu(n)}{n}\frac{dx}{x} \sim \frac{dx}{x}$.

Remark 6: ((LaE) §56 p. 214): the $n$-th prime number is asymptotically equal to $n \cdot logn$.

Building an arithmetical function for the term $F(u,x) := \pi(2n-p)$ is challenging because of

Lemma 4, ((LaE) §56, p. 215):

from a certain number $x$ on, there are more primes in the interval $(1,x)$ than in the interval $(x,2x)$.

Remark 7: Lemma 4 cannot be proven with the PNT; the proof requires the asymptotics

$$\pi(x) = \frac{x}{logx} + \frac{x}{log^2x} + o\left(\frac{x}{log^2x}\right)$$

Remark 8: The functions ${}_1F_1\left(\frac{1}{2},\frac{3}{2};-\mathrm{z}\right)$ and ${}_1F_1\left(1,\frac{3}{2};\mathrm{z}\right)$ possesses the same zeros as the related Whittaker function $z^{-3/4}M_{\frac{1}{4},\frac{1}{4}}(z)$, which is the solution of the corresponding self-adjoint (ODE) Whittaker operator ((BuH), §17.1). The latter one also plays key role for the radial Schrödinger operator with a Coulomb potential or with a morse potential on the half line (DeJ), (LaJ1).

Corresponding integrals of oscillatory type for ${}_1F_1\left(\frac{1}{2},\frac{3}{2};x\right)$ (when the variables are real) are the Fresnel integrals $C(x), S(x)$ with $\lim_{x\to\infty} C(x) = \lim_{x\to\infty} S(x) = \frac{1}{2}$ with the exact error formula (OlF) p. 67,

$$C\left(\sqrt{\frac{2x}{\pi}}\right) + i \cdot C\left(\sqrt{\frac{2x}{\pi}}\right) = \sqrt{\frac{2x}{\pi}}\ {}_1F_1\left(\frac{1}{2},\frac{3}{2};ix\right) = \int_0^x \frac{costdt}{\sqrt{2\pi t}} + i \cdot \int_0^x \frac{sintdt}{\sqrt{2\pi t}} = \left(\frac{1}{2} + i\frac{1}{2}\right) - \frac{ie^{ix}}{\sqrt{2\pi x}} \sum_{n=0}^{\infty} \frac{1\cdot 3\cdot \ldots (2n-1)}{(2ix)^n} .$$

The formula allows an alternative *„2-semicircle (even/odd integer) method*" with an underlying „major/minor arcs I/II" concept based on two appropriately defined Fresnel integral distribution functions ($\cos(t) > 0$, $\sin(t) > 0$), enabled by the real part values $\omega_n$ of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$. With respect to the link to exponentials Riesz bases we refer to (KoG), (NaA) and the references cited there.

Lemma 5:

For the real part values $\omega_n$ of ${}_1F_1\left(\frac{1}{2},\frac{3}{2};2\pi iz\right)$ it holds (SeA)

i) $2n - 1 < 2\omega_n, \omega_n + \omega_{n+1} - 1 < 2n < 2\omega_n + 1, \omega_n + \omega_{n+1} < 2n + 1$

ii) $n - \frac{3}{4} < \lambda_n^{(1)} - \frac{1}{4} := \omega_n - \frac{1}{4}, \lambda_n^{(2)} - \frac{1}{4} := \frac{\omega_n + \omega_{n+1} - 1}{2} - \frac{1}{4} < n - \frac{1}{4} < \omega_n + \frac{1}{4}, \frac{\omega_n + \omega_{n+1} - 1}{2} + \frac{1}{4} < n + \frac{1}{4}$

iii) $n - \frac{1}{2} < \omega_n < n$ , $\frac{1}{2} < \omega_1, \frac{\omega_1 + \omega_2 - 1}{2} < 1$ , $s_n := \frac{\omega_n}{n} \to 1$ $n \in N$

iv) the non-integer sequences $2\omega_n$ and $\omega_n + \omega_{n+1}$ fulfill a kind of Hadamard gap condition

$$\frac{\omega_{n+1}}{\omega_n} > \frac{n+\frac{1}{2}}{n} = 1 + \frac{1}{2n} > q > 1 \qquad \text{resp.} \qquad \frac{\omega_{n+1}+\omega_{n+2}}{\omega_n+\omega_{n+1}} > \frac{2n+2}{2n+1} = 1 + \frac{1}{2n+1} > q > 1$$

v) $\theta := \frac{1}{4} < \omega_{n+1} - \omega_n < 1 - \frac{1}{4} = 1 - \theta$

