

ORTHOGONAL POLYNOMIALS AND LAURENT POLYNOMIALS RELATED TO THE HAHN-EXTON q -BESSEL FUNCTION

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ABSTRACT. Laurent polynomials related to the Hahn-Exton q -Bessel function, which are q -analogues of the Lommel polynomials, have been introduced by Koelink and Swarttouw. The explicit strong moment functional with respect to which the Laurent q -Lommel polynomials are orthogonal is given. The strong moment functional gives rise to two positive definite moment functionals. For the corresponding sets of orthogonal polynomials the orthogonality measure is determined using the three-term recurrence relation as a starting point. The relation between Chebyshev polynomials of the second kind and the Laurent q -Lommel polynomials and related functions is used to obtain estimates for the latter.

1. INTRODUCTION AND MOTIVATION

The Lommel polynomials are orthogonal polynomials closely related to the Bessel function. Although the Lommel polynomials have a representation involving a hypergeometric ${}_2F_3$ -series, they do not fit into Askey's scheme of hypergeometric orthogonal polynomials. The reason for this is that the orthogonality measure for the Lommel polynomials is supported on the set consisting of one over the zeros of a Bessel function, which are not explicitly known in general. So there is no Rodrigues formula or difference equation for the Lommel polynomials.

The Bessel function $J_\nu(z)$ of order ν and argument z is given by the absolutely convergent series expansion

$$(1.1) \quad J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k! \Gamma(\nu + k + 1)}.$$

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The properties of this special function are well understood, see e.g. the book on Bessel functions by Watson [22]. A simple recurrence relation for the Bessel functions is, cf. [22, §3.2(1)],

$$(1.2) \quad J_{\nu+1}(z) = \frac{2\nu}{z}J_{\nu}(z) - J_{\nu-1}(z).$$

From iteration of (1.2) we see that we can express $J_{\nu+m}(z)$ in terms of $J_{\nu}(z)$ and $J_{\nu-1}(z)$ and the coefficients of $J_{\nu}(z)$ and $J_{\nu-1}(z)$ are polynomials in z^{-1} . This was first observed by Lommel in 1871. Explicitly, we have, cf. Watson [22, §9.6],

$$(1.3) \quad J_{\nu+m}(z) = h_{m,\nu}\left(\frac{1}{z}\right)J_{\nu}(z) - h_{m-1,\nu+1}\left(\frac{1}{z}\right)J_{\nu-1}(z),$$

where $h_{m,\nu}(z)$ are the Lommel polynomials, which are also known as associated Lommel polynomials. The Lommel polynomials satisfy the three-term recurrence relation

$$(1.4) \quad h_{m+1,\nu}(z) = 2z(m+\nu)h_{m,\nu}(z) - h_{m-1,\nu}(z), \quad h_{-1,\nu}(z) = 0, \quad h_{0,\nu}(z) = 1.$$

Favard's theorem, cf. Chihara [7, Ch. II, thm. 6.4], implies that the Lommel polynomials are orthogonal polynomials with respect to a positive weight function for $\nu > 0$. The explicit orthogonality relations are, cf. Chihara [7, Ch. VI, §6], Dickinson [9], Dickinson, Pollak and Wannier [10], Ismail [15], Schwartz [19],

$$(1.5) \quad \sum_{k=1}^{\infty} \frac{1}{(j_k^{\nu-1})^2} h_{m,\nu}\left(\frac{\pm 1}{j_k^{\nu-1}}\right) h_{n,\nu}\left(\frac{\pm 1}{j_k^{\nu-1}}\right) = \frac{\delta_{n,m}}{2(\nu+n)},$$

where j_k^{ν} , $\nu > -1$, are the positive zeros of the Bessel function $J_{\nu}(z)$ numbered increasingly, cf. Watson [22, Ch. 15].

Another relation between the Lommel polynomials and the Bessel function is given by Hurwitz's asymptotic formula, cf. [22, 9.65(1)],

$$(1.6) \quad \frac{(2z)^{1-\nu-m} h_{m,\nu}(z)}{\Gamma(\nu+m)} \longrightarrow J_{\nu-1}\left(\frac{1}{z}\right), \quad m \rightarrow \infty.$$

For the Bessel function (1.1) there exist several q -analogues. The oldest q -analogues for the Bessel function were introduced by Jackson in a series of papers in 1903-1905, see the references in [15]. For the Jackson q -Bessel function Ismail [15] introduced the associated q -Lommel polynomials, which turned out to satisfy an orthogonality relation similar to (1.5), but now involving the zeros of the Jackson q -Bessel function. Ismail used these q -Lommel polynomials to prove that the zeros of the Jackson q -Bessel functions behave similarly as the zeros of the Bessel function.

A more recent q -analogue of the Bessel function has been introduced by Hahn in a special case and by Exton in full generality, see the references in Koornwinder and Swarttouw [18]. The zeros of the Hahn-Exton q -Bessel function and several associated q -analogues of the Lommel polynomial have been studied by Koelink and Swarttouw [17]. The zeros of

the Hahn-Exton q -Bessel function behave in a similar fashion as the zeros of the Bessel function. In that paper [17] a q -analogue of the Lommel polynomials was introduced. However, this q -analogue of the Lommel polynomial is no longer a polynomial, but a Laurent polynomial. One of the goals of this paper is to give an explicit orthogonality measure for these orthogonal Laurent q -Lommel polynomials.

The Laurent q -Lommel polynomials are defined by, cf. [17, prop. 4.3 with $R_{m,\nu}(z^{-1}; q) = h_{m,\nu}(z; q)$],

$$(1.7) \quad h_{m+1,\nu}(x; q) = \left(\frac{1}{x} + x(1 - q^{\nu+m})\right)h_{m,\nu}(x; q) - h_{m-1,\nu}(x; q),$$

with initial conditions $h_{-1,\nu}(x; q) = 0$, $h_{0,\nu}(x; q) = 1$. A second independent solution of (1.7) is given by $h_{m-1,\nu+1}(x; q)$. Note that taking the limit $q \uparrow 1$ in (1.7) after replacing x by $2z/(1 - q)$ gives (1.4). The Laurent q -Lommel polynomials originate from a relation similar to (1.3), see proposition 3.1.

The explicit orthogonality relations for the Laurent q -Lommel polynomials $h_{m,\nu}(x; q)$ defined in (1.7) is derived in §3. The method of proof is based on the existence of asymptotically well-behaved solutions of (1.7) reminiscent of $J_{\nu+m}(x)$, cf. (1.3). The method of Dickinson [9] to prove (1.5) can then be adapted to our situation. The orthogonality measure gives rise to a strong moment functional \mathcal{L} , i.e. a functional on the space of Laurent polynomials so that all moments $\mathcal{L}(x^n)$, $n \in \mathbb{Z}$, exist. From \mathcal{L} we obtain two moment functionals \mathcal{L}_\pm , as considered in e.g., [7, Ch. 1], by putting $\mathcal{L}_+(x^n) = \mathcal{L}(x^n)$, $n \in \mathbb{Z}_+$, and $\mathcal{L}_-(x^n) = -\mathcal{L}(x^{-2-n})$, $n \in \mathbb{Z}_+$. (The 2 has to do with the fact that all moment functionals are symmetric.) It turns out that both \mathcal{L}_+ and \mathcal{L}_- are positive definite moment functionals.

The orthogonal polynomials for \mathcal{L}_+ are q -analogues of the Lommel polynomials and the support of orthogonality measure consists of zero and one over the zeros of a Hahn-Exton q -Bessel function. This is worked out in detail in §4, where we use Dickinson's method [9] once more. In §5 we study the orthogonal polynomials for \mathcal{L}_- . We give explicit expressions for these polynomials in terms of Al-Salam–Chihara polynomials, which can be used to determine the asymptotic behaviour as the degree tends to infinity. The asymptotic behaviour is expressed in terms of a function $j_\nu(x; q)$ closely related to the Hahn-Exton q -Bessel function. Since we can do this for the associated polynomials as well, we have the Stieltjes transform of the orthogonality measure from which the orthogonality follows. Using the results of §5 we can simplify the expression for the strong moment functional \mathcal{L} using a Wronskian type formula. This is done in §6.

For $q = 0$, or for $\nu \rightarrow \infty$, we see that $U_m((x+x^{-1})/2)$, where U_m denotes the Chebyshev polynomial of the second kind, satisfies (1.7) with the same initial conditions. So we can view the Laurent q -Lommel polynomials $h_{m,\nu}(x; q)$ as a perturbation of the Chebyshev polynomials. This point of view allows us to obtain estimates for the Laurent q -Lommel polynomials, the Hahn-Exton q -Bessel function and the related function $j_\nu(x; q)$. This is done in §7.

Finally, in §2 we show that the general theory of orthogonal Laurent polynomials presents us with an existence theorem for the strong moment functional \mathcal{L} . We also state a result concerning the zeros of the Laurent q -Lommel polynomials.

To end this introduction we briefly recall the notation for basic (or q)-hypergeometric series. We follow the standard notation of Gasper and Rahman [11, Ch. 1]. We take $0 < q < 1$ for the rest of the paper. A q -shifted factorial is a product defined by

$$(a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i), \quad a \in \mathbb{C}, \quad k \in \mathbb{Z}_+,$$

where the empty product equals 1 by definition. Since $0 < q < 1$ we can take $k \rightarrow \infty$ to get $\lim_{k \rightarrow \infty} (a; q)_k = (a; q)_\infty$. A basic (or q)-hypergeometric series is

$$(1.8) \quad \begin{aligned} {}_r\varphi_s \left(\begin{matrix} a_1, \dots, a_r \\ b_1, \dots, b_s \end{matrix}; q, z \right) &= {}_r\varphi_s(a_1, \dots, a_r; b_1, \dots, b_s; q, z) \\ &= \sum_{k=0}^{\infty} \frac{(a_1; q)_k \dots (a_r; q)_k}{(q; q)_k (b_1; q)_k \dots (b_s; q)_k} \left((-1)^k q^{\frac{1}{2}k(k-1)} \right)^{1+s-r} z^k \end{aligned}$$

For generic values of the parameters the radius of convergence of the series in (1.8) is 0, 1, ∞ corresponding to $r > s + 1$, $r = s + 1$, $r < s + 1$.

2. ORTHOGONAL LAURENT POLYNOMIALS

In this section we apply some of the theory of orthogonal Laurent polynomials to the Laurent polynomials $h_{m,\nu}(x; q)$ to obtain the existence of a strong moment functional \mathcal{L} , i.e. a linear functional on the space of Laurent polynomials for which the moments $\mathcal{L}(x^m)$ exist for all $m \in \mathbb{Z}$, for which the Laurent q -Lommel polynomials are orthogonal. We use the paper by Hendriksen and van Rossum [14] as the main reference for this section. The recurrence relation as in (2.1) has been generalised to a wider class of recurrence relations by Ismail and Masson [16] by replacing x in front of the $V_{m-1,\nu}(x)$ by $(x - a_m)$, for which they prove a Favard type theorem. Specialisation to the case considered here yields the Favard type theorem contained in Hendriksen and van Rossum [14]. For further information concerning this section the reader may consult the introductory paper by Cochran and Cooper [8].

From the recurrence relation (1.7) it follows that $h_{m,\nu}(x; q)$ is an even function for even m and an odd function for odd m . Consequently, $x^m h_{m,\nu}(x; q)$ is a polynomial in x^2 , which we denote by $V_{m,\nu}(x^2) = x^m h_{m,\nu}(x; q)$. For V_m we obtain from (1.7) the recurrence relation

$$(2.1) \quad V_{m+1,\nu}(x) = (1 + x(1 - q^{\nu+m}))V_{m,\nu}(x) - xV_{m-1,\nu}(x)$$

with initial conditions $V_{-1,\nu}(x) = 0$, $V_{0,\nu}(x) = 1$, cf. [14, (2.2)]. The Favard-type theorem, cf. [14, thm. 1.1], implies that for the Laurent polynomials $Q_n(x)$, defined by

$$\begin{aligned} Q_{2n}(x) &= x^{-n} V_{2n,\nu}(x) = h_{2n,\nu}(\sqrt{x}; q) \\ Q_{2n+1}(x) &= x^{-n-1} V_{2n+1,\nu}(x) = x^{-\frac{1}{2}} h_{2n+1,\nu}(\sqrt{x}; q) \end{aligned}$$

there exists a strong moment functional \mathcal{L}_1 such that $\mathcal{L}_1(Q_n Q_m) = 0$ for $n \neq m$.

If we form the lacunary Laurent polynomials, cf. [14, (1.16)], we get the Laurent polynomials $P_{2m}(x) = h_{m,\nu}(x; q)$, $P_{2m+1}(x) = x^{-1}h_{m,\nu}(x; q)$. The lacunary Laurent polynomials are orthogonal with respect to the strong moment functional \mathcal{L} defined by $\mathcal{L}(x^{2n}) = \mathcal{L}_1(x^n)$, $\mathcal{L}(x^{2n+1}) = 0$ for $n \in \mathbb{Z}$, cf. [14, prop. III]. So the orthogonality relations for the even lacunary Laurent polynomials gives

$$(2.2) \quad \mathcal{L}(h_{n,\nu}(x; q)h_{m,\nu}(x; q)) \begin{cases} = 0, & n \neq m, \\ \neq 0, & n = m. \end{cases}$$

But we also have the orthogonality for the odd lacunary Laurent polynomials,

$$(2.3) \quad \mathcal{L}(x^{-2}h_{n,\nu}(x; q)h_{m,\nu}(x; q)) \begin{cases} = 0, & n \neq m, \\ \neq 0, & n = m. \end{cases}$$

The space Λ_n of Laurent polynomials of the form $\sum_{p=-n}^n c_p x^p$ is $(2n+1)$ -dimensional, $n \in \mathbb{Z}_+$. The Laurent polynomials $h_{m,\nu}(x; q)$, $m = 0, 1, \dots, n$, form a $(n+1)$ -dimensional subspace of Λ_n . Moreover, they form an orthogonal basis for this subspace with respect to \mathcal{L} . Equation (2.3) states that this orthogonal basis can be complemented with $x^{-1}h_{m,\nu}(x; q)$, $m = 0, 1, \dots, n-1$, to give an orthogonal basis of Λ_n with respect to \mathcal{L} . $\mathcal{L}(x^{-1}h_{m,\nu}(x; q)h_{n,\nu}(x; q)) = 0$ is immediate from $\mathcal{L}(x^{2p+1}) = 0$.

