A simple way of constructing the function p_s is linear extrapolation of p in $\Delta_o(s) - \Delta_o$ with $p_s = 0$ on $\partial \Delta_o(s)$. Since the distance of $\partial \Delta_o(s)$ from Δ_o as well as the area of $\Delta_o(s) - \Delta_o$ are of order s the estimates of Proposition 5 clearly hold with the $L_\infty(\Delta_o)$-norm of p and hence also with the $L_2(\Delta_o)$-norm because of the finite dimension of the space of polynomials considered.

2. To any $\Delta \in \Gamma_h$ and $\chi \in S_h$ define

$$\chi_\Delta = \begin{cases} \chi & \text{in } \Delta \\ 0 & \text{otherwise} \end{cases} .$$

Since the triangles Δ are $\varkappa$-regular there is a linear transformation T_Δ mapping Δ onto Δ_o the Jacobian of which resp. of the inverse mapping is bounded by $c\, h^{-2}$ resp. $c\, h^2$ with c depending only on $\varkappa$. Proposition 5 gives

<u>Corollary 5:</u> Given $s > 0$. To $\chi \in S_h$ there is an approximating function $\chi_\Delta^s \in \overset{o}{H}_1(R^2)$ according to

$$\|\chi_\Delta - \chi_\Delta^s\|^2 \le c\, \frac{s}{h}\, \|\chi_\Delta\|_\Delta \quad ,$$

$$\|\nabla \chi_\Delta^s\|^2 \le c\, \frac{1}{sh}\, \|\chi_\Delta\|_\Delta^2$$

with supp $(\chi_\Delta^s) \subseteq \Delta(hs)$.

3. Following the lines of the proof of Theorem 3 we get directly

<u>Theorem 4:</u> Let $\vartheta \in (0, 1/2)$. Any $\chi \in S_h$ belongs to H_ϑ and

$$\|\chi\|_{H_\vartheta} \le c\, h^{-\vartheta}\, \|\chi\|_{L_2} .$$

finally

$$\|u\|^2_{H_\vartheta} \le c \sum_{i=1}^{I} h_i^{-2} \lambda_i \quad .$$

Thus we have proved

<u>Theorem 3:</u> Let $\vartheta \in (0, 1/2)$. Any $u \in H'(\Gamma)$ belongs to H_ϑ and

$$\|u\|^2_{H_\vartheta} \le c \sum_{\Delta\in\Gamma} h_\Delta^{-2\vartheta} \left\{\|u\|^2_\Delta + h^2_\Delta \|\nabla u\|^2_\Delta\right\}$$

with c depending only on ϑ and $\varkappa$.

An application of this theorem is

<u>Corollary:</u> Let Γ be a $\varkappa$-regular triangulation and S_Γ be of class $(-1. m)$. To any $u \in H^1_k(\Gamma)$ with $k \le m$ and $\vartheta \in (0, 1/2)$ there is a $\chi \in S_\Gamma$ with

$$\|u-\chi\|^2_{H_\vartheta} \le c \sum_{\Delta\in\Gamma} h_\Delta^{2(k-\vartheta)} \|\nabla^k u_{|\Delta}\|^2_\Delta \ .$$

<u>Remark:</u> With $\bar{h} = \text{Max}\ h_\Delta$ we may rewrite this also in the form

$$\|u-\chi\|_{H_\vartheta} \le c\ \bar{h}^{k-\vartheta} \|\nabla^k u\|' \quad .$$

<u>Proof:</u> To any $\Delta \in \Gamma$ there is a polynomial $p = p_\Delta$ of degree less than $k \le m$ according to

$$\|u_{|\Delta} - p_\Delta\|_\Delta \le c\ h^k_\Delta \|\nabla^k u_{|\Delta}\|_\Delta \quad ,$$

$$\|\nabla p_\Delta\|_\Delta \le c\ h^{k-1}_\Delta \|\nabla^k u_{|\Delta}\|_\Delta \quad .$$

The function χ defined by $\chi_{|\Delta} = p_\Delta$ is an element of S_Γ and obviously $u - \chi$ belongs to H_1' . Theorem 3 then gives $u - \chi \in H_\vartheta$ and in connection with the above bounds of $u - \chi$ in Δ the corollary.