vi) putting $a_n := \frac{\omega_n}{n} - \frac{1}{2}$ , $b_n := \frac{\omega_1 + \omega_2 - 1}{2n} - \frac{1}{2}$ it follows

$$0 < a_1 = \omega_1 - \frac{1}{2} \le a_n \to \frac{1}{2}, \quad \frac{1}{2} \leftarrow b_n < b_1 = \frac{\omega_1 + \omega_2 - 1}{2} < 1, \quad a_n, b_n - a_n \in (0, \tfrac{1}{2}) \quad , \quad b_n, 1 - a_n \in (\tfrac{1}{2}, 1).$$

Lemma 6, (BaR), (KaD):

The following inequalities are valid

$$c_1 \cdot \left[{}_1F_1\left(\frac{1}{2}, c; x\right)\right]^2 \le {}_1F_1(a_n, c; x) \cdot {}_1F_1(1 - a_n, c; x) \le \left[{}_1F_1\left(\frac{1}{2}, c; x\right)\right]^2.$$

Remark 9: For the sequences $a_n := \frac{\omega_n}{n} - \frac{1}{2}$ , $b_n := \frac{\omega_n + \omega_{n+1} - 1}{2n} - \frac{1}{2}$ it holds $a_n \in (0, \frac{1}{2})$ resp. $b_n \in (\frac{1}{2}, 1)$ with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1/2$, and the domain of each sequence has Snirelmann density $\frac{1}{2}$. Therefore, it holds

$$-x\ {}_1F_1(a_n; a_n + 1, -log\, x), -x\ {}_1F_1(b_n; b_n + 1, -log\, x) \ \to\ li(x) = -x\ {}_1F_1(1; 1, -log\, x).$$

Remark 10: In the context of non-harmonic Fourier series theory the counterpart of $\gamma := \{11,1,1,\ldots\ldots\}$ can be defined by $(a,b)_n := \{a_1, b_1. a_2, b_2.\ldots.\}$. It enables „two check points", while running once through the unit circle, one „check point" for each prime number $p$, $q$.

Remark 11: The sequences $s_n := \frac{\omega_n}{n}$ and $t_n := \frac{\omega_n + \omega_{n+1} - 1}{2n}$ fulfill the Hardy-Littlewood condition $|s_{n+1} - s_n| < \frac{K}{n}$ , e.g. $s_n$ has a defined Abel average ((EdH) 12.7) $\lim_{r\uparrow 0} \frac{s_1 r + s_1 r^2 + s_3 r^3 + \cdots \ldots}{r + r^2 + r^3 + \cdots \ldots} = L$.

Remark 12 ((KaM), (KoA), (ZyA)): Let $\{n_k\}$ be a sequence of integers satisfying the Hadamard gap condition, i.e. $\frac{n_{k+1}}{n_k} > q > 1$, then trigonometric gap series $\sum_{k=1}^{\infty} c_k \sin(2\pi n_k x)$ converges almost everywhere iff $\sum_{k=1}^{\infty} c_k^2 < \infty$.

Remark 13: We note that for $\tilde{T}(log\, x) := \sum_{n\le x} \frac{1}{n} \log\left(\frac{x}{n}\right)$, $x \ge 1$, the inverse mapping is given by ([ScW] lemma 3.3)

$$\sigma(x) := \tilde{T}^{-1}(log\, x) = \sum_{n\le x} \frac{\mu(n)}{n} \log\left(\frac{x}{n}\right) .$$

Remark 14: With respect to Remark 7 we note that what also cannot derived from the PNT is the convergence of the series

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n}log(\frac{1}{n})=1 \quad .$$

*„The corresponding theorem goes deeper than the PNT, and from it the PNT can be easily derived"* ((LaE) §160). From (ApT) p. 66 we recall that the PNT can be derived from $\sum_{n=1}^{\infty}\frac{\mu(n)}{n}=0$.