Remark 2.1. For orthonormal polynomials the three-term recurrence relation can be used to prove that the zeros of the orthonormal polynomials correspond precisely to the eigenvalues of a truncated Jacobi matrix. A similar approach can be used here. Define coefficients by

$$(2.4) \quad xV_{n,\nu}(x) = \sum_{k=0}^{n+1} c_{n,k} V_{k,\nu}(x),$$

then the matrix $H_n = (c_{i,j})_{0 \leq i, j \leq n-1}$ is Hessenberg matrix, i.e. $c_{i,j} = 0$ for $i+1 < j$. Using (2.4) in (2.1) gives recurrence relations for the matrix elements $c_{i,j}$, which can be solved to give

$$(2.5) \quad c_{n,k} = \begin{cases} \frac{1}{1 - q^{\nu+n}}, & \text{if } k = n+1, \\ \frac{(q^\nu; q)_{k-1} q^{\nu+k-1}}{(q^\nu; q)_{n+1}}, & \text{if } 0 < k \leq n, \\ \frac{-1}{(q^\nu; q)_{n+1}}, & \text{if } k = 0. \end{cases}$$

Note that each row sum of H_n , except the last, equals zero.

Introduce the vector $w_n(x) = (V_{0,\nu}(x), V_{1,\nu}(x), \dots, V_{n-1,\nu}(x))^t$, then we see from (2.1) that $H_n w_n(x) = x w_n(x)$ if $V_{n,\nu}(x) = 0$. So a zero x of $V_{n,\nu}$ implies that H_n has an eigenvector for the eigenvalue x . It is also possible to prove that an eigenvalue x of H_n implies that $V_{n,\nu}(x) = 0$, which can be proved by showing that the characteristic polynomial of H_n times the normalisation constant $(-1)^n (q^\nu; q)_n$ satisfies (2.1). So we conclude that the zeros of $V_{n,\nu}(x)$, and hence the zeros of the Laurent q -Lommel polynomials $h_{n,\nu}(x; q)$, are completely determined by the spectrum of the Hessenberg matrix H_n .

3. MINIMAL SOLUTIONS AND ORTHOGONALITY RELATIONS

In this section we give an explicit formulation for the strong moment functional \mathcal{L} introduced in the previous section. We describe \mathcal{L} in terms of contour integrals where the integrands depend on the Hahn-Exton q -Bessel function and on a function closely related to the Hahn-Exton q -Bessel function. These functions give rise to two other solutions of the recurrence relation (1.7), but now with prescribed behaviour for $m \rightarrow \infty$. The proof of orthogonality of the Laurent q -Lommel polynomials for \mathcal{L} uses a method already introduced by Dickinson [9] to prove the orthogonality relations (1.5) for the Lommel polynomials.

Using a generating function argument the following explicit expressions for the Laurent q -Lommel polynomials have been derived in [17, (4.23)] from the recurrence relation (1.7)

$$(3.1) \quad h_{m,\nu}(x; q) = \sum_{n=0}^m x^{m-2n} \frac{(q^{n+1}; q)_{\infty} (q^{\nu}; q)_{\infty}}{(q; q)_{\infty} (q^{\nu+m-n}; q)_{\infty}} {}_2\varphi_1 \left(\begin{matrix} q^{-n}, q^{\nu+m-n} \\ q^{\nu} \end{matrix}; q, q^{n+1} \right)$$

$$(3.2) \quad = \sum_{n=0}^m x^{m-2n} {}_2\varphi_1 \left(\begin{matrix} q^{n-m}, q^{n+1} \\ q \end{matrix}; q, q^{\nu+m-n} \right).$$

The Hahn-Exton q -Bessel function is defined by

$$(3.3) \quad J_{\nu}(x; q) = \frac{(q^{\nu+1}; q)_{\infty}}{(q; q)_{\infty}} x^{\nu} {}_1\varphi_1 \left(\begin{matrix} 0 \\ q^{\nu+1} \end{matrix}; q, qx^2 \right)$$

and then the following q -analogue of Hurwitz's formula (1.6) holds

$$(3.4) \quad \lim_{m \rightarrow \infty} x^{-m} h_{m,\nu}(x; q) = \frac{(q; q)_{\infty}}{(x^{-2}; q)_{\infty}} x^{\nu-1} J_{\nu-1} \left(\frac{1}{x}; q \right), \quad |x| > 1.$$

Relation (3.4) has been proved formally in Koelink and Swarttouw [17, (4.24)] from (3.1), but it follows from their proof that it is valid only for $|x| > 1$.

In order to state the asymptotic behaviour of the Laurent q -Lommel polynomials inside the circle we introduce the function

$$(3.5) \quad j_{\nu}(x; q) = x^{\nu} (qx^2; q)_{\infty} {}_1\varphi_1(0; qx^2; q, q^{\nu+1}x^2) = x^{\nu} (q^{\nu+1}x^2; q)_{\infty} {}_1\varphi_1(0; q^{\nu+1}x^2; q, qx^2),$$

where we use $(x; q)_{\infty} {}_1\varphi_1(0; x; q, y) = (y; q)_{\infty} {}_1\varphi_1(0; y; q, x)$, cf. [18, (2.3)]. This function is related to the Hahn-Exton q -Bessel function in the following way

$$x^{-\nu} j_{\nu}(x; q) = (q; q)_{\infty} \left(x^{-\mu} J_{\mu}(x; q) \right) \Big|_{\mu=\nu+2 \ln x / \ln q}.$$

Now we can use (3.2) to obtain

$$\begin{aligned} x^m h_{m,\nu}(x; q) &= \sum_{n=0}^m x^{2m-2n} {}_2\varphi_1 \left(\begin{matrix} q^{n-m}, q^{n+1} \\ q \end{matrix}; q, q^{\nu+m-n} \right) \\ &= \sum_{n=0}^m x^{2n} {}_2\varphi_1 \left(\begin{matrix} q^{-n}, q^{m-n+1} \\ q \end{matrix}; q, q^{\nu+n} \right) \end{aligned}$$

and by dominated convergence we obtain

$$\begin{aligned} \lim_{m \rightarrow \infty} x^m h_{m,\nu}(x; q) &= \sum_{n=0}^{\infty} x^{2n} {}_2\varphi_1 \left(\begin{matrix} q^{-n}, 0 \\ q \end{matrix}; q, q^{\nu+n} \right) \\ &= \sum_{l=0}^{\infty} \frac{q^{\nu l}}{(q; q)_l (q; q)_l} \sum_{n=l}^{\infty} (q^{-n}; q)_l q^{nl} x^{2n}, \end{aligned}$$

where the last equality follows from interchanging the summations, which is allowed for $|x| < 1$. The inner sum can be written as

$$\begin{aligned} \sum_{p=0}^{\infty} (q^{-p-l}; q)_l x^{2(p+l)} q^{l(p+l)} &= x^{2l} (-1)^l q^{\frac{1}{2}l(l-1)} \sum_{p=0}^{\infty} (q^{p+l}; q^{-1})_l x^{2p} \\ &= x^{2l} (-1)^l q^{\frac{1}{2}l(l-1)} (q; q)_l \sum_{p=0}^{\infty} \frac{(q^{l+1}; q)_p}{(q; q)_p} x^{2p} = x^{2l} (-1)^l q^{\frac{1}{2}l(l-1)} \frac{(q; q)_l}{(x^2; q)_{l+1}}, \end{aligned}$$

by the q -binomial theorem, cf. [11, (1.3.2)]. This leads to the result

$$(3.6) \quad \lim_{m \rightarrow \infty} x^m h_{m,\nu}(x; q) = \frac{1}{1-x^2} {}_1\varphi_1 \left(\begin{matrix} 0 \\ qx^2 \end{matrix}; q, q^{\nu} x^2 \right) = \frac{x^{1-\nu}}{(x^2; q)_{\infty}} j_{\nu-1}(x; q), \quad |x| < 1.$$

Proposition 3.1. *The functions $J_{\nu+m}(x^{-1}; q)$ and $j_{\nu+m}(x; q)$ satisfy the recurrence relation (1.7). Moreover,*

$$\begin{aligned} J_{\nu+m}(x^{-1}; q) &= h_{m,\nu}(x; q) J_{\nu}(x^{-1}; q) - h_{m-1,\nu+1}(x; q) J_{\nu-1}(x^{-1}; q), \\ j_{\nu+m}(x; q) &= h_{m,\nu}(x; q) j_{\nu}(x; q) - h_{m-1,\nu+1}(x; q) j_{\nu-1}(x; q). \end{aligned}$$

Proof. Since $h_{m,\nu}(x; q)$ and $h_{m-1,\nu+1}(x; q)$ are linearly independent solutions of the recurrence relation (1.7), the last statement of the proposition implies the first. Also, if $J_{\nu+m}(x^{-1}; q)$ and $j_{\nu+m}(x; q)$ satisfy (1.7), then they must be a linear combination of $h_{m,\nu}(x; q)$ and $h_{m-1,\nu+1}(x; q)$ from which the second statement follows by considering the cases $m = 0$ and $m = -1$.

The last statement for $J_{\nu+m}$ has already been proved in [17, (4.12)], so it remains to consider $j_{\nu+m}$. The second order q -difference equation for the ${}_1\varphi_1$ -series, or by taking a suitable limit in [17, (4.14)] in combination with (3.6), reveals that

$$j_{\nu+1}(x; q) = \left(\frac{1}{x} + x(1-q^{\nu}) \right) j_{\nu}(x; q) - j_{\nu-1}(x; q).$$

Replacing ν by $\nu + m$ proves the statement. \square

Remark. (i) The solutions $J_{\nu+m}(x^{-1}; q)$ and $j_{\nu+m}(x; q)$ of (1.7) have the following asymptotic behaviour for $m \rightarrow \infty$ valid for $x \in \mathbb{C}$;

$$(3.7) \quad \begin{aligned} \lim_{m \rightarrow \infty} x^{m+\nu} J_{\nu+m}(x^{-1}; q) &= \frac{(qx^{-2}; q)_{\infty}}{(q; q)_{\infty}}, \\ \lim_{m \rightarrow \infty} x^{-m-\nu} j_{\nu+m}(x; q) &= (qx^2; q)_{\infty}. \end{aligned}$$

Note that $x^{\pm m}$ are solutions of (1.7) for $m \rightarrow \infty$ (or for $q = 0$). So the solutions $J_{\nu+m}(x^{-1}; q)$ and $j_{\nu+m}(x; q)$ behave as $x^{\mp m}$ up to a factor independent of m as $m \rightarrow \infty$. (ii) The functions $J_{\nu+m}(x^{-1}; q)$ and $j_{\nu+m}(x; q)$ are related to a minimal solution $X_m(x)$ of (2.1), i.e. $X_m(x)$ is a solution such that $\lim_{m \rightarrow \infty} X_m(x)/V_{m,\nu}(x) = 0$, where $V_{m,\nu}(x)$ is the polynomial solution of (2.1). Using the limit transitions (3.4) and (3.6) and the relations in proposition 3.1 we obtain

$$X_m(x) = \begin{cases} j_{\nu}(\sqrt{x}; q)V_{m,\nu}(x) - x^{\frac{1}{2}}j_{\nu-1}(\sqrt{x}; q)V_{m-1,\nu+1}(x) \\ \qquad \qquad \qquad = x^{\frac{1}{2}m}j_{\nu+m}(\sqrt{x}; q), & |x| < 1, \\ J_{\nu}(1/\sqrt{x}; q)V_{m,\nu}(x) - x^{\frac{1}{2}}J_{\nu-1}(1/\sqrt{x}; q)V_{m-1,\nu+1}(x) \\ \qquad \qquad \qquad = x^{\frac{1}{2}m}J_{\nu+m}(1/\sqrt{x}; q), & |x| > 1. \end{cases}$$

With the functions $J_{\nu}(x; q)$ and $j_{\nu}(x; q)$ and their relation with the Laurent q -Lommel polynomials described in proposition 3.1 at hand, we can give an explicit expression for the strong moment functional \mathcal{L} . The proof we give is an adaption to the Laurent case of Dickinson's proof of the orthogonality (1.5) for the Lommel polynomials [9].

First we investigate the quotient of two Hahn-Exton q -Bessel functions.

Lemma 3.2. *For $\nu > 0$ the following expansion holds around 0 for $n \in \mathbb{Z}_+$*

$$\frac{J_{\nu+n}(x; q)}{J_{\nu-1}(x; q)} = \frac{x^{n+1}}{(q^{\nu}; q)_{n+1}} \sum_{k=0}^{\infty} c_k x^{2k}$$

with the coefficients c_k recursively defined by $c_0 = 1$ and

$$(3.8) \quad c_k = \frac{(-1)^k q^{\frac{1}{2}k(k+1)}}{(q^{\nu+n+1}; q)_k (q; q)_k} - \sum_{p=0}^{k-1} c_p \frac{(-1)^{k-p} q^{\frac{1}{2}(k-p)(k-p+1)}}{(q^{\nu}; q)_k (q; q)_{k-p}}.$$

Proof. From (3.3) we immediately get

$$\frac{J_{\nu+n}(x; q)}{J_{\nu-1}(x; q)} = \frac{x^{n+1}}{(q^{\nu}; q)_{n+1}} \frac{{}_1\varphi_1(0; q^{\nu+n+1}; q, qx^2)}{{}_1\varphi_1(0; q^{\nu}; q, qx^2)}$$

so we have to solve for the coefficients c_k by comparing powers of x on both sides of

$$\sum_{k=0}^{\infty} c_k x^{2k} \sum_{p=0}^{\infty} \frac{(-1)^p q^{\frac{1}{2}p(p+1)} x^{2p}}{(q^{\nu}; q)_p (q; q)_p} = \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}m(m+1)} x^{2m}}{(q^{\nu+n+1}; q)_m (q; q)_m}$$

from which the recurrence relation (3.8) for the coefficients c_k is obtained.

A rude estimate gives

$$\left| \frac{(-1)^k q^{\frac{1}{2}k(k+1)}}{(q^{\nu+n+1}; q)_r (q; q)_r} \right| \leq A = \frac{1}{(q^{\nu}; q)_{\infty} (q; q)_{\infty}}$$

for $\nu > 0$. The same estimate applies to the factor in front of c_n on the right hand side of (3.8), so that we obtain

$$|c_k| \leq A + \sum_{n=0}^{k-1} A|c_n|.$$

A discrete version of Gronwall's inequality, cf. e.g. [21, p. 440]

$$(3.9) \quad a_k \leq A + \sum_{n=0}^{k-1} d_n a_n, \text{ and } A, a_n, d_n \geq 0 \implies a_n \leq A \exp\left(\sum_{n=0}^{k-1} d_n\right),$$

yields $|c_k| \leq Ae^{kA}$ so that the series on the right hand side of the statement of the lemma is absolutely convergent for $|x| < e^{-A/2}$. \square

Choose $0 < R < j_1^{\nu-1}$, where $j_1^{\nu-1}$ denotes the smallest positive zero of $J_{\nu-1}(x; q)$, $\nu > 0$, cf. [17, §3]. Using lemma 3.2 we obtain for $\nu > 0$, $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$

$$(3.10) \quad \frac{1}{2\pi i} \oint_{|z|=\frac{1}{R}} z^m \frac{J_{\nu+n}(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz = \begin{cases} 0, & m - n \text{ odd or } m < n, \\ (q^\nu; q)_{n+1}^{-1}, & m = n. \end{cases}$$

Note that the coefficients c_k of lemma 3.2 for $n = 0$ are in fact the moments of the linear functional \mathcal{L}_+ defined by

$$(3.11) \quad \mathcal{L}_+(x^m) = \frac{1}{2\pi i} \oint_{|z|=\frac{1}{R}} z^m \frac{J_\nu(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz = \begin{cases} 0, & m \in \mathbb{Z}_+ \text{ odd}, \\ c_{m/2}, & m \in \mathbb{Z}_+ \text{ even}. \end{cases}$$

We return to this moment functional in section 4 and we calculate the corresponding orthogonal polynomials, which turn out to be q -analogues of the Lommel polynomials.