6. Inverse Properties of Finite Elements

In this section we restrict ourselves to uniform $\varkappa$-regular triangulations $\Gamma = \Gamma_h$ of a fixed polygonal domain $\bar{T}$. Further let S_Γ be of class $(-1,m)$ for m fixed. We will show

<u>Theorem 4:</u> $S_\Gamma \subseteq H_\vartheta$ for $\vartheta < 1/2$ and for $\chi \in S_\Gamma$ inverse properties hold

$$\|\chi\|_{H_\vartheta} \le c\ h^{-\vartheta} \|\chi\|_{L_2} .$$

The proof is given by means of three steps.

1. Let Δ_o be the reference triangle with vertices (0,0), (1,0), and (0,1). Further let p denote a function which is a polynomial of degree less than m in Δ_o and zero outside Δ_o . Then the assertion holds

<u>Proposition 5:</u> For $s > 0$ there is a $p_s \in \overset{o}{H}_1(R^2)$ according to

$$\|p-p_s\|^2 \le c\ s\ \|p\|^2_{\Delta_o} \quad ,$$

$$\|\nabla p_s\|^2 \le c\ s^{-1} \|p\|^2_{\Delta_o} \quad .$$

Further we define

$$\hat{u}_t = \Sigma \, \hat{u}_t^i \quad .$$

Because of the finite intersection property (Lemma 2) we get

$$\|u-\hat{u}_t\|^2 + t^2\|\nabla\hat{u}_t\|^2 \leq$$

$$c\Big\{\sum_{J_t^-} \|u^i\|^2 + \sum_{J_t^+} \big[\|u^i-\hat{u}_t^i\|^2 + t^2\|\nabla u_t^i\|^2\big]\Big\} \quad .$$

With the help of Theorem 2 - using the abbreviation

$$\lambda_i = \|u^i\|^2_{A_i} + h_i^2\|\nabla u^i\|^2_{A_i}$$

we find

$$K^2(u,t) \leq \|u-\hat{u}_t\|^2 + t^2\|\nabla\hat{u}_t\|^2$$

$$\leq c\Big\{\sum_{J_t^-} \lambda_i + \sum_{J_t^+} t\, h_i^{-1}\, \lambda_i\Big\} \quad .$$

For $t \in (h_j, h_{j+1})$ with $h_o = 0$ and $h_{I+1} = 1$ the Index-sets J_t^- , J_t^+ are given by

$$J_t^- = \{1, 2, \ldots, j\} \quad ,$$

$$J_t^+ = \{j+1\, , j+2\, , \ldots, I\} \quad .$$

Therefore we get for $\theta \in (0, 1/2)$

$$\int_{h_j}^{h_{j+1}} K^2(u,t)\, t^{-1-2\theta}\, dt \leq c\, c_\theta\Big\{(h_j^{-2\theta} - h_{j+1}^{-2\theta}) \sum_{i=1}^{j} \lambda_i \; +$$

$$+ \; (h_{j+1}^{1-2\theta} - h_j^{1-2\theta}) \sum_{i=j+1}^{I} h_i^{-1}\, \lambda_i\Big\}$$

with $c_\theta = \text{Max}\, (1/(2\theta)\; , \; 1/(1-2\theta))$. This gives

$$\|u\|^2_{H_\theta} = \int_0^1 K^2(u,t)\, t^{-1-2\theta} dt$$

$$\leq c\, c_\theta \sum_{j=1}^{I} (h_j^{-2\theta} - h_{j+1}^{-2\theta}) \sum_{i=1}^{j} \lambda_i$$

$$+ \; c\, c_\theta \sum_{j=0}^{I-1} (h_{j+1}^{1-2\theta} - h_j^{1-2\theta}) \sum_{i=j+1}^{I} h_i^{-1}\, \lambda_i \quad .$$

Interchanging the order of summation and using

$$\sum_{j=1}^{I} \sum_{i=1}^{j} = \sum_{i=1}^{I} \sum_{j=i}^{I}$$

$$\sum_{j=0}^{I-1} \sum_{i=j+1}^{I} = \sum_{i=1}^{I} \sum_{j=0}^{i-1} \quad .$$

We get because of

$$\sum_{j=i}^{I} (h_j^{-2\theta} - h_{j+1}^{-2\theta}) = h_i^{-2\theta} - 1 < h_i^{-2\theta} \quad ,$$

$$\sum_{j=0}^{i-1} (h_{j+1}^{1-2\theta} - h_j^{1-2\theta}) = h_i^{1-2\theta}$$

The angle-condition implies that at most $n_1 = [2\pi/\alpha]$ triangles have one vertex in common. Therefore $n_2 = 3(n_1-2)$ is an upper bound for the number of elements of $\mathfrak{N}(\Delta)$.