The related point measure is given by $\vartheta(x)=\sum_{n\le x}\frac{\mu(n)}{n}log\,(\frac{x}{n}))$ with $d\vartheta=\frac{1}{x}\left[\sum_{n\le x}\frac{\mu(n)}{n}\right]dx$. Because of $\frac{d}{dx}\log\left(\frac{x}{a}\right)=\frac{1}{x}$ it can be replaced by

$$\vartheta^{*}(x)=\sum_{\substack{n\le x\\ n\,odd}}\frac{\mu(n)}{n}\log\,(\frac{2x}{2\omega_n-\frac{1}{2}})+\sum_{\substack{n\le x\\ n\,even\\ n=1}}\frac{\mu(n)}{n}\log\left(\frac{2x}{\omega_n+\omega_{n+1}-\frac{1}{2}}\right)$$

with $d\vartheta=d\vartheta^{*}\sim\frac{dx}{x}=d(logx)$ resp.

$$\sum_{n\,odd}^{\infty}\frac{\mu(n)}{n}\log\left(\frac{2}{2\omega_n-\frac{1}{2}}\right)+\sum_{\substack{n\,even\\ n=1}}^{\infty}\frac{\mu(n)}{n}\log\left(\frac{2}{\omega_n+\omega_{n+1}-\frac{1}{2}}\right)\le 1$$

The additional term $\omega_n+\omega_{n+1}-\frac{1}{2}$ for the *„n even"* series (i.e. not starting the series $\omega_n+\omega_{n+1}+\frac{1}{2}$) is to get „$n=1$" as an element of the underlying domain, i.e. both domains get a Snirelmann density of ½. Additionally both related sequences, $\left(\frac{2}{2\omega_n-\frac{1}{2}}\right)^{-1}=\omega_n-\frac{1}{4}$ and $\left(\frac{2}{\omega_n+\omega_{n+1}-\frac{1}{2}}\right)^{-1}=\frac{\omega_n+\omega_{n+1}}{2}-\frac{1}{4}$ fulfill the prerequisite of the ((YoR) p. 36)

Kadec $\frac{1}{4}$–Theorem: If $\sigma_\nu$ is a sequence of real numbers for which

$$|\sigma_\nu-\nu|\le L<\frac{1}{4} \quad ,\ \nu=0,\pm1,\pm2,\dots,$$

then $\{e^{i\sigma_\nu x}\}_{\nu=0,\pm1,\pm2,\dots}$ satisfies the Paley-Wiener criterion and so forms a Riesz basis for $L_2^{\#}(-\pi,\pi)=H_0^{\#}(-\pi,\pi)$.

The below strongly increasing sequences $(A_n^{(1)},\bar{A}_n^{(1)})$ and $(A_n^{(2)},\bar{A}_n^{(2)})$ (lemma 9) enable the definition of a twofold finer sieve method (HaH): there is the split above between „odd" and „even" series, both with underlying Snirelmann densitiy ½, governing two different arithmetical functions on the first level; on a second level per each series there can be built specific sieve methods per underlying considered „prime interval" domains $[1,p]$ and $[p,2p]$, to overcome the mentioned challenge in lemma 4 to build an arithmetical function for $F(u,x):=\pi(2n-p)$ based on lemma 2.

From (LaE1) we note that $\sum_{k\text{ odd}}^{2x}G_k\sim\frac{2x}{\log(2x)}$, while $\sum_{k=1}^{x}G_{2k}\sim\frac{1}{2}(\frac{2x}{\log(2x)})^2$, because the odd $k$ can be neglected as every odd number can be represented as sum of two primes, if $k-2$ is prime, otherwise not.

Lemma 7:

For the sequences pair $(\lambda_k^{(1)},\lambda_k^{(2)})$ it holds $\lambda_k^{(1)}\in\left(k-\frac{1}{2},k\right)$, $\lambda_k^{(2)}\in\left(k,k+\frac{1}{2}\right)$, and therefore,

$$\frac{2k-1}{2k+1}<\frac{\lambda_k^{(1)}}{k}\cdot\frac{\lambda_k^{(2)}}{k+\frac{1}{2}}<1 \quad \text{resp.} \quad \frac{1}{2k}\cdot\frac{1}{2(k+1)}<\frac{1}{2k}\cdot\frac{2k-1}{2k+1}<\frac{1}{2k}\cdot\frac{\lambda_k^{(1)}}{k}\cdot\frac{\lambda_k^{(2)}}{k+\frac{1}{2}}<\frac{1}{2k}$$

resp.

$$\frac{n}{2}n<\sum_{k=1}^{n}\lambda_k^{(1)}<\frac{n}{2}(n+1)<\sum_{k=1}^{n}\lambda_k^{(2)}<\frac{n}{2}(n+2).$$

Lemma 8:

Putting $a_k^{(1)}:=\lambda_k^{(1)}-\frac{1}{2}$ , $a_k^{(2)}:=\lambda_k^{(3)}-\frac{1}{2}$ one gets

$$(2n-1)<A_n^{(1)}:=\frac{1}{n}\sum_{k=1}^{2n}a_k^{(1)} \ ,\ \bar{A}_n^{(1)}:=\frac{1}{n}\sum_{k=1}^{2n}\left[2n-a_k^{(2)}\right]<2n<$$

$$<A_n^{(2)}:=\frac{1}{n}\sum_{k=1}^{2n}a_k^{(2)} \ ,\ \bar{A}_n^{(2)}:=\frac{1}{n}\sum_{k=1}^{2n}\left[2n-a_k^{(1)}\right]<(2n+1)$$

and each of the two domains of the two merged, strongly increasing sequences $(A_n^{(1)},\bar{A}_n^{(1)})$ and $(A_n^{(2)},\bar{A}_n^{(2)})$ both have Snirelmann density ½.

The link to the proposed Kummer function based Zeta function theory is given by the theory of quasi-asymptotics in distributional Hilbert scales, defined via the eigenvalues of the specific Whittaker operator with the fundamental solutions, (AbM) 13.1.31, (GrI) 9.212,

$${}_1F_1\left(\frac{1}{2},\frac{3}{2};z\right) = z^{-\frac{3}{4}}e^{\frac{z}{2}}M_{\frac{1}{4},\frac{1}{4}}(z) \quad \text{and} \quad {}_1F_1\left(1,\frac{3}{2};-z\right) = z^{-\frac{3}{4}}e^{-\frac{z}{2}}M_{\frac{1}{4},\frac{1}{4}}(z) \ .$$

From the (A1) lemma we recall the asymptotics

$$d\left[{}_1F_1\left(\frac{1}{2},\frac{3}{2};\log x\right)\right]^2 \sim \left(\frac{x}{\log x}\right)^2 dx \ .$$

For the construction of a Kummer function related arithmetical function one gets from lemma 5 vii)) with $s = 1/2$

$$\int_0^\infty t^{1/4}\,{}_1F_1\left(\frac{1}{2},\frac{3}{2};-t\right)\frac{dt}{t} = 2\Gamma\left(\frac{1}{4}\right),$$

i.e. ($t \to -\log x = \log\left(\frac{1}{x}\right)$) it holds

$$\int_0^1 h(x)dx = 1 \quad \text{for} \quad h(x) := \frac{1}{x}\frac{\log^{\frac{1}{4}}\left(\frac{1}{x}\right)}{2\Gamma\left(\frac{1}{4}\right)}\frac{{}_1F_1\left(\frac{1}{2},\frac{3}{2};\log x\right)}{\log x} = \frac{1}{2\Gamma\left(\frac{1}{4}\right)}\frac{1}{\sqrt{x\log x}}\frac{M_{\frac{1}{4},\frac{1}{4}}(\log x)}{\log x} \ .$$

Remark: Recalling ${}_1F_1(a, a+1; x) \sim \frac{1}{\Gamma(a)}\frac{e^x}{x}$ and $\frac{1}{2}\frac{d}{dx}F_a^2(x) = \frac{a}{a+1}F_a(x)\cdot F_{a+1}(x)$ two (strongly increasing resp. strongly decreasing) sequences $\vartheta_n^{(i)}$ with $0 < \theta_n < 1$ and $\lim_{n\to\infty}\vartheta_n^{(i)} = \frac{1}{2}$ ( $i = 1,2$) allow to approximate the $d\left[{}_1F_1'\left(\frac{1}{2},\frac{3}{2};\log t\right)\right]^2$ integral density in the following form

$$\lim_{n\to\infty}\left[{}_1F_1\left(\vartheta_n^{(1)},\vartheta_n^{(1)}+1;\log x\right)\cdot\ {}_1F_1\left(\vartheta_n^{(2)}+1,\vartheta_n^{(2)}+2;\log x\right)\right] = \frac{1}{2}\frac{d}{dx}\left[{}_1F_1\left(\frac{1}{2},\frac{3}{2};\log x\right)\right]^2$$

$$\frac{\pi}{2}\,{}_1F_1\left(\vartheta_n^{(1)},\vartheta_n^{(1)}+1;\log x\right)\cdot\ {}_1F_1\left(\vartheta_n^{(2)}+1,\vartheta_n^{(2)}+2;\log x\right) \sim \frac{\pi}{2}\frac{1}{\Gamma\left(\vartheta_n^{(1)}\right)}\frac{1}{\Gamma\left(\vartheta_n^{(2)}\right)}\left[\frac{x}{\log x}\right]^2 \sim \frac{1}{2}\left[\frac{x}{\log x}\right]^2.$$

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