The following lemma is the analogue of lemma 3.2 for the functions $j_\nu(x; q)$ instead of the Hahn-Exton q -Bessel function.

Lemma 3.3. *For $\nu \in \mathbb{R}$ the following expansion holds around 0 for $n \in \mathbb{Z}_+$*

$$\frac{j_{\nu+n}(x; q)}{j_{\nu-1}(x; q)} = x^{n+1} \sum_{k=0}^{\infty} d_k x^{2k}$$

with the coefficients d_k recursively defined by $d_0 = 1$ and

$$(3.12) \quad d_k = {}_2\varphi_1(q^{-k}, 0; q; q, q^{\nu+n+1+k}) - \sum_{p=0}^{k-1} d_p {}_2\varphi_1(q^{p-k}, 0; q; q, q^{\nu+k-p})$$

Proof. The proof is completely analogous to the proof of lemma 3.2 and we only give the differences. Here use the expansion

$$\begin{aligned} \frac{1}{(1-x^2)} {}_1\varphi_1(0; qx^2; q, q^{\nu+1}x^2) &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)}}{(q; q)_k (x^2; q)_{k+1}} q^{(\nu+1)k} x^{2k} \\ &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k(k-1)}}{(q; q)_k} q^{(\nu+1)k} x^{2k} \frac{(q^{k+1}; q)_l}{(q; q)_l} x^{2l} \\ &= \sum_{p=0}^{\infty} x^{2p} {}_2\varphi_1(q^{-p}, 0; q; q, q^{\nu+1+p}) \end{aligned}$$

by the q -binomial theorem valid for $|x| < 1$ and rearranging the absolutely convergent sum using $l = p - k$. From this we obtain the recurrence relation (3.12). The general estimate

$$|{}_2\varphi_1(q^{-p}, 0; q; q, xq^p)| \leq \frac{(-q, -|x|; q)_\infty}{(q; q)_\infty}$$

and Gronwall's inequality (3.9) prove that sum is absolutely convergent around 0. \square

Choose $r > 0$ so that $j_{\nu-1}(x; q)$ has no non-zero zeros in the ball with radius r and the origin as centre, which is possible since ${}_1\varphi_1(0; qx^2; q, q^\nu x^2)$ equals 1 at $x = 0$ and defines an analytic function for $|x| < q^{-1/2}$. Using lemma 3.3 we obtain for $m \in \mathbb{Z}$ and $n \in \mathbb{Z}_+$

$$(3.13) \quad \frac{1}{2\pi i} \oint_{|z|=r} z^m \frac{j_{\nu+n}(z; q)}{j_{\nu-1}(z; q)} dz = \begin{cases} 0, & m - n \text{ odd or } m > -n - 2, \\ 1, & m = -n - 2. \end{cases}$$

The coefficients d_k of lemma 3.3 for $n = 0$ can be interpreted as the moments of the moment functional \mathcal{L}_- defined by

$$(3.14) \quad \mathcal{L}_-(x^m) = \frac{1}{2\pi i} \oint_{|z|=\frac{1}{r}} z^m \frac{j_\nu(z^{-1}; q)}{j_{\nu-1}(z^{-1}; q)} dz = \begin{cases} 0, & m \in \mathbb{Z}_+ \text{ odd}, \\ d_{m/2}, & m \in \mathbb{Z}_+ \text{ even}. \end{cases}$$

In §5 we consider the orthogonal polynomials for \mathcal{L}_- from which some properties for $j_\nu(x; q)$ can be derived.

Define the following strong moment functional \mathcal{L} for $\nu > 0$ on the space of Laurent polynomials by

$$(3.15) \quad \mathcal{L}(p) = \frac{1}{2\pi i} \oint_{|z|=\frac{1}{R}} p(z) \frac{J_\nu(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz - \frac{1}{2\pi i} \oint_{|z|=r} p(z) \frac{j_\nu(z; q)}{j_{\nu-1}(z; q)} dz$$

for any Laurent polynomial $p(z) = \sum_{p=n}^m c_p z^p$, $n \leq m$, $n, m \in \mathbb{Z}$. Note that \mathcal{L} is independent of the choice of R , respectively r , as long as $J_{\nu-1}(x; q)$, respectively $j_{\nu-1}(x; q)$, has no non-zero zeros in the ball with radius R , respectively r . All moments of \mathcal{L} , both positive and negative, are well-defined due to lemmas 3.2 and 3.3.

The moments of strong moment functional \mathcal{L} and the moments of the moment functionals \mathcal{L}_\pm defined in (3.11) and (3.14) are related by $\mathcal{L}_+(x^n) = \mathcal{L}(x^n)$, $n \in \mathbb{Z}_+$, and by $\mathcal{L}_-(x^n) = -\mathcal{L}(x^{-2-n})$, $n \in \mathbb{Z}_+$.

Theorem 3.4. *Let $\nu > 0$. The Laurent q -Lommel polynomials $h_{n,\nu}(x; q)$ defined by (1.7) are orthogonal Laurent polynomials with respect to the strong moment functional \mathcal{L} , cf. (3.15). Moreover, also the Laurent polynomials $x^{-1}h_{n,\nu}(x; q)$ are orthogonal with respect to \mathcal{L} . Explicitly,*

$$\mathcal{L}(h_{n,\nu}(x; q)h_{m,\nu}(x; q)) = \frac{\delta_{n,m}}{1 - q^{\nu+m}}, \quad \mathcal{L}(x^{-1}h_{n,\nu}(x; q)x^{-1}h_{m,\nu}(x; q)) = -\delta_{n,m}.$$

Remark. (i) This result corresponds nicely with the fact that the Laurent q -Lommel polynomials correspond to a sequence of lacunary orthogonal Laurent polynomials, cf. (2.2), (2.3).

(ii) Since $\mathcal{L}(x^{-2}) = -1$ we see that \mathcal{L} is not a positive definite strong moment functional.

Proof. The asymptotically well-behaved solutions $J_{\nu+n}(x^{-1}; q)$ and $j_{\nu+n}(x; q)$ of the recurrence relation (1.7) are expressible in terms of the Laurent polynomials $h_{n,\nu}(x; q)$ and the associated Laurent polynomials $h_{n-1,\nu+1}(x; q)$, cf. proposition 3.1. From this we obtain for any $m \in \mathbb{Z}$ the expressions

$$(3.16) \quad x^m \frac{J_{\nu+n}(x^{-1}; q)}{J_{\nu-1}(x^{-1}; q)} = x^m \frac{J_{\nu}(x^{-1}; q)}{J_{\nu-1}(x^{-1}; q)} h_{n,\nu}(x; q) - x^m h_{n-1,\nu+1}(x; q),$$

$$(3.17) \quad x^m \frac{j_{\nu+n}(x; q)}{j_{\nu-1}(x; q)} = x^m \frac{j_{\nu}(x; q)}{j_{\nu-1}(x; q)} h_{n,\nu}(x; q) - x^m h_{n-1,\nu+1}(x; q).$$

Since we obviously have

$$\frac{1}{2\pi i} \oint_{|z|=\frac{1}{R}} z^m h_{n-1,\nu+1}(z; q) dz = \frac{1}{2\pi i} \oint_{|z|=r} z^m h_{n-1,\nu+1}(z; q) dz,$$

we get from the combination of (3.16), (3.17), (3.10) and (3.13) the relations

$$\begin{aligned} \mathcal{L}(x^m h_{n,\nu}(x; q)) &= \begin{cases} 0, & -n \leq m < n, \\ (q^\nu; q)_{n+1}^{-1}, & m = n, \end{cases} \\ \mathcal{L}(x^m x^{-1} h_{n,\nu}(x; q)) &= \begin{cases} 0, & -n < m \leq n, \\ -1, & m = -n - 1, \end{cases} \end{aligned}$$

This proves the orthogonality.

It remains to calculate the norm. From (3.1), (3.2) we see that the coefficient of x^n in $h_{n,\nu}(x; q)$ equals $(q^\nu; q)_n$ and that the coefficient of x^{-n-1} in $x^{-1} h_{n,\nu}(x; q)$ equals 1. \square

4. ORTHOGONAL q -LOMMEL POLYNOMIALS ASSOCIATED WITH THE POSITIVE MOMENTS

In this section we consider the orthogonal polynomials for the moment functional \mathcal{L}_+ , cf. (3.11), which corresponds to the positive moments of the strong moment functional \mathcal{L} . These polynomials are q -analogues of the Lommel polynomials $h_{n,\nu}(z)$, cf. (1.4).

We consider the following three-term recurrence relation,

$$(4.1) \quad p_{n+1}(x) = x(1 - q^{\nu+n})p_n(x) - \lambda_n p_{n-1}(x), \quad \lambda_{2n} = q^n, \quad \lambda_{2n+1} = q^{\nu+3n+1}$$

with initial conditions $p_{-1}(x) = 0$, $p_0(x) = 1$. Note that we can write the recurrence coefficient λ_n in closed form as $q^{(\nu+n)(\lfloor(n+1)/2\rfloor - \lfloor n/2\rfloor) + \lfloor n/2\rfloor}$, where $\lfloor a \rfloor$ denotes the greatest integer less than or equal to $a \in \mathbb{R}$. So the recurrence relation (4.1) depends on whether n is odd or even. Favard's theorem implies that these polynomials are orthogonal with respect to a positive definite moment functional for $\nu > 0$. Taking $q \uparrow 1$ in (4.1) after replacing x by $2z/(1-q)$ yields the three-term recurrence relation (1.4) for the Lommel polynomials, so that we have q -analogues of the Lommel polynomials. The recurrence relation (4.1) has been guessed using the explicit form for the positive moments of \mathcal{L} , i.e.

the moments of \mathcal{L}_+ , obtainable from lemma 3.2 and calculating the first few terms of the recurrence relation (4.1) using Mathematica.

The monic orthogonal polynomials satisfy a recurrence relation of the type

$$r_{n+1}(x) = xr_n(x) - \mu_n r_{n-1}(x), \quad r_{-1}(x) = 0, \quad r_0(x) = 1,$$

with $\mu_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \mu_n < \infty$. This type of orthogonal polynomials has been studied by Dickinson, Pollak and Wannier [10], by Goldberg [13], who corrected some of the results of [10], and, from the point of view of continued fractions, by Schwartz [19]. See also Chihara [7, Ch. IV, thm. 3.5]. The support of the corresponding orthogonality measure, which is uniquely determined, is a purely discrete denumerable bounded set with only one accumulation point at zero. This result can also be obtained by remarking that the Jacobi matrix J for the corresponding orthonormal polynomials defines a self-adjoint operator $J: \ell^2(\mathbb{Z}_+) \rightarrow \ell^2(\mathbb{Z}_+)$, which is an operator of trace class. Since the spectral measure of J is the orthogonality measure for the orthogonal polynomials r_n , the result follows from standard facts on the spectral measure of a self-adjoint trace class operator. Moreover, for the orthogonal polynomials in this class we have the asymptotic behaviour of the form $\lim_{n \rightarrow \infty} x^{-n} r_n(x) = f(x)$ for an analytic function f in $\mathbb{C} \setminus \{0\}$, cf. [10], [13], [19].

By $p_n^{(1)}$ we denote the associated orthogonal polynomials, i.e. the polynomials satisfying

$$(4.2) \quad p_n^{(1)}(x) = x(1 - q^{\nu+n})p_{n-1}^{(1)}(x) - \lambda_n p_{n-2}^{(1)}(x), \quad \lambda_{2n} = q^n, \quad \lambda_{2n+1} = q^{\nu+3n+1}$$

with initial conditions $p_{-1}^{(1)}(x) = 0$, $p_0^{(1)}(x) = 1$.

The following proposition is a q -analogue of the identity (1.3) relating the Bessel functions and Lommel polynomials.

Proposition 4.1. *For $n \in \mathbb{Z}_+$ the polynomials defined by (4.1) and (4.2) satisfy*

$$p_n\left(\frac{1}{x}\right)J_\nu(x; q) - p_{n-1}^{(1)}\left(\frac{1}{x}\right)J_{\nu-1}(x; q) = q^{\frac{1}{2}[(n+1)/2](n+\nu)}J_{\nu+n}(xq^{\frac{1}{2}[(n+1)/2]}; q)$$

where $J_\nu(x; q)$ denotes the Hahn-Exton q -Bessel function (3.3).

Proof. The left hand side is a solution of the three-term recurrence relation

$$(4.3) \quad a_{n+1} = \frac{1 - q^{\nu+n}}{x}a_n - \lambda_n a_{n-1}.$$

The right hand side satisfies the same recurrence relation (4.3). To see this we use for even n the relation

$$(4.4) \quad \frac{1 - q^\nu}{x}J_\nu(x; q) - J_{\nu-1}(x; q) = q^{\frac{1}{2}(\nu+1)}J_{\nu+1}(xq^{\frac{1}{2}}; q),$$

and for odd n we use the relation

$$(4.5) \quad \frac{1 - q^\nu}{x}J_\nu(x; q) - q^{\frac{1}{2}(\nu-1)}J_{\nu-1}(xq^{-\frac{1}{2}}; q) = J_{\nu+1}(x; q).$$

These identities can be checked straightforwardly by comparing the coefficients of the powers of x on both sides of (4.4) and (4.5).

Since $p_n(x^{-1})$ and $p_{n-1}^{(1)}(x^{-1})$ are linearly independent solutions of (4.3) we obtain the proposition after checking the equality for $n = 0$, which is trivial, and for $n = 1$, which is (4.4). \square

The polynomials defined by (4.1) turn out to be the orthogonal polynomials with respect to the moment functional \mathcal{L}_+ . For more information concerning the zeros of the Hahn-Exton q -Bessel function, which play a role in the following theorem, the reader is referred to [17, §3].