Now let Δ' be a triangle sharing one side with Δ. Since $\kappa^{-1}h_\Delta$, $\kappa^{-1}h_{\Delta'}$ are lower bounds and κh_Δ, $\kappa h_{\Delta'}$ are upper bounds for the length of this side we get in this case

$$\kappa^{-2} \le h_\Delta / h_{\Delta'} \le \kappa^2 \quad .$$

Therefore there is a $c_1 = c_1(\kappa)$ such that the width of any $\Delta' \in \mathfrak{N}(\Delta)$ is bounded by

$$c_1^{-1}h_\Delta \le h_{\Delta'} \le c_1 h_\Delta \quad .$$

Next let Δ be a κ-regular triangle. Any circle with center in one vertex of Δ and radius less than $\kappa^{-1}h_\Delta$ does not intersect the opposite side. Similarily any parallel line to one of the sides with distance less than $\kappa^{-1}h_\Delta$ intersects the two other sides. From this it follows that the domain $\Delta(\sigma)$ with $\sigma \le \kappa^{-1}c_1^{-1}h_\Delta$ is contained in $\mathfrak{N}(\Delta)$ which proves Lemma 1.

As a consequence of this lemma we get

<u>Lemma 2:</u> Let Γ be a κ-regular triangulation and c the constant of Lemma 1.

For any $\Delta \in \Gamma$ the domain $\Delta(c\, h_\Delta)$ has a non-void intersection with $\Delta'(c\, h_{\Delta'})$ for $\Delta' \in \Gamma$ for at most $\bar{n} = \bar{n}(\kappa)$ elements Δ' of Γ.

5. Norm Estimates

Now we turn to the proof of the teorem stated in Section 1. Let Γ be a κ-regular triangulation. Further let the triangles $\Delta \in \Gamma$ be numbered such that the sequence $\{h_i | h_i = h_{\Delta_i} | i = 1,2,\ldots$ is monotone not decreasing. For $t > 0$ we subdivide the index-set $\{1,2,\ldots,I\}$ into two classes according to

$$J_t^- = \{i \mid h_i < t\} \quad .$$

$$J_t^+ = \{i \mid h_i \ge t\} \quad .$$

We will also use the splitting

$$u = \sum_1^I u^i$$

with

$$u^i = \begin{cases} u & \text{in } \Delta_i \\ 0 & \text{otherwise} \end{cases} \quad .$$

In order to get an approximation $w = w_t$ needed for the K-functional we define

$$\hat{u}_t^i = \begin{cases} u^i_{(ct)} & \text{for } i \in J_t^+ \\ 0 & \text{for } i \in J_t^- \end{cases} \quad .$$

Here $u^i_{(ct)}$ denotes the smoothed functions considered in Section 3 and c is the constant from Lemma 1.

which gives with Proposition 2'

$$\|\partial_x \hat{\omega}_s(u)\| \le \|\partial_x \omega_s^1(u)\| \quad .$$

Because of

$$\|u\|^2_{\partial\Delta_o} \le c\|u\|_{\Delta_o}(\|u\|_{\Delta_o} + \|\nabla u\|_{\Delta_o})$$
$$\le c(\|u\|^2_{\Delta_o} + \|\nabla u\|^2_{\Delta_o})$$

we have shown

Theorem 1: For $s > 0$ there is to any u with $u \in H_1(\Delta_o)$ and $u = 0$ outside Δ_o a function $u_s = \hat{\omega}_s(u) \in \overset{o}{H}_1(R^2)$ with $\mathrm{supp}\,(u_s) \le \Delta_o(\sqrt{2}\,s)$ according to

$$\|u-u_s\|^2 \le c\; s\{\|u\|^2_{\Delta_o} + \|\nabla u\|^2_{\Delta_o}\} \quad ,$$

$$\|\nabla u_s\|^2 \le c\; s^{-1}\{\|u\|^2_{\Delta_o} + \|\nabla u\|^2_{\Delta_o}\} \quad .$$

Next let Δ be a $\varkappa$-regular triangle with width h. Then there is a linear mapping $\Delta \to \Delta_o$ whose Jacobian resp. its inverse is bounded by $c(\varkappa)h^{-2}$ resp. $c(\varkappa)h^2$. Replacing s by $s/\sqrt{2}\,h$ we get from Theorem 1

Theorem 2: Let Δ be a $\varkappa$-regular triangle with width h. To any u with $u \in H_1(\Delta)$ and $u = 0$ outside Δ, there is a $u_s \in \overset{o}{H}_1(R^2)$ with $\mathrm{supp}(u_s) \le \Delta(hs)$ according to

$$\|u-u_s\|^2 \le c\,\frac{s}{h}\{\|u\|^2_\Delta + h^2\|\nabla u\|^2_\Delta\} \quad ,$$

$$\|\nabla u_s\|^2 \le c\,\frac{1}{sh}\{\|u\|^2_\Delta + h^2\|\nabla u\|^2_\Delta\} \quad .$$

4. $\varkappa$-Regular Triangulations

Let Γ be a $\varkappa$-regular triangulation of Ω. Though we have assumed nothing on the widths h_Δ of $\Delta \in \Gamma$ they cannot change too fast as we will discuss to some extend in this section. Especially we will show

Lemma 1: Let Γ be a $\varkappa$-regular triangulation. Then there is a $c = c(\varkappa)$ such that for any $\Delta \in \Gamma$ the domain $\Delta(\sigma)$ has a non-void intersection with any $\Delta' \in \Gamma$ for at most $n(\varkappa)$ triangles provided $\sigma \le c\,h_\Delta$.

By elementary geometrical consideration we see that any angle of $\Delta \in \Gamma$ is bounded from below by α defined by

$$\sin\frac{\alpha}{2} = \frac{1}{2\varkappa^2-1} \quad .$$

To $\Delta \in \Gamma$ let $\mathfrak{N}(\Delta)$ be the set of all neighbours, i.e. of all $\Delta' \in \Gamma$ with $\bar{\Delta} \cap \bar{\Delta}' \neq \emptyset$, and let $\Delta(\mathfrak{N})$ be defined by

$$\Delta(\mathfrak{N}) = \mathrm{int}\{\bar{\Delta} \cup \bar{\Delta}' \,|\, \Delta' \in \mathfrak{N}(\Delta)\} \quad .$$

Using $u(x+s) = u(0) + \int_0^{x+s} u_1 \, d\xi$ resp. $-u(x) = -u(a) + \int_x^a u_1 d\xi$ in the first resp. the third case Proposition 4 is proved similar to Proposition 3 .

3. Smoothing Operators in Two Variables

In the following Δ_o denotes the reference triangle with vertices (0,0), (1,0) and (0,1).

Let ω_s^1, ω_s^2 be the operators considered in Section 2 with respect to the first or second variable and $\hat{\omega}_s = \omega_s^1\omega_s^2 = \omega_s^2\omega_s^1$. Further let for $\delta > 0$ and any domain $\Omega \subseteq R^2$ the domain $\Omega(\delta)$ be defined by

$$\Omega(\delta) = \{(x,y) \mid \text{dist}((x,y),\bar{\Omega}) < \delta\} .$$

Now we consider functions $u = u(x,y)$ with $u = 0$ for $(x,y) \notin \Delta_o$ and $u_{|\Delta_o} \in H_1(\Delta_o)$. Again u_1, u_2 are the functions coinciding with u_x, u_y in Δ_o and zero otherwise.

A direct consequence of Propositions 1 and 2 are

Proposition 1': For $\omega = \omega^1, \omega^2, \hat{\omega}$

$$\omega_s(u) \in \overset{o}{H}_1(R^2)$$

with

$$\text{supp}(\omega_s(u)) \le \Delta_o(\sqrt{2}\, s) .$$

Proposition 2': For $\omega = \omega^1, \omega^2, \hat{\omega}$

$$\|\omega_s(u)\| \le \|u\| .$$

The norms here are two-dimensional.