Theorem 4.2. *We have the following orthogonality relations for $\nu > 0$ for the polynomials defined by (4.1);*

$$\begin{aligned} \sum_{k=1}^{\infty} p_n\left(\frac{\pm 1}{j_k^{\nu-1}}\right) p_m\left(\frac{\pm 1}{j_k^{\nu-1}}\right) \frac{-J_\nu(j_k^{\nu-1}; q)}{(j_k^{\nu-1})^2 J'_{\nu-1}(j_k^{\nu-1}; q)} \\ + p_n(0)p_m(0) = \delta_{n,m} \frac{q^{(n+\nu)\lfloor(n+1)/2\rfloor}}{1 - q^{n+\nu}}. \end{aligned}$$

Here $j_k^{\nu-1}$ are the positive simple zeros of the Hahn-Exton q -Bessel function $J_{\nu-1}(x; q)$ numbered increasingly. All weights are positive.

Proof. We start by establishing a complex orthogonality similarly to the previous section following Dickinson's method [9]. For this we need the expansion

$$(4.6) \quad q^{\frac{1}{2}\lfloor(n+1)/2\rfloor(n+\nu)} \frac{J_{\nu+n}(xq^{\frac{1}{2}\lfloor(n+1)/2\rfloor}; q)}{J_{\nu-1}(x; q)} = q^{\lfloor(n+1)/2\rfloor(n+\nu)} \frac{x^{n+1}}{(q^\nu; q)_{n+1}} \sum_{k=0}^{\infty} c_k x^{2k},$$

which is absolutely convergent for small x . Moreover, $c_0 = 1$. This is proved similarly as in lemmas 3.2 and 3.3.

Let $R > 0$ be smaller than the smallest positive zero $j_1^{\nu-1}$ of $J_{\nu-1}(x; q)$, then we obtain from proposition 4.1 and (4.6) for $0 \leq m \leq n$

$$\begin{aligned} \oint_{|z|=\frac{1}{R}} z^m p_n(z) \frac{J_\nu(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz &= \oint_{|z|=\frac{1}{R}} z^m q^{\frac{1}{2}\lfloor(n+1)/2\rfloor(n+\nu)} \frac{J_{\nu+n}(z^{-1}q^{\frac{1}{2}\lfloor(n+1)/2\rfloor}; q)}{J_{\nu-1}(z^{-1}; q)} dz \\ &= \begin{cases} 0, & 0 \leq m < n, \\ 2\pi i q^{(\nu+n)\lfloor(n+1)/2\rfloor} (q^\nu; q)_{n+1}^{-1}, & m = n, \end{cases} \end{aligned}$$

since $\oint_{|z|=\frac{1}{R}} z^m p_{n-1}^{(1)}(z) dz = 0$. The leading coefficient of p_n is $(q^\nu; q)_n$, as can be read off from (4.1), and so we get the complex orthogonality relations

$$(4.7) \quad \mathcal{L}_+(p_n p_m) = \frac{1}{2\pi i} \oint_{|z|=\frac{1}{R}} p_m(z) p_n(z) \frac{J_\nu(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz = \delta_{n,m} \frac{q^{(n+\nu)\lfloor(n+1)/2\rfloor}}{(1 - q^{\nu+n})}.$$

The considerations given at the beginning of this section show that we can rewrite (4.7) as a sum over the zeros of the Hahn-Exton q -Bessel function $J_{\nu-1}(z; q)$ and possibly zero. The residues at the pole $(j_k^{\nu-1})^{-1}$ of the left hand side of (4.7) equals

$$p_n\left(\frac{1}{j_k^{\nu-1}}\right)p_m\left(\frac{1}{j_k^{\nu-1}}\right)\frac{-J_\nu(j_k^{\nu-1}; q)}{(j_k^{\nu-1})^2 J'_{\nu-1}(j_k^{\nu-1}; q)}$$

To see this we note that $J'_{\nu-1}(j_k^{\nu-1}; q) \neq 0$ since the zeros of $J_{\nu-1}(x; q)$ are simple, cf. [17, lemma 3.3], and that $J_\nu(j_k^{\nu-1}; q) \neq 0$ by the interlacing property of the zeros of the Hahn-Exton q -Bessel function [17, thm. 3.7]. The positivity of the corresponding mass follows from the fact that $J_\nu(j_k^{\nu-1}; q)$ and $J'_{\nu-1}(j_k^{\nu-1}; q)$ have opposite signs, which follows from the Fourier-Bessel orthogonality relations for the Hahn-Exton q -Bessel function [17, prop. 3.6], or from the fact that the zeros of the Hahn-Exton q -Bessel functions $J_\nu(x; q)$ and $J_{\nu+1}(x; q)$ are interlaced as described in [17, thm. 3.7]. The mass at $-(j_k^{\nu-1})^{-1}$ yields the same weight.

The set of mass points $(j_k^{\nu-1})^{-1}$, $k \in \mathbb{N}$, has zero as the only point of accumulation, so that zero may occur as a mass point as well. This happens if $\sum_{k=0}^{\infty} |\tilde{p}_k(0)|^2 < \infty$, where \tilde{p}_n are the corresponding orthonormal polynomials, cf. e.g. [5, thm. 2.8]. Now the orthonormal polynomials \tilde{p}_n are given by

$$(4.8) \quad \tilde{p}_n(x) = \left(\frac{1 - q^{n+\nu}}{1 - q^\nu}\right)^{1/2} q^{-\frac{1}{2}[(n+1)/2](n+\nu)} p_n(x).$$

Moreover, $\mathcal{M}(\tilde{p}_n \tilde{p}_m) = \delta_{n,m}$, where \mathcal{M} is the moment functional given by

$$\mathcal{M}(p) = \frac{1 - q^\nu}{2\pi i} \oint_{|z|=\frac{1}{R}} p(z) \frac{J_\nu(z^{-1}; q)}{J_{\nu-1}(z^{-1}; q)} dz = (1 - q^\nu) \mathcal{L}_+(p).$$

From (4.1) with $x = 0$ we see that $p_{2n+1}(0) = 0$ and that $p_{2n}(0)$ satisfies a simple two-term recurrence relation from which we get $p_{2n}(0) = (-1)^n q^{n(\nu+1) + \frac{3}{2}n(n-1)}$. Combining this with (4.8) shows that $\tilde{p}_{2n+1}(0) = 0$ and

$$\tilde{p}_{2n}(0) = (-1)^n \left(\frac{1 - q^{\nu+2n}}{1 - q^\nu}\right)^{1/2} q^{\frac{1}{2}n\nu + \frac{1}{2}n(n-1)}.$$

Hence,

$$\rho = \sum_{k=0}^{\infty} |\tilde{p}_k(0)|^2 = \frac{1}{1 - q^\nu} \sum_{n=0}^{\infty} (1 - q^{\nu+2n}) q^{n\nu + n(n-1)}$$

and this sum is an absolutely convergent telescoping series so that $\rho = (1 - q^\nu)^{-1}$. Consequently, \mathcal{M} has a mass point at zero with weight ρ^{-1} and \mathcal{L}_+ has a mass point at zero with weight 1. \square

From the explicit orthogonality relations of theorem 4.2 we see that the orthogonality measure for $p_n(x)$ is supported in $[-1/j_1^{\nu-1}, 1/j_1^{\nu-1}]$. On the other hand, from the explicit values of the recurrence coefficients for the orthonormal polynomials \tilde{p}_n , which are easily obtained from (4.1) and (4.8), and the bound on the spectrum from [21, (1.3) with $n \rightarrow \infty$], which is Gershgorin's theorem for the Jacobi matrix, we see that the orthogonality measure is supported in $[-N, N]$ with $N \leq 2/(1 - q^\nu)$. So we obtain the following corollary after shifting ν by 1.

Corollary 4.3. *For $\nu > -1$ the first positive zero j_1^ν of $J_\nu(x; q)$ satisfies $j_1^\nu \geq (1 - q^{\nu+1})/2$.*

5. ORTHOGONAL POLYNOMIALS ASSOCIATED WITH THE NEGATIVE MOMENTS

In this section we consider the orthogonal polynomials for the moment functional \mathcal{L}_- related to the negative moments of the strong moment functional \mathcal{L} introduced in (3.14). In subsection 5.1 we introduce the three-term recurrence relation for the polynomials which we study. The three-term recurrence relation has been obtained by calculating the first few recurrence coefficients using lemma 3.3 with $n = 0$ using Mathematica and then guessing the general result. In subsection 5.1 we give explicit expressions for these orthogonal polynomials and the associated orthogonal polynomials in terms of Al-Salam–Chihara polynomials. From the explicit expressions we can determine the asymptotic behaviour of the (associated) polynomials as the degree tends to infinity in terms of the function $j_\nu(x; q)$. In particular, we obtain the Stieltjes transform of the orthogonality measure. In subsection 5.2 we use the Stieltjes transform to obtain information on the zeros of $j_\nu(x; q)$ in a similar way as in Ismail [15, §4], see also [2, §4], and to give explicit orthogonality relations. In subsection 5.3 we give a different derivation of some of these results in the special case $\nu = 1/2$, which turns out to be related to known orthogonal polynomials [2], [20]. Comparison of these two approaches yields a summation formula for a one-parameter terminating ${}_3\varphi_2$ -series. A special case of this summation formula is the evaluation of the continuous q -Hermite polynomials at a special point outside the spectrum.

5.1. Explicit expressions for orthogonal polynomials. We investigate the monic orthogonal polynomials satisfying the three-term recurrence relation

$$(5.1) \quad P_{n+1}(x) = xP_n(x) - \lambda_n P_{n-1}(x), \quad \lambda_{2n} = q^n, \quad \lambda_{2n+1} = q^{n+\nu},$$

with initial conditions $P_{-1}(x) = 0$, $P_0(x) = 1$. By Favard’s theorem the polynomials P_n are orthogonal with respect to a positive definite moment functional for $\nu \in \mathbb{R}$. Moreover, the polynomials P_n fit into the same class of Dickinson, Pollak and Wannier [10], Goldberg [13] and Schwartz [19] described at the beginning of the previous section.

The polynomials P_n are even functions of x for even n , and odd functions of x for odd n . Introduce

$$P_{2n}(x) = R_n(x^2), \quad P_{2n+1}(x) = xS_n(x^2),$$

so that the monic polynomials R_n and S_n satisfy the three-term recurrence relations (see, e.g., Chihara [7, p. 45])

$$\begin{aligned} R_{n+1}(x) &= (x - \lambda_{2n} - \lambda_{2n+1})R_n(x) - \lambda_{2n-1}\lambda_{2n}R_{n-1}(x), \\ S_{n+1}(x) &= (x - \lambda_{2n+1} - \lambda_{2n+2})S_n(x) - \lambda_{2n}\lambda_{2n+1}S_{n-1}(x), \end{aligned}$$

with initial conditions $R_0(x) = 1$, $R_1(x) = x - q^\nu$ and $S_{-1}(x) = 0$, $S_0(x) = 1$. A simple computation from (5.1) gives the recurrence coefficients for the polynomials R_n

$$\lambda_{2n} + \lambda_{2n+1} = \begin{cases} (1 + q^\nu)q^n & \text{if } n > 0, \\ q^\nu & \text{if } n = 0. \end{cases}, \quad \lambda_{2n}\lambda_{2n-1} = q^{2n-1+\nu}, \quad n \geq 0.$$

For the recurrence coefficients of S_n we find similarly

$$\lambda_{2n+1} + \lambda_{2n+2} = (q + q^\nu)q^n, \quad \lambda_{2n}\lambda_{2n+1} = q^{2n+\nu}, \quad n \geq 0.$$

The recurrence coefficients of R_n and S_n decrease exponentially.

Consider the monic polynomials $u_n(x; a, b; q)$ satisfying the recurrence relation

$$(5.2) \quad u_{n+1}(x; a, b; q) = (x - aq^n)u_n(x; a, b; q) - b^2q^{2n-2}u_{n-1}(x; a, b; q),$$

$u_{-1}(x) = 0$, $u_0(x) = 1$, which are studied in [20], then $S_n(x) = u_n(x; q + q^\nu, q^{(\nu+2)/2}; q)$.

For R_n we have to be a little bit more careful, since for $n = 0$ one of the recurrence coefficients behaves differently. However, R_n is still a solution of the recurrence relation (5.2) with $a = 1 + q^\nu$ and $b^2 = q^{\nu+1}$, but it satisfies the different initial condition $R_1(x) = x - q^\nu = u_1(x) + 1$. Such polynomials are known as co-recursive polynomials [6] and can be expressed as

$$R_n(x) = u_n(x; 1 + q^\nu, q^{(\nu+1)/2}; q) + u_{n-1}^{(1)}(x; 1 + q^\nu, q^{(\nu+1)/2}; q).$$

The associated polynomials corresponding to the recurrence relation (5.2) are given by $u_n^{(1)}(x; a, b; q) = u_n(x; aq, bq; q) = q^n u_n(x/q; a, b; q)$, so that

$$R_n(x) = u_n(x; 1 + q^\nu, q^{(\nu+1)/2}; q) + q^{n-1}u_{n-1}(x/q; 1 + q^\nu, q^{(\nu+1)/2}; q).$$

An explicit expression of the polynomials $u_n(x; a, b; q)$ in terms of Al-Salam–Chihara polynomials is given by Van Assche [20, thm. 2];

$$u_n(x; a, b; q) = \sum_{k=0}^n \frac{x^{n-k} q^{\frac{1}{2}k(k-1)}}{(q; q)_k} P_k(-a; q; -aq^{n-k+1}, b^2q^{2(n-k)+1}, b^2/q).$$

Here $P_n(x; q; a, b, c)$ are Al-Salam–Chihara polynomials, cf. [1, (6.1)], which satisfy the recurrence relation

$$(5.3) \quad P_{n+1}(x; q; a, b, c) = (x - aq^n)P_n(x; q; a, b, c) - (c - bq^{n-1})(1 - q^n)P_{n-1}(x; q; a, b, c).$$

More information, including the orthogonality relations, concerning the Al-Salam–Chihara polynomials can be found in Askey and Ismail [5, §3].

So we obtain the explicit expressions

$$(5.4) \quad S_n(x) = \sum_{k=0}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} P_k(-(q + q^\nu); q; -(1 + q^{\nu-1})q^{n-k+2}, q^{2(n-k)+\nu+3}, q^{\nu+1})$$

and

$$\begin{aligned} R_n(x) &= \sum_{k=0}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} P_k(-(1 + q^\nu); q; -(1 + q^\nu)q^{n-k+1}, q^{2(n-k)+\nu+2}, q^\nu) \\ &\quad + \sum_{k=0}^{n-1} \frac{x^{n-1-k} q^{k(k+1)/2}}{(q; q)_k} P_k(-(1 + q^\nu); q; -(1 + q^\nu)q^{n-k}, q^{2(n-1-k)+\nu+2}, q^\nu) \\ &= x^n + \sum_{k=1}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} \left[P_k(-(1 + q^\nu); q; -(1 + q^\nu)q^{n-k+1}, q^{2(n-k)+\nu+2}, q^\nu) \right. \\ &\quad \left. + (1 - q^k) P_{k-1}(-(1 + q^\nu); q; -(1 + q^\nu)q^{n-k+1}, q^{2(n-k)+\nu+2}, q^\nu) \right]. \end{aligned}$$

A generating function for the Al-Salam–Chihara polynomials is, cf. [1, p. 23],

$$\Phi(x, z) = \sum_{n=0}^{\infty} P_n(x; q; a, b, c) \frac{z^n}{(q; q)_n} = \frac{(\alpha z; q)_{\infty} (\beta z; q)_{\infty}}{(\gamma z; q)_{\infty} (\delta z; q)_{\infty}},$$

where $1 - az + bz^2 = (1 - \alpha z)(1 - \beta z)$ and $1 - xz + cz^2 = (1 - \gamma z)(1 - \delta z)$. Take $x = -(1 + q^{\nu})$ and $c = q^{\nu}$ so that $\gamma = -1$ and $\delta = -q^{\nu}$. Consequently $(1 + z)\Phi(z, -(1 + q^{\nu}))$ is the generating function for $x = -(q + q^{\nu})$ and $c = q^{\nu+1}$. Hence,

$$P_n(-(1 + q^{\nu}); q; a, b, q^{\nu}) + (1 - q^n)P_{n-1}(-(1 + q^{\nu}); q; a, b, q^{\nu}) = P_n(-(q + q^{\nu}); q; a, b, q^{\nu+1}),$$

and thus

$$(5.5) \quad R_n(x) = \sum_{k=0}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} P_k(-(q + q^{\nu}); q; -(1 + q^{\nu})q^{n-k+1}, q^{2(n-k)+\nu+2}, q^{\nu+1}).$$

Now that we have the explicit expression for the polynomials P_n defined in (5.1) at hand, we can determine the asymptotic behaviour, which is related to the function $j_{\nu}(x; q)$ introduced in (3.5).

Proposition 5.1. *For the orthogonal polynomials $P_n(x)$ defined by (5.1) we have for every $x \in \mathbb{C}$*

$$\lim_{n \rightarrow \infty} x^n P_n(1/x) = x^{1-\nu} j_{\nu-1}(x; q).$$

Proof. We follow the proof of theorem 2 of [20]. For this we need the continuous q -Hermite polynomials $H_n(x | q)$ introduced by Rogers in 1894. The three-term recurrence relation is

$$(5.6) \quad H_{n+1}(x | q) = 2xH_n(x | q) - (1 - q^n)H_{n-1}(x | q)$$

with initial conditions $H_{-1}(x | q) = 0$, $H_0(x | q) = 1$, cf. Askey and Ismail [4, §6]. From (5.3) and (5.6) we obtain, cf. [20, thm. 2],

$$(5.7) \quad \lim_{n \rightarrow \infty} P_k(-a; q; -aq^{n-k+1}, b^2 q^{2(n-k)+1}, b^2/q) = (-1)^k b^k q^{-\frac{k}{2}} H_k\left(\frac{aq^{1/2}}{2b} | q\right).$$

Using this limit relation and dominated convergence we obtain

$$\begin{aligned} \lim_{n \rightarrow \infty} x^n R_n(1/x) &= \lim_{n \rightarrow \infty} x^n S_n(1/x) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k} x^k q^{\frac{k}{2}(\nu+1)} H_k\left(\frac{1}{2}(q^{\frac{1}{2}(1-\nu)} + q^{\frac{1}{2}(\nu-1)}) | q\right), \end{aligned}$$

and hence

$$(5.8) \quad \lim_{n \rightarrow \infty} x^n P_n(1/x) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k} x^{2k} q^{\frac{k}{2}(\nu+1)} H_k\left(\frac{1}{2}(q^{\frac{1}{2}(1-\nu)} + q^{\frac{1}{2}(\nu-1)}) | q\right).$$

To see that the right hand side of (5.8) equals $x^{1-\nu}j_{\nu-1}(x; q)$ we insert the explicit expression, cf. [4, (6.1), (3.1)],

$$H_k\left(\frac{1}{2}(x+x^{-1}) \mid q\right) = \sum_{l=0}^k \frac{(q; q)_k}{(q; q)_l (q; q)_{k-l}} x^{k-2l}$$

for $x = q^{(\nu-1)/2}$ in (5.8). Interchanging summations and introducing $m = k - l$ shows that (5.8) equals

$$\begin{aligned} & \sum_{l=0}^{\infty} \frac{(-1)^l q^{\frac{1}{2}l(l+1)} x^{2l}}{(q; q)_l} \sum_{m=0}^{\infty} \frac{(-1)^m q^{\frac{1}{2}m(m-1)}}{(q; q)_m} x^{2m} q^{m(l+\nu)} = \\ & \sum_{l=0}^{\infty} \frac{(-1)^l q^{\frac{1}{2}l(l+1)} x^{2l}}{(q; q)_l} (x^2 q^{l+\nu}; q)_{\infty} = (x^2 q^{\nu}; q)_{\infty} {}_1\varphi_1(0; x^2 q^{\nu}; q, qx^2) = x^{1-\nu} j_{\nu-1}(x; q) \end{aligned}$$

by use of [11, (1.3.16)]. \square

Observe that the continuous q -Hermite polynomials are orthogonal on the interval $[-1, 1]$, so that the inequality $2 \leq q^{\frac{1}{2}(1-\nu)} + q^{\frac{1}{2}(\nu-1)}$ shows that the variable of the continuous q -Hermite polynomial in (5.8) lies outside the support of the orthogonality measure for the continuous q -Hermite polynomials, except when $\nu = 1$ in which case it is an endpoint of the interval.

The Stieltjes transform of the orthogonality measure μ for the orthogonal polynomials P_n can be obtained from

$$(5.9) \quad \int_{\mathbb{R}} \frac{d\mu(t)}{z-t} = \lim_{n \rightarrow \infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)}$$

where $P_n^{(1)}$ are the associated polynomials, cf. Askey and Ismail [5, thm. 2.4] and further references therein.

So let us now consider the associated monic polynomials $P_n^{(1)}$ satisfying

$$(5.10) \quad P_{n+1}^{(1)}(x) = xP_n^{(1)}(x) - \gamma_n P_{n-1}^{(1)}(x), \quad P_{-1}^{(1)}(x) = 0, \quad P_0^{(1)}(x) = 1,$$

where $\gamma_n = \lambda_{n+1}$ is defined in (5.1). These polynomials can be determined in a similar fashion as before. Because of the parity of these polynomials, we again set

$$P_{2n}^{(1)}(x) = T_n(x^2), \quad P_{2n+1}^{(1)}(x) = xU_n(x^2),$$

and the monic polynomials T_n and U_n then satisfy the recurrence relations

$$\begin{aligned} T_{n+1}(x) &= (x - \gamma_{2n} - \gamma_{2n+1})T_n(x) - \gamma_{2n-1}\gamma_{2n}T_{n-1}(x), \\ U_{n+1}(x) &= (x - \gamma_{2n+1} - \gamma_{2n+2})U_n(x) - \gamma_{2n}\gamma_{2n+1}U_{n-1}(x), \end{aligned}$$

$T_0(x) = 1$, $T_1(x) = x - q$ and $U_{-1}(x) = 0$, $U_0(x) = 1$, where

$$\gamma_{2n} + \gamma_{2n+1} = \begin{cases} (q + q^\nu)q^n & \text{if } n > 0, \\ q & \text{if } n = 0. \end{cases}, \quad \gamma_{2n}\gamma_{2n-1} = q^{2n+\nu}, \quad n \geq 0,$$

and

$$\gamma_{2n+1} + \gamma_{2n+2} = (1 + q^\nu)q^{n+1}, \quad \gamma_{2n}\gamma_{2n+1} = q^{2n+\nu+1}, \quad n \geq 0.$$

Hence

$$\begin{aligned} U_n(x) &= u_n(x; q(1 + q^\nu), q^{(\nu+3)/2}; q) \\ &= \sum_{k=0}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} P_k(-q(1 + q^\nu); q; -(1 + q^\nu)q^{n-k+2}, q^{2(n-k)+\nu+4}, q^{\nu+2}). \end{aligned}$$

The polynomials T_n are again co-recursive polynomials for the recurrence relation (5.2) with $a = q + q^\nu$ and $b = q^{(\nu+2)/2}$, with $T_1(x) = u_1(x) + q^\nu$, and thus

$$T_n(x) = u_n(x; q + q^\nu, q^{(\nu+2)/2}; q) + q^{\nu+n-1} u_{n-1}(x/q; q + q^\nu, q^{(\nu+2)/2}; q).$$

From the generating function of the Al-Salam–Chihara polynomials we find

$$\begin{aligned} P_n(-q(1 + q^\nu); q; a, b, q^{\nu+2}) &= \\ &P_n(-(q + q^\nu); q; a, b, q^{\nu+1}) + q^\nu(1 - q^n)P_{n-1}(-(q + q^\nu); q; a, b, q^{\nu+1}), \end{aligned}$$

so that

$$T_n(x) = \sum_{k=0}^n \frac{x^{n-k} q^{k(k-1)/2}}{(q; q)_k} P_k(-q(1 + q^\nu); q; -(q + q^\nu)q^{n-k+1}, q^{2(n-k)+\nu+3}, q^{\nu+2}).$$

The proof of the following proposition is analogous to the proof of proposition 5.1.

Proposition 5.2. *For every $x \in \mathbb{C}$ we have*

$$\lim_{n \rightarrow \infty} x^n P_n^{(1)}(1/x) = x^{-\nu} j_\nu(x; q).$$

5.2. Zeros of $j_\nu(x; q)$ and orthogonality relations. Combining propositions 5.1 and 5.2 and (5.9) shows that the Stieltjes transform of the orthogonality measure μ for the polynomials is

$$(5.11) \quad \int_{\mathbb{R}} \frac{d\mu(t)}{z - t} = \frac{j_\nu(1/z; q)}{j_{\nu-1}(1/z; q)}$$

for all $z \notin \text{supp}(d\mu)$, cf. [5, thm. 2.4]. From the Stieltjes transform we can derive the orthogonality relations for the orthogonal polynomials P_n defined in (5.1). We start with an investigation of the zeros of $j_\nu(x; q)$. It turns out that the zeros of the function $j_\nu(x; q)$ behave similarly to the zeros of the (Hahn-Exton q -)Bessel function for $\nu > -1$. The method of proof largely follows Ismail's investigation [15] of the roots of the Jackson q -Bessel function, see also [2, §4].

Theorem 5.3. *Let $\nu \in \mathbb{R}$ and let the function $j_\nu(x; q)$ be defined by (3.5).*

- (i) *The functions $j_\nu(x; q)$ and $j_{\nu+1}(x; q)$ have no common zeros, except possibly $x = 0$.*
- (ii) *The zeros of $x^{-\nu}j_\nu(x; q)$ are real, simple and symmetric with respect to $x = 0$. There are infinitely many of them and their only point of accumulation is ∞ .*
- (iii) *The zeros of $x^{-\nu}j_\nu(x; q)$ and $x^{-\nu-1}j_{\nu+1}(x; q)$ interlace. Moreover, the smallest positive zero of $x^{-\nu}j_\nu(x; q)$ is smaller than the smallest positive zero of $x^{-\nu-1}j_{\nu+1}(x; q)$.*

Proof. First we prove (i) by use of an equality for ${}_1\varphi_1$ -series. The relation

$$(5.12) \quad {}_1\varphi_1(0; c; q, z) - {}_1\varphi_1(0; c; q, qz) = \frac{-z}{1-c} {}_1\varphi_1(0; cq; q, qz)$$

can be proved directly or it can be obtained from one of Heine's contiguous relations for the ${}_2\varphi_1$ -series, cf. [11, ex. 1.9(iv)]. Take $c = qx^2$ and $z = q^{\nu+1}x^2$ in (5.12) to get from (3.5)

$$(5.13) \quad j_\nu(x; q) - x^{-1}j_{\nu+1}(x; q) = -q^{1+\nu/2}x^2j_\nu(x\sqrt{q}; q).$$

Substituting $c = q^{\nu+2}x^2$, $z = qx^2$ in (5.12) and using (3.5) gives

$$(5.14) \quad j_{\nu+1}(x; q) - q^{-\frac{1}{2}\nu}xj_\nu(x\sqrt{q}; q) = -q^{\frac{1}{2}(1-\nu)}x^2j_{\nu+1}(x\sqrt{q}; q).$$

If $0 \neq a$ is a zero of $j_\nu(x; q)$ and $j_{\nu+1}(x; q)$, then (5.13) implies that $a\sqrt{q}$ is a zero of $j_\nu(x; q)$. Next (5.14) implies that $a\sqrt{q}$ is a zero of $j_{\nu+1}(x; q)$ as well. So $aq^{k/2}$, $k \in \mathbb{Z}_+$, are zeros of the analytic function $x^{-\nu}j_\nu(x; q)$, which implies that this function is zero. This contradiction proves (i).

To prove (ii) we recall that the orthogonality measure $d\mu$ is supported on a bounded denumerable discrete set with zero as the only point of accumulation. So let $d\mu$ have mass A_k at the points $\{t_k\}_{k=1}^\infty$, then (5.11) is

$$(5.15) \quad \sum_{k=1}^{\infty} \frac{A_k}{z - t_k} = \frac{j_\nu(1/z; q)}{j_{\nu-1}(1/z; q)}, \quad z \neq t_k.$$

The zeros of $x^{1-\nu}j_{\nu-1}(1/x; q)$ correspond precisely to the non-zero poles t_k of the left hand side. So the zeros are real and simple. Since $\{t_k\}_{k=1}^\infty$ has zero as the only point of accumulation, the only point of accumulation of the zeros of $j_{\nu-1}(x; q)$ is infinity.

To prove (iii) we consider the (positive) mass of $d\mu$ at a non-zero t_k ,

$$0 < A_k = -t_k^2 \frac{j_\nu(1/t_k; q)}{j'_{\nu-1}(1/t_k; q)}.$$

So $j_\nu(a; q)$ and $j'_{\nu-1}(a; q)$ have opposite signs for $0 \neq a$ a zero of $j_{\nu-1}(x; q)$. If $0 < a < b$ are two consecutive zeros of $j_{\nu-1}(x; q)$, then $j'_{\nu-1}(a; q)j'_{\nu-1}(b; q) < 0$. Hence also $j_\nu(a; q)j_\nu(b; q) < 0$ and $j_\nu(x; q)$ has at least one zero in (a, b) . In the interval $(1/b, 1/a)$ both sides of (5.15) are differentiable, and the derivative of the left hand side is strictly negative. If $j_\nu(1/z; q)$ has more than one zero in $(1/b, 1/a)$, then the derivative has a zero

in that interval. Thus $j_\nu(x; q)$ has precisely one zero in (a, b) . This proves the interlacing property.

Denote by x_k^ν the positive zeros of $j_\nu(x; q)$ numbered increasingly;

$$0 < x_1^\nu < x_2^\nu < \dots < x_j^\nu < x_{j+1}^\nu < \dots$$

Then it remains to prove that $x_1^{\nu-1} < x_1^\nu$. Since $x^\nu j_\nu(x; q)$ equals 1 for $x = 0$ we get that $j'_{\nu-1}(x_1^{\nu-1}; q) < 0$ and thus $j_\nu(x_1^{\nu-1}; q) > 0$. So $j_\nu(x; q)$ has an even number of zeros in $(0, x_1^{\nu-1})$, and the same argument as in the previous paragraph shows that this number is zero. \square

The following proposition is the analogue of proposition 4.1 for the orthogonal polynomials P_n and the functions $j_\nu(x; q)$.

Proposition 5.4. *For $n \in \mathbb{Z}_+$ the polynomials P_n and $P_n^{(1)}$ defined by (5.1) and (5.10) satisfy*

$$P_n\left(\frac{1}{x}\right)j_\nu(x; q) - P_{n-1}^{(1)}\left(\frac{1}{x}\right)j_{\nu-1}(x; q) = \begin{cases} q^{m(m+\frac{1}{2}\nu)}x^{2m}j_\nu(xq^{\frac{1}{2}m}; q), & n = 2m, \\ q^{m(m+\frac{1}{2}(\nu-1))}x^{2m}j_{\nu-1}(xq^{\frac{1}{2}m}; q), & n = 2m - 1, \end{cases}$$

where $j_\nu(x; q)$ is defined in (3.5).

Proof. It suffices to show that the right hand side satisfies (5.1) with x replaced by x^{-1} , since the left hand side satisfies this equation and the cases $n = 0$ (trivial) and $n = 1$ (from (5.13)) are easily proved. For $n = 2m$ this follows from (5.13) with x, ν replaced by $xq^{m/2}, \nu - 1$, and for $n = 2m - 1$ this follows from (5.14) with x, ν replaced by $xq^{(m-1)/2}, \nu - 1$. \square

In the proof of theorem 5.3 we obtained information on the orthogonality measure for the polynomials P_n defined in (5.1). In the next theorem we describe the full orthogonality relations. This theorem can also be proved from proposition 5.4 analogously to the proof of theorem 4.2 from proposition 4.1.

Theorem 5.5. *Let $\nu \in \mathbb{R}$, denote by $x_k^{\nu-1}$, $k \in \mathbb{N}$, the positive zeros of the function $j_{\nu-1}(x; q)$ defined in (3.5). Then for the polynomials P_n defined by (5.1) we have the orthogonality relations*

$$\sum_{k=1}^{\infty} P_n\left(\frac{\pm 1}{x_k^{\nu-1}}\right)P_m\left(\frac{\pm 1}{x_k^{\nu-1}}\right)\frac{-j_\nu(x_k^{\nu-1}; q)}{(x_k^{\nu-1})^2 j'_{\nu-1}(x_k^{\nu-1}; q)} + (1 - q^{\nu-1})P_n(0)P_m(0) = \delta_{n,m} \begin{cases} q^{\frac{1}{2}l(l+\nu)}, & n = 2l, \\ q^{\frac{1}{2}(l+1)(l+\nu)}, & n = 2l + 1, \end{cases}$$

where the mass at $x = 0$ only occurs for $\nu > 1$. All weights are positive.

Proof. The only statements to be proved concern the norm and the weight at $x = 0$. Denote the squared norm of P_n by $\|P_n\|^2$, then (5.1) implies, cf. [10, (7)],

$$\|P_n\|^2 = \lambda_n \|P_{n-1}\|^2 \implies \|P_n\|^2 = \lambda_n \dots \lambda_1 \|1\|^2.$$

Together with the explicit value for λ_n in (5.1) the statement on the norm follows if we prove $\|1\|^2 = 1$. The value of $\|1\|^2$ can be read off from the Stieltjes transform (5.11) as the coefficient of z^{-1} on the right hand side and lemma 3.3 for $n = 0$ and $x = z^{-1}$ shows that it equals 1.

The weight at $x = 0$ equals ρ , where $\rho^{-1} = \sum_{n=0}^{\infty} \tilde{P}_n(0)^2$ and \tilde{P}_n denote the orthonormal polynomials, cf. [5, thm. 2.8]. From (5.1) we compute $P_{2n+1}(0) = 0$, $P_{2n}(0) = (-1)^n q^{n\nu+n(n-1)/2}$ so that for the orthonormal polynomials we have

$$\tilde{P}_{2n}(0) = \frac{P_{2n}(0)}{\sqrt{\lambda_1 \lambda_2 \dots \lambda_{2n}}} = (-1)^n q^{n(\nu-1)/2},$$

and thus

$$\sum_{n=0}^{\infty} \tilde{P}_n^2(0) = \sum_{n=0}^{\infty} q^{n(\nu-1)} = \begin{cases} \infty & \text{if } \nu \leq 1, \\ (1 - q^{\nu-1})^{-1} & \text{if } \nu > 1, \end{cases}$$

so that there is a mass $1 - q^{\nu-1}$ at the origin whenever $\nu > 1$. \square

Again, as in the proof of corollary 4.3, using [21, (1.3) with $n \rightarrow \infty$] shows that the orthogonality measure for the P_n is contained in $[-N, N]$ with $N \leq 1 + q^{\nu/2}$. Shifting ν to $\nu + 1$ we get the following corollary.

Corollary 5.6. *The first positive zero x'_1 of $j_\nu(x; q)$ satisfies $x'_1 \geq (1 + q^{(\nu+1)/2})^{-1}$.*

5.3. The case $\nu = 1/2$. In the simple case $\nu = 1/2$ we have $\lambda_n = q^{n/2}$. For simplicity we take $p = q^{1/2}$ so that the recurrence relation (5.1) can be rewritten as

$$(5.16) \quad P_{n+1}(x) = xP_n(x) - p^n P_{n-1}(x).$$

We consider the generating function $G(z, x) = \sum_{n=0}^{\infty} P_n(x)z^n$. Multiply (5.16) by z^{n+1} and add all the terms from $n = 0$ to infinity, then we get

$$G(z, x) - 1 = xzG(z, x) - z^2 p G(pz, x) \implies G(z, x) = \frac{1}{1 - xz} - \frac{z^2 p}{1 - xz} G(pz, x).$$

Solving the p -difference equation with respect to the condition $G(0, x) = 1$ gives by iteration

$$(5.17) \quad G(z, x) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k} p^{k^2}}{(zx; p)_{k+1}}.$$

We use the p -binomial theorem,

$$\frac{1}{(zx; p)_{k+1}} = \sum_{n=0}^{\infty} \frac{(p^{k+1}; p)_n}{(p; p)_n} (zx)^n,$$

in (5.17). Changing the summation index n to $j - 2k$ gives

$$\begin{aligned} G(z, x) &= \sum_{k=0}^{\infty} \sum_{j=2k}^{\infty} (-1)^k z^j x^{j-2k} p^{k^2} \frac{(p^{k+1}; p)_{j-2k}}{(p; p)_{j-2k}} \\ &= \sum_{j=0}^{\infty} z^j \sum_{k=0}^{\lfloor j/2 \rfloor} (-1)^k x^{j-2k} p^{k^2} \frac{(p^{k+1}; p)_{j-2k}}{(p; p)_{j-2k}}. \end{aligned}$$

Next identify the coefficient of z^n and use $(p^{k+1}; p)_{j-2k} = (p; p)_{j-k}/(p; p)_k$ to find

$$(5.18) \quad P_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k x^{n-2k} p^{k^2} \frac{(p; p)_{n-k}}{(p; p)_k (p; p)_{n-2k}}.$$

These polynomials are a special case of orthogonal polynomials associated with the Rogers-Ramanujan continued fraction; they correspond to the case $a = 0$, $b = p$ and $q = p$ in Al-Salam and Ismail [2] and (5.18) corresponds to [2, (3.7)]. These polynomials are also the special case $u_n(x)$ in Van Assche [20] with $a = 0$, $b = q$ and $q^2 = p$, and (5.18) corresponds to [20, (2.7)] after observing that for the Al-Salam–Chihara polynomials in (5.3) we have

$$P_{2n+1}(0; q; 0, b, c) = 0, \quad P_{2n}(0; q; 0, b, c) = (-1)^n c^n \left(\frac{b}{c}; q^2\right)_n (q; q^2)_n.$$

The associated polynomials $P_n^{(1)}$ satisfy the recurrence relation

$$(5.19) \quad P_{n+1}^{(1)}(x) = xP_n^{(1)}(x) - p^{n+1}P_{n-1}^{(1)}(x),$$

with $P_{-1}^{(1)} = 0$ and $P_0^{(1)}(x) = 1$. Replace x by x/\sqrt{p} in (5.19), then the monic polynomials $p^{n/2}P_n(x/\sqrt{p})$ satisfy the recurrence relation (5.19) so that $P_n^{(1)}(x) = p^{n/2}P_n(x/\sqrt{p})$.

In case $\nu = 1/2$ we have two different expressions for the same polynomials. From (5.4) and (5.18) we obtain the following summation formula for the Al-Salam–Chihara polynomials, $0 \leq k \leq n$,

$$(5.20) \quad P_k(-(q+q^{\frac{1}{2}}); q; -(1+q^{-\frac{1}{2}})q^{n-k+2}, q^{2(n-k)+\frac{7}{2}}, q^{\frac{3}{2}}) = \frac{(-q^{\frac{1}{2}})^k (q; q)_k (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2n+1-k}}{(q^{\frac{1}{2}}; q^{\frac{1}{2}})_k (q^{\frac{1}{2}}; q^{\frac{1}{2}})_{2n+1-2k}}.$$

The Al-Salam–Chihara polynomials are expressible in a ${}_3\varphi_2$ -series as proved by Askey and Ismail [5, §3.8]. Explicitly, the following connection between the original notation of Al-Salam and Chihara [1] and the notation of Askey and Ismail [5] holds;

$$(5.21) \quad \begin{aligned} \frac{\alpha^{-k}}{(q; q)_k} P_k(2\alpha x; q; (\gamma + \delta)\alpha, \gamma\delta\alpha^2, \alpha^2) &= S_k(x; \gamma, \delta \mid q) \\ &= \frac{(\gamma\delta; q)_k}{(q; q)_k} \gamma^{-k} {}_3\varphi_2 \left(\begin{matrix} q^{-k}, \gamma y, \gamma/y \\ \gamma\delta, 0 \end{matrix}; q, q \right), \end{aligned}$$

where $x = (y + y^{-1})/2$.

Corollary 5.7. *The summation formula*

$$(c^2; q)_k {}_3\varphi_2 \left(\begin{matrix} q^{-k}, cq^{-1/2}, c \\ c^2, 0 \end{matrix}; q, q \right) = (cq^{-1/2})^k (-q^{1/2}; q^{1/2})_k (c; q^{1/2})_k$$

holds for $k \in \mathbb{Z}_+$ and $c \in \mathbb{C}$.

Proof. In (5.20) we use (5.21) with the parameters $\alpha = -q^{\frac{3}{4}}$, $x = (q^{\frac{1}{4}} + q^{-\frac{1}{4}})/2$, $\gamma = q^{n-k+\frac{3}{4}}$, $\delta = q^{n-k+\frac{5}{4}}$ to get the result of the proposition for $c = q^{n-k+1}$. Replace $n - k$ by m in this result, so that we have proved the corollary for $c = q^{m+1}$, $m \in \mathbb{Z}_+$. Since both sides are polynomial in c , the result follows for arbitrary values of c . \square

Remark. Comparison of (5.5) with (5.18) instead of (5.4) with (5.18) leads to the same corollary. The same result is also obtained if we work out the different expressions for the associated polynomials in case $\nu = 1/2$.

Proposition 5.8. *Consider the monic orthogonal polynomials given by (5.16) and the associated polynomials given by (5.19). Then for every $x \in \mathbb{C}$ we have*

$$\lim_{n \rightarrow \infty} x^n P_n(1/x) = F(x), \quad \lim_{n \rightarrow \infty} x^n P_n^{(1)}(1/x) = F(x\sqrt{p})$$

where

$$F(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k} p^{k^2}}{(p; p)_k} = {}_0\varphi_1 \left(\begin{matrix} - \\ 0 \end{matrix}; p, -x^2 p \right).$$

Proof. Straightforward by letting $n \rightarrow \infty$ in (5.18) after changing x to $1/x$ and multiplication by x^n . \square

From proposition 5.8 and propositions 5.1 and 5.2 for $\nu = 1/2$ we obtain the equalities

$$(5.22) \quad {}_0\varphi_1(-; 0; q^{\frac{1}{2}}, -x^2 q^{\frac{1}{2}}) = x^{\frac{1}{2}} j_{-1/2}(x; q), \quad {}_0\varphi_1(-; 0; q^{\frac{1}{2}}, -x^2 q) = x^{-\frac{1}{2}} j_{1/2}(x; q),$$

which gives two transformations of a ${}_0\varphi_1$ -series of base $q^{1/2} = p$ in terms of ${}_1\varphi_1$ -series of base q . In two special cases the left hand sides of (5.22) can be summed by the Rogers-Ramanujan identities, cf. e.g. [11, (2.7.3), (2.7.4)] and this gives explicit values for $j_{-1/2}(x; q)$ for $x = \pm i, \pm i q^{1/4}$ and for $j_{1/2}(x; q)$ for $x = \pm i, \pm i q^{-1/4}$.

On comparing powers of x in the first equality of (5.22) and using the expansion (5.8) with $\nu = 1/2$ for $x^{1-\nu} j_{\nu-1}(x; q)$ we find for the continuous q -Hermite polynomial the evaluation

$$H_k \left(\frac{1}{2} (q^{\frac{1}{4}} + q^{-\frac{1}{4}}) \mid q \right) = q^{-k/4} (-q^{1/2}; q^{1/2})_k.$$

The same result is obtained if we compare powers of x in the second equality of (5.22). Moreover, this result is a limit case of corollary 5.7, cf. (5.7).

6. ORTHOGONALITY FOR THE LAURENT q -LOMMEL POLYNOMIALS

In this section we give a different form for the strong moment functional introduced in §3. The limit transitions (3.4) and (3.6) suggest to rewrite the strong moment functional \mathcal{L} defined in (3.15) as a contour integral over the unit circle. This can be done if $j_{\nu-1}(1; q) \neq 0$, since we have sufficient knowledge on the location of the zeros of $j_{\nu-1}(x; q)$, cf. theorem 5.3, and of $J_{\nu-1}(x; q)$, cf. [17, §3] and §4. A Wronskian type formula can be used to simplify the integrand.

Lemma 6.1. *Let $r_m(x)$, $s_m(x)$ be solutions of the recurrence relation (1.7), then the Wronskian $r_m(x)s_{m+1}(x) - s_m(x)r_{m+1}(x)$ is independent of $m \in \mathbb{Z}$.*

Proof. Multiply the recurrence formula for $r_m(x)$ by $s_m(x)$ and multiply the recurrence relation for $s_m(x)$ by $r_m(x)$. Subtract the resulting identities to find the result. \square

Lemma 6.2.

$$J_{\nu}(1/x; q)j_{\nu-1}(x; q) - J_{\nu-1}(1/x; q)j_{\nu}(x; q) = x^{-1} \frac{(qx^{-2}; q)_{\infty}(x^2; q)_{\infty}}{(q; q)_{\infty}}.$$

Proof. $J_{\nu+m}(1/x; q)j_{\nu+m-1}(x; q) - J_{\nu+m-1}(1/x; q)j_{\nu+m}(x; q)$ is independent of m by proposition 3.1 and lemma 6.1. Take $m = 0$ to obtain the left hand side of the lemma and use (3.7) and $m \rightarrow \infty$ to see that it also equals

$$(x^{-1} - x) \frac{(qx^{-2}; q)_{\infty}(qx^2; q)_{\infty}}{(q; q)_{\infty}},$$

which proves the lemma. \square

Lemma 6.2 implies that $x(q; q)_{\infty} [J_{\nu}(1/x; q)j_{\nu-1}(x; q) - J_{\nu-1}(1/x; q)j_{\nu}(x; q)]$ is a theta product, cf. Askey [3, §1].

Now we can rewrite the strong moment functional \mathcal{L} with respect to which the Laurent q -Lommel polynomials are orthogonal, cf. theorem 3.4.

Theorem 6.3. *Let $s > 0$ such that s is not a zero of $J_{\nu-1}(1/x; q)$ and $j_{\nu-1}(x; q)$. For $\nu > 0$ the strong moment functional \mathcal{L} defined in (3.15) equals*

$$\begin{aligned} \mathcal{L}(p) = & \frac{1}{2\pi i (q; q)_{\infty}} \oint_{|z|=s} p(z) \frac{(qz^{-2}; q)_{\infty}(z^2; q)_{\infty}}{J_{\nu-1}(1/z; q)j_{\nu-1}(z; q)} \frac{dz}{z} + \\ & \sum_{k=1}^N \left(p\left(\frac{1}{j_k^{\nu-1}}\right) + p\left(\frac{-1}{j_k^{\nu-1}}\right) \right) \frac{-J_{\nu}(j_k^{\nu-1}; q)}{(j_k^{\nu-1})^2 J'_{\nu-1}(j_k^{\nu-1}; q)} + \sum_{l=1}^M \left(p(x_l^{\nu-1}) + p(-x_l^{\nu-1}) \right) \frac{j_{\nu}(x_l^{\nu-1}; q)}{j'_{\nu-1}(x_l^{\nu-1}; q)}, \end{aligned}$$

where p is an arbitrary Laurent polynomial. Here $j_k^{\nu-1}$, respectively $x_l^{\nu-1}$, denote the positive zeros of $J_{\nu-1}(x; q)$, respectively $j_{\nu-1}(x; q)$, numbered increasingly. N is defined by $j_N^{\nu-1} < s < j_{N+1}^{\nu-1}$ and $N = 0$, and so the sum over the zeros of $J_{\nu-1}(x; q)$ is empty, if $j_1^{\nu-1} > s$. M is defined by $x_M^{\nu-1} < s < x_{M+1}^{\nu-1}$ and $M = 0$, and so the sum over the zeros of

$j_{\nu-1}(x; q)$ is empty, if $x_1^{\nu-1} > s$. The discrete weights in the first sum over k are positive and the discrete weights in the second sum over l are negative.

Remark. (i) By choosing $s = r$, respectively $s = 1/R$, with r, R as in §3, we get $M = 0$, respectively $N = 0$. In §7 we show that for ν sufficiently large we have $N = M = 0$ for a suitable choice of s .

(ii) The non-zero poles of the integrand in theorem 6.3 are simple. Indeed, if $0 \neq a$ satisfies $J_{\nu-1}(1/a; q) = 0 = j_{\nu-1}(a; q)$, then lemma 6.2 implies that the numerator is zero as well. Moreover, $a = q^{p/2}$ for some $p \in \mathbb{Z}$, which is a simple zero of the numerator. There exist only finitely many of such values in the (possibly empty) interval $[x_1^{\nu-1}, 1/j_1^{\nu-1}]$.

Proof. In the first contour integral in (3.15) we shift the contour integration from $|z| = 1/R$ to $|z| = s$ and in general we assume $s < 1/R$. We pick up residues at the simple poles $z = \pm 1/j_k^{\nu-1}$, $k = 1, \dots, N$, cf. theorem 4.2. For $1/R \leq s$ we are in the case $N = 0$. The second contour integral in (3.15) is shifted from $|z| = r$ to $|z| = s$. In general we assume $r < s$, otherwise we are in case $M = 0$. Here we pick up residues at the simple poles $z = \pm x_l^{\nu-1}$, $l = 1, \dots, M$. The residues are easily calculated. Next we take together the integrands of the contour integrals over $|z| = s$ using lemma 6.2 to prove the expression for $\mathcal{L}(p)$ in this case. The last statement follows from theorem 4.2 and theorem 5.5. \square

Remark. The most natural choice for s in theorem 6.3 seems $s = 1$. This is motivated by the fact that on the unit circle there is a transition in the asymptotic behaviour of the Laurent q -Lommel polynomials, cf. (3.4), (3.6). Moreover, numerical experiments indicate that for $m \rightarrow \infty$ the non-real zeros of the Laurent q -Lommel polynomials, cf. remark 2.1, are possibly dense on the unit circle. Of course, from (3.4), respectively (3.6), we see that the real zeros outside, respectively inside, the unit circle tend to the zeros of $J_{\nu-1}(x^{-1}; q)$, respectively $j_{\nu-1}(x; q)$. This corresponds precisely with the discrete set in the orthogonality measure of theorem 6.3 for $s = 1$.

7. LAURENT q -LOMMEL POLYNOMIALS AS PERTURBATIONS OF CHEBYSHEV POLYNOMIALS

Let us now return to the recurrence relation (1.7) which we rewrite as

$$(7.1) \quad h_{n+1, \nu}(x; q) - (x^{-1} + x)h_{n, \nu}(x; q) + h_{n-1, \nu}(x; q) = -xq^{\nu+n}h_{n, \nu}(x; q).$$

In this way, as $q \rightarrow 0$ or as $\nu \rightarrow \infty$ the Laurent polynomials $h_{n, \nu}(x; q)$ should be close to a solution of the three term recurrence relation

$$(7.2) \quad h_{n+1}(x; 0) - (x^{-1} + x)h_n(x; 0) + h_{n-1}(x; 0) = 0.$$

The solution of this recurrence, with initial values $h_0(x; 0) = 1$ and $h_{-1}(x; 0) = 0$ is given by $h_n(x; 0) = (x^{n+1} - x^{-n-1})/(x - x^{-1})$, which in terms of Chebyshev polynomials of the second kind can be written as

$$h_n(x; 0) = U_n \left(\frac{x + x^{-1}}{2} \right), \quad n \in \mathbb{Z}_+.$$

In this way the Laurent polynomials $h_{n,\nu}(x; q)$ can be considered as perturbations of the Chebyshev polynomials. We now do a perturbation analysis, much as is done for perturbations of orthogonal polynomials in [21]. In the spirit of the Liouville-Green approximation (WKB method), we will consider (7.1) as a second order recurrence relation with non-homogeneous term $-xq^{\nu+n}h_{n,\nu}(x; q)$, even though this term depends on the desired solution $h_{n,\nu}(x; q)$.

We solve this non-homogeneous recurrence relation by Green's method. We need the Green function $G_1(n, m)$, which is the solution of the recurrence relation with non-homogeneous term $\delta_{n,m}$, i.e.,

$$(7.3) \quad G_1(n+1, m) - (x^{-1} + x)G_1(n, m) + G_1(n-1, m) = \delta_{n,m}$$

with boundary conditions

$$(7.4) \quad G_1(n, m) = 0, \quad n \geq m.$$

Clearly $G_1(m, m) = G_1(m+1, m) = 0$ and thus from (7.3) we find $G_1(m-1, m) = 1$. For $k \geq 0$ we find that $r_k(x) = G_1(m-k-1, m)$ is a solution of the homogeneous recurrence relation (7.2) with the same initial conditions $r_0(x) = 1$ and $r_{-1}(x) = 0$, hence

$$G_1(n, m) = U_{m-n-1} \left(\frac{x + x^{-1}}{2} \right), \quad n < m.$$

Now multiply (7.1) by $G_1(n, m)$ and (7.3) by $h_{n,\nu}(x; q)$ and subtract the obtained equations to find

$$\begin{aligned} & h_{n+1,\nu}(x; q)G_1(n, m) - h_{n,\nu}(x; q)G_1(n-1, m) + h_{n,\nu}(x; q)\delta_{n,m} \\ & = h_{n,\nu}(x; q)G_1(n+1, m) - h_{n-1,\nu}(x; q)G_1(n, m) - xq^{\nu+n}h_{n,\nu}(x; q)G_1(n, m). \end{aligned}$$

Add all the equations from $n = 0$ to $n = m$ and use the boundary conditions (7.4) to find

$$h_{0,\nu}(x; q)G_1(-1, m) = h_{m,\nu}(x; q) + x \sum_{n=0}^{m-1} q^{\nu+n} G_1(n, m) h_{n,\nu}(x; q).$$

This gives

$$(7.5) \quad h_{m,\nu}(x; q) = U_m \left(\frac{x + x^{-1}}{2} \right) - x \sum_{n=0}^{m-1} q^{\nu+n} U_{m-n-1} \left(\frac{x + x^{-1}}{2} \right) h_{n,\nu}(x; q).$$

From this relation we can deduce some useful properties.

Lemma 7.1. *Suppose $x = e^{i\theta}$ with $\theta \in [0, 2\pi)$, then*

$$(7.6) \quad |h_{n,\nu}(x; q)| \leq (n+1) \exp \left(\frac{q^\nu}{(1-q)^2} \right),$$

and

$$(7.7) \quad |\sin \theta h_{n,\nu}(x; q)| \leq 1 + \frac{q^\nu}{(1-q)^2} \exp\left(\frac{q^\nu}{(1-q)^2}\right).$$

For $|x| \neq 1$ we have

$$\begin{aligned} |x^n h_{n,\nu}(x; q)| &\leq \frac{2}{|1-x^2|} \exp\left(\frac{2}{|1-x^2|} \frac{q^\nu}{1-q}\right), & |x| < 1, \\ |x^{-n} h_{n,\nu}(x; q)| &\leq \frac{2}{|1-x^{-2}|} \exp\left(\frac{2}{|1-x^{-2}|} \frac{q^\nu}{1-q}\right), & |x| > 1. \end{aligned}$$

Proof. We use Gronwall's inequality, cf. (3.9); for non-negative A, c_n, d_n , ($n \geq 0$) we have

$$c_n \leq A + \sum_{k=0}^{n-1} d_k c_k \implies c_n \leq A \exp\left(\sum_{k=0}^{n-1} d_k\right).$$

From the bound $|U_n(\cos \theta)| \leq n+1$ and (7.5) we find

$$|h_{n,\nu}(x; q)| \leq n+1 + \sum_{k=0}^{n-1} q^{\nu+k} (n-k) |h_{k,\nu}(x; q)|.$$

Hence taking $c_n = |h_{n,\nu}(x; q)|/(n+1)$ in Gronwall's inequality gives

$$\frac{|h_{n,\nu}(x; q)|}{n+1} \leq \exp\left(\sum_{k=0}^{n-1} (k+1) q^{\nu+k}\right).$$

The desired inequality (7.6) then follows from $\sum_{k=0}^{\infty} (k+1) q^k = (1-q)^{-2}$. If we use this inequality (7.6) and $|\sin \theta U_n(\cos \theta)| \leq 1$ in (7.5), then

$$|\sin \theta h_{n,\nu}(x; q)| \leq 1 + \exp\left(\frac{q^\nu}{(1-q)^2}\right) \sum_{k=0}^{n-1} (k+1) q^{\nu+k},$$

which gives (7.7). The bounds away from the unit circle follow by using

$$\left|x^n U_n\left(\frac{x+x^{-1}}{2}\right)\right| = \left|x^n \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}\right| \leq \frac{2}{|1-x^2|}, \quad |x| < 1,$$

and

$$\left|x^{-n} U_n\left(\frac{x+x^{-1}}{2}\right)\right| = \left|x^{-n} \frac{x^{n+1} - x^{-n-1}}{x - x^{-1}}\right| \leq \frac{2}{|1-x^{-2}|}, \quad |x| > 1,$$

and by using Gronwall's inequality. \square

From these bounds we see that the Laurent Lommel polynomials have an exponentially increasing upper bound both inside the unit circle and outside the unit circle, and that on the unit circle the Laurent polynomials are bounded, except when $x = \pm 1$, in which case $|h_{n,\nu}(x; q)| = O(n)$. This strongly suggests that in theorem 6.3 the choice $s = 1$ for the strong moment functional \mathcal{L} is the most natural.

The Laurent polynomial solution of (7.1) is not the only interesting solution. In §3 we already obtained the minimal solutions $j_{\nu+n}(x; q)$ and $J_{\nu+n}(x^{-1}; q)$ on respectively the open unit disk and the exterior of the closed unit disk. The minimal solutions $h_n^-(x; 0)$ and $h_n^+(x; 0)$ of the recurrence relation (7.2) on respectively the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and the exterior of the closed unit disk $\{z \in \mathbb{C} : |z| > 1\}$ are given by $h_n^-(x; 0) = x^n$ and $h_n^+(x; 0) = x^{-n}$. Our intention now is to find similar solutions $h_{n,\nu}^\pm(x; q)$ satisfying

$$\lim_{n \rightarrow \infty} h_{n,\nu}^\pm(x; q)x^{\pm n} = 1,$$

on $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\}$ respectively. Such functions clearly exist, since by proposition 3.1 and (3.7) we see that

$$(7.8) \quad h_{n,\nu}^+(x; q) = \frac{(q; q)_\infty}{(qx^{-2}; q)_\infty} x^\nu J_{\nu+n}(x^{-1}; q), \quad h_{n,\nu}^-(x; q) = \frac{1}{(qx^2; q)_\infty} x^{-\nu} j_{\nu+n}(x; q),$$

fulfil the required conditions.

We will now do a perturbation analysis of these minimal solutions in a similar way as is done for orthogonal polynomials [12]. Again we write the recurrence relation as

$$(7.9) \quad h_{n+1,\nu}^\pm - (x^{-1} + x)h_{n,\nu}^\pm(x; q) + h_{n-1,\nu}^\pm(x; q) = -xq^{\nu+n}h_{n,\nu}^\pm(x; q),$$

and look at this equation as a non-homogeneous second order recurrence relation with non-homogeneous term $-xq^{\nu+n}h_{n,\nu}^\pm(x; q)$. The homogeneous equation has two simple solutions, $h_n^\pm(x; 0) = x^{\mp n}$. We solve the non-homogeneous recurrence relation using Green functions, but now the Green function $G_2(n, m)$ is the solution of

$$(7.10) \quad G_2(n+1, m) - (x^{-1} + x)G_2(n, m) + G_2(n-1, m) = \delta_{n,m}$$

with boundary conditions

$$(7.11) \quad G_2(n, m) = 0, \quad n \leq m.$$

Since $G_2(m, m) = G_2(m-1, m) = 0$ we find $G_2(m+1, m) = 1$ and in general

$$G_2(n, m) = U_{n-m-1} \left(\frac{x + x^{-1}}{2} \right), \quad n > m.$$

Multiply the recurrence (7.9) by $G_2(n, m)$ and (7.10) by $h_{n,\nu}^\pm(x; q)$ and subtract to find

$$\begin{aligned} & h_{n+1,\nu}^\pm(x; q)G_2(n, m) - h_{n,\nu}^\pm(x; q)G_2(n-1, m) + h_{n,\nu}^\pm(x; q)\delta_{n,m} \\ &= h_{n,\nu}^\pm(x; q)G_2(n+1, m) - h_{n-1,\nu}^\pm(x; q)G_2(n, m) - xq^{\nu+n}h_{n,\nu}^\pm(x; q)G_2(n, m). \end{aligned}$$

Add the equations from $n = m$ to $n = M$, with $m < M$ and use the boundary conditions (7.11) to find

$$\begin{aligned} h_{M+1,\nu}^{\pm}(x; q)G_2(M, m) - h_{M,\nu}^{\pm}(x; q)G_2(M + 1, m) \\ = -h_{m,\nu}^{\pm}(x; q) - x \sum_{n=m+1}^M q^{\nu+n} h_{n,\nu}^{\pm}(x; q)G_2(n, m). \end{aligned}$$

From (7.8) together with (3.7) we obtain

$$\lim_{M \rightarrow \infty} h_{M+1,\nu}^+(x; q)G_2(M, m) - h_{M,\nu}^+(x; q)G_2(M + 1, m) = -x^{-m}, \quad |x| > 1,$$

and

$$\lim_{M \rightarrow \infty} h_{M+1,\nu}^-(x; q)G_2(M, m) - h_{M,\nu}^-(x; q)G_2(M + 1, m) = -x^m, \quad |x| < 1,$$

so by letting $M \rightarrow \infty$ we have

$$(7.12) \quad h_{n,\nu}^{\pm}(x; q) = x^{\mp n} - x \sum_{k=n+1}^{\infty} q^{\nu+k} h_{k,\nu}^{\pm}(x; q)U_{k-n-1} \left(\frac{x + x^{-1}}{2} \right).$$

Compare these relations to (7.5). We can find appropriate bounds on these solutions and from this we can obtain bounds for $J_{\nu+n}(x^{-1}; q)$ and $j_{\nu+n}(x; q)$.

Lemma 7.2. *If $x \neq \pm 1$ then*

$$(7.13) \quad \begin{aligned} |x^n h_{n,\nu}^+(x; q)| &\leq \exp \left(\frac{2}{|1 - x^{-2}|} \frac{q^{\nu+n+1}}{1 - q} \right), & |x| \geq 1, \\ |x^{-n} h_{n,\nu}^-(x; q)| &\leq \exp \left(\frac{2}{|1 - x^2|} \frac{q^{\nu+n+1}}{1 - q} \right), & |x| \leq 1, \end{aligned}$$

and

$$(7.14) \quad \begin{aligned} |x^n h_{n,\nu}^+(x; q)| &\leq \exp \left(\frac{nq^{\nu+n+1}}{1 - q} + \frac{q^{\nu+n+1}}{(1 - q)^2} \right), & |x| \geq 1, \\ |x^{-n} h_{n,\nu}^-(x; q)| &\leq \exp \left(\frac{nq^{\nu+n+1}}{1 - q} + \frac{q^{\nu+n+1}}{(1 - q)^2} \right), & |x| \leq 1. \end{aligned}$$

Proof. We now use a backward version of Gronwall's inequality: for non-negative A, c_n, d_n ($n \geq 0$) we have

$$c_n \leq A + \sum_{k=n+1}^{\infty} d_k c_k < \infty \implies c_n \leq A \exp \left(\sum_{k=n+1}^{\infty} d_k \right).$$

The inequalities (7.13) then follow from (7.12) and the inequalities $|x^{\pm n}U_n((x+x^{-1})/2)| \leq 2/|1-x^{\pm 2}|$, which hold for $|x| \leq 1$ (for the + sign) and $|x| \geq 1$ (for the - sign).

Inequality (7.14) uses the inequality $|x^{\pm n}U_n((x+x^{-1})/2)| \leq n+1$ on $|x| \leq 1$ and $|x| \geq 1$ respectively. So from (7.12) we get

$$\begin{aligned} |x^n h_{n,\nu}^+(x; q)| &\leq 1 + \sum_{k=n+1}^{\infty} q^{\nu+k} |x^k h_{k,\nu}^+(x; q)| (k-n) \\ &\leq 1 + \sum_{k=n+1}^{\infty} k q^{\nu+k} |x^k h_{k,\nu}^+(x; q)|, \quad |x| \geq 1, \end{aligned}$$

from which the first inequality of (7.14) follows by Gronwall's inequality. \square

We are now ready to give some information about the zeros of the functions $h_{n,\nu}^{\pm}(x; q)$ inside and outside the open unit disk.

Theorem 7.3. *The zeros of $h_{n,\nu}^{\pm}(x; q)$ are all real. The function $h_{n,\nu}^+(x; q)$ has no zeros in $\{x \in \mathbb{C} : |x| \geq 1\}$ and $h_{n,\nu}^-(x; q)$ has no zeros inside $\{x \in \mathbb{C} : |x| \leq 1\}$, whenever $n \geq M(\nu, q)$, where*

$$(7.15) \quad M(\nu, q) = -\nu - 1 + 2 \frac{\ln(1-q)}{\ln q} - \frac{1}{\ln q}.$$

In particular $h_{-1,\nu}^+(x; q)$ has at most $2M(\nu, q) + 2$ zeros in $\{x \in \mathbb{C} : |x| \geq 1\}$ and $h_{-1,\nu}^-(x; q)$ has at most $2M(\nu, q) + 2$ zeros in $\{x \in \mathbb{C} : |x| \leq 1\}$.

Proof. The reality of the zeros follows from the explicit representation (7.8) and the reality of the zeros of the Hahn-Exton Bessel function [17, §3] and the zeros of $j_{\nu}(x; q)$, cf. theorem 5.3. For an upper bound on the number of zeros, we use (7.12) to find

$$1 - x^{\pm n} h_{n,\nu}^{\pm}(x; q) = \sum_{k=n+1}^{\infty} q^{\nu+k} x^{\pm k} h_{k,\nu}^{\pm}(x; q) x^{\pm n \mp k+1} U_{k-n-1} \left(\frac{x+x^{-1}}{2} \right).$$

Use the inequality (7.14) and $|x^{\pm n}U_n((x+x^{-1})/2)| \leq n+1$ to find for $|x| \geq 1$

$$|1 - x^n h_{n,\nu}^+(x; q)| \leq \sum_{k=n+1}^{\infty} (k-n) q^{\nu+k} \exp \left(\frac{kq^{\nu+k+1}}{1-q} + \frac{q^{\nu+k+1}}{(1-q)^2} \right),$$

and similarly for $|x| \leq 1$

$$|1 - x^{-n} h_{n,\nu}^-(x; q)| \leq \sum_{k=n+1}^{\infty} (k-n) q^{\nu+k} \exp \left(\frac{kq^{\nu+k+1}}{1-q} + \frac{q^{\nu+k+1}}{(1-q)^2} \right).$$

The right hand side can be bounded by

$$\begin{aligned} \sum_{k=n+1}^{\infty} (k-n)q^{\nu+k} \exp\left(\frac{kq^{\nu+k+1}}{1-q} + \frac{q^{\nu+k+1}}{(1-q)^2}\right) &\leq \exp\left(\frac{q^{\nu+1}}{(1-q)^2}\right) \sum_{k=n+1}^{\infty} (k-n)q^{\nu+k} \\ &= \frac{q^{\nu+n+1}}{(1-q)^2} \exp\left(\frac{q^{\nu+n+1}}{(1-q)^2}\right). \end{aligned}$$

Choose $M = M(\nu, q)$ such that

$$\frac{q^{\nu+M+1}}{(1-q)^2} \exp\left(\frac{q^{\nu+M+1}}{(1-q)^2}\right) < 1,$$

then $h_{n,\nu}^+(x; q)$ for $n \geq M$ cannot be zero for any x such that $|x| \geq 1$. An appropriate $M(\nu, q)$ is given by (7.15). The same reasoning holds for $h_{n,\nu}^-(x; q)$ on the closed unit disk. So now we have established that for $n \geq M$ the function $h_{n,\nu}^+$ has no zeros for $|x| \geq 1$ and $h_{n,\nu}^-$ has no zeros for $|x| \leq 1$. The zeros of $h_{n,\nu}^+$ are equal to the zeros of $J_{\nu+n}(1/x; q)$. If j_k^ν , $k = 1, 2, 3, \dots$, are the zeros of $J_\nu(x; q)$ numbered increasingly, then from the interlacing property of theorem 3.7 in [17] we have $j_k^\nu < j_k^{\nu+1} < j_{k+1}^\nu$, hence when the parameter ν is decreased by one, then the k th positive zero moves to the left. This means that the k th positive zero (counted from the right) of $h_{n-1,\nu}^+(x; q)$ is to the right of the k th positive zero of $h_{n,\nu}^+(x; q)$. Since $h_{M,\nu}^+(x; q)$ has no zeros $x \geq 1$, this means that $h_{M-1,\nu}^+(x; q)$ can have one zero $x \geq 1$, namely $1/j_1^{\nu+M-1}$, and it cannot have two zeros $x > 1$ since $1/j_2^{\nu+M-1} < 1/j_1^{\nu+M} < 1$. Decreasing the degree of $h_{n,\nu}^+(x; q)$ by one thus increases the number of zeros in $|x| \geq 1$ by at most 2 (one positive zero and one negative zero). Therefore $h_{-1,\nu}^+(x; q)$ has at most $2M + 2$ zeros in $|x| \geq 1$. A similar reasoning works for the zeros of $h_{n,\nu}^-(x; q)$ in $|x| \leq 1$ by using the interlacing property of the zeros of $j_\nu(x; q)$ and $j_{\nu+1}(x; q)$ given by theorem 5.3. \square

The upper bound on the number of zeros of $h_{-1,\nu}^\pm(x; q)$ gives a useful upper bound on the number of discrete mass points of the strong moment functional \mathcal{L} as given in theorem 6.3 when $s = 1$. Indeed, the zeros of $h_{-1,\nu}^+(x; q)$ correspond with the zeros of $J_{\nu-1}(1/x; q)$ and thus $N \leq M(\nu, q) + 1$. Similarly the zeros of $h_{-1,\nu}^-(x; q)$ correspond with the zeros of $j_{\nu-1}(x; q)$ and thus $M \leq M(\nu, q) + 1$. In particular, $M = N = 0$ in theorem 6.3 for $s = 1$ for ν satisfying $M(\nu, q) < 0$.

Finally let us give another derivation of the orthogonality of the Laurent polynomials $h_{n,\nu}(x; q)$ by using the minimal solutions $h_{n,\nu}^\pm(x; q)$. Observe that from (3.3), (3.5) and (7.8) it follows that $h_{n,\nu}^\pm(x; q)$ have a power expansion of the form

$$x^n h_{n,\nu}^+(x; q) = 1 + \sum_{k=0}^{\infty} K^+(n, k) x^{-2k}, \quad |x| > 1,$$

and

$$x^{-n} h_{n,\nu}^-(x; q) = 1 + \sum_{k=1}^{\infty} K^-(n, k) x^{2k}, \quad |x| < 1.$$

We can get some information on the coefficients $K^\pm(n, k)$ by introducing Banach algebras. If f is analytic in the open unit disk with Taylor series

$$f(z) = \sum_{k=0}^{\infty} f_k z^k,$$

then we define

$$\|f\|_- = \sum_{k=0}^{\infty} \nu_k |f_k|,$$

and we denote by A^- all the functions f for which $\|f\|_- < \infty$. Here $\nu_k, k \in \mathbb{Z}_+$, is a positive increasing sequence for which $\nu_0 = 1$ and $\nu_n \leq \nu_m \nu_{n-m}$ for every $n \geq m \geq 0$. Similarly, when g is analytic near infinity with Laurent series

$$g(z) = \sum_{k=0}^{\infty} g_k z^{-k},$$

then we define

$$\|g\|_+ = \sum_{k=0}^{\infty} \nu_k |g_k|,$$

and denote by A^+ all the functions g for which $\|g\|_+ < \infty$. One easily verifies that for two functions $f_1, f_2 \in A^\pm$ one has

$$\|f_1 f_2\|_\pm \leq \|f_1\|_\pm \|f_2\|_\pm,$$

so that we are dealing with Banach algebras.

Observe that

$$\begin{aligned} \|x^n h_{n,\nu}^+(x; q)\|_+ &= 1 + \sum_{k=0}^{\infty} \nu_{2k} |K^+(n, k)|, \\ \|x^{-n} h_{n,\nu}^-(x; q)\|_- &= 1 + \sum_{k=1}^{\infty} \nu_{2k} |K^-(n, k)|. \end{aligned}$$

Taking norms in (7.12) gives

$$\|x^n h_{n,\nu}^+(x; q)\|_+ \leq 1 + \sum_{k=n+1}^{\infty} q^{\nu+k} \|x^k h_{k,\nu}^+(x; q)\|_+ \|x^{n-k+1} U_{k-n-1} \left(\frac{x+x^{-1}}{2} \right)\|_+.$$

Now

$$\|x^{-n} U_n \left(\frac{x+x^{-1}}{2} \right)\|_+ = \left\| \sum_{j=0}^n x^{-2j} \right\|_+ = \sum_{j=0}^n \nu_{2j} \leq (n+1) \nu_{2n},$$

so that Gronwall