Proposition 3': For $\omega = \omega^1, \omega^2, \hat{\omega}$

$$\|\omega_s(u) - u\|^2 \le c\, s\{\|u\|^2_{\partial\Delta_o} + s\|\nabla u\|^2_{\Delta_o}\} .$$

In case of ω^1 or ω^2 this follows from Proposition 3 by integration with respect to y or x . Using Proposition 2' we get

$$\|\hat{\omega}_s(u) - u\| \le \|\omega_s^1(\omega_s^2(u) - u)\| + \|\omega_s^1(u) - u\|$$
$$\le \|\omega_s^2(u) - u\| + \|\omega_s^1(u) - u\| .$$

Proposition 4':

$$\left.\begin{array}{l} \|\partial_x(\omega_s^1(u))\|^2 \\ \|\partial_y(\omega_s^2(n))\|^2 \\ \|\nabla(\hat{\omega}_s(u))\|^2 \end{array}\right\} \le c s^{-1}\{\|u\|^2_{\partial\Delta_o} + s\|\nabla u\|^2_{\Delta_o}\} .$$

The first two inequalities are got by integration of their one-dimensional analoga. In case of the third we have for example

$$\partial_x(\hat{\omega}_s(u)) = \omega_s^2(\partial_x(\omega_s^1(u))$$

<u>Case i:</u> Then we have

$$\omega_s(u) - u = s^{-1} \int_0^{s+x} u d\xi$$

$$= s^{-1}(s+x)\, u(0) + s^{-1} \int_0^{s+x} d\xi (u(\xi)-u(0))$$

$$= f_1 + f_2 \; .$$

Integration gives

$$\int_{-s}^{0} f_1^2 \, dx \; = \; \frac{1}{3} s \, u(0)^2 \; .$$

Using $u(\xi) - u(0) = \int_0^{\xi} u_1 \, d\eta$ and interchanging the order of integration gives

$$f_2(x) = s^{-1} \int_0^{s+x} u_1 d\eta \int_{\eta}^{s+x} d\xi$$

$$= s^{-1} \int_0^{s+x} (s+x-\eta) \, u_1 \, d\eta$$

and for $-s \le x \le 0$ therefore

$$|f_2(x)| \le \int_0^{s+x} |u_1| \, d\eta \; .$$

This gives

$$\int_{-s}^{0} f_2^2 \, dx \le \int_{-s}^{0} dx \, (s+x) \int_0^{s+x} u_1^2 \, d\eta$$

$$\le \frac{1}{2} s^2 \int_0^{s} u_1^2 \, d\eta \; .$$

<u>Case ii:</u> We have here

$$\omega_s(u) - u = s^{-1} \int_x^{x+s} (u(\xi) - u(x)) \, d\xi$$

$$= s^{-1} \int_x^{x+s} d\xi \int_x^{\xi} u_1 \, d\eta$$

$$= s^{-1} \int_x^{x+s} (x+s-\eta) \, u_1 \, d\eta$$

and therefore

$$|\omega_s(u) - u| \le \int_x^{x+s} |u_1| \, d\eta \; .$$

Similar to the estimate for f_2 we get,

$$\int_0^{\mathrm{Max}(0,a-s)} |\omega_s(u) - u|^2 \, dx \le s^2 \int_0^{a} u_1^2 \, dx \; .$$

The third case is analogue to the first and omitted here.

<u>Proposition 4:</u>

$$\|\omega_s(u)'\|^2 \le c \, s^{-1}\left\{u(0)^2 + u(1)^2 + s\|u^1\|^2\right\} \; .$$

<u>Proof:</u> We have - assuming $s < a$ -

$$\omega_s(u)' = s^{-1} \begin{cases} u(x+s) & -s \le x < 0 \\ u(x+s) - u(x) \quad \text{for} & 0 \le x < a-s \\ -u(x) & a-s \le x < a \end{cases}$$

For $\partial\Omega$ sufficiently smooth the Sobolev spaces $H_k(\Omega)$ are parts of a full scale H_α of Hilbert spaces. For our purposes we can restrict ourselves to the interval $[0,1]$ for α. The spaces $H_k'(\Gamma)$ belong to H_0 but in general not to H_1. Probably well known to all working in this field is the inclusion

$$H_k'(\Gamma) \subseteq H_\vartheta$$

for $\vartheta \in (0, 1/2)$ and $k \geq 1$ provided the $\varkappa$-regularity of Γ. But obviously there seems to be no reasonable reference for this fact. The aim of this report is a proof for this both simple and elementary. An application to the approximability in H_ϑ-norms as well as inverse properties of finite elements in these norms are added.

In Section 2 by $\|.\|$ the L_2-norm is R^1 and in the next sections this norm in R^2 is denoted.

With respect to the interpolation theory we refer to

Brenner, P.; V. Thomée, and L.B. Wahlbin, 'Besov Spaces and Applications to Difference Methods for Initial Value Problems', Lecture Notes in Mathematics 434, Springer, New York, 1975.

2. Smoothing Operators in one Variable

In this section we will give some approximation properties of the Stecklov-operator defined by

$$\omega_s(u)(x) = \frac{1}{s}\int_x^{x+s} u(\xi)d\xi$$

with $s > 0$. Especially we will consider functions u vanishing outside a fixed interval $I = (0,a)$ and sufficiently smooth in I - actually only $u_{|I} \in H_1(I)$ is needed - . We will write u_1 for the function coinciding with the first derivative of u in I and zero otherwise. We have at once

Proposition 1: $\omega_s(u) \in \overset{o}{H}_1(R^1), \operatorname{supp}(\omega_s(u)) \subseteq (-s, a)$.

Proposition 2: $\|\omega_s(u)\| \leq \|u\|$.

Proof: Using Schwarz's inequality we get

$$\|\omega_s(u)\|^2 = \int_{-s}^{s+a} s^{-2}\left\{\int_x^{x+s} u d\xi\right\}^2 dx$$

$$\leq s^{-1}\int_{-s}^{s+a} dx \int_x^{x+s} u^2 d\xi \leq \int_0^a u^2 d\xi \quad .$$

Proposition 3: $\|\omega_s(u) - u\|^2 \leq s\{u(0)^2 + u(a)^2 + s\|u_1\|^2\}$

Proof: We have to consider the three intervals - the second possibly being empty -

i) $-s < x \leq 0$,

ii) $0 < x \leq \operatorname{Max}(0,a-s)$,

iii) $\operatorname{Max}(0,a-s) < x < a$.

THE REGULARITY OF PIECEWISE DEFINED FUNCTIONS WITH RESPECT TO SCALES OF SOBOLEV SPACES

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Typical in the theory of finite elements is the local character of the functions to be used. For simplicity let Ω be a bounded domain in R^2 and Γ a subdivision into (possibly curved) triangles Δ. Then finite element spaces $S = S_\Gamma$ consist of functions χ with the properties

i. $\chi \in L_2(\Omega)$ resp. $\chi \in C_k(\Omega)$ for some fixed k.

ii. The restriction $\chi_\Delta = \chi|_\Delta$ to any $\Delta \in \Gamma$ is an element of a finite dimensional subspace, especially χ_Δ is a polynomial of degree less than m.

We will say then S_Γ is of class (k,m) with $k \geq -1$. Parallel to this we denote by $H_k' = H_k'(\Gamma,\Omega)$ the space of functions $u \in L_2(\Omega)$ with $u_\Delta \in H_k(\Delta)$ for $\Delta \in \Gamma$.

To any Δ we will associate the regularity $\varkappa_\Delta$ and the width h_Δ defined in the following way: Let $\bar{r}_\Delta$ be the radius of the smallest circle containing $\bar{\Delta}$ and $\underline{r}_\Delta$ that of the largest one contained in $\bar{\Delta}$. Then

$$\varkappa_\Delta = (\bar{r}_\Delta / \underline{r}_\Delta)^{1/2} ,$$

$$h_\Delta = (\bar{r}_\Delta \underline{r}_\Delta)^{1/2} .$$

If for a given subdivision $\varkappa_\Delta \leq \varkappa$ then we say that Γ is $\varkappa$-regular. Γ is said to be uniform $\varkappa$-regular if there is a constant c with $c^{-1}h \leq h_\Delta \leq c\, h$ for all $\Delta \in \Gamma$ with some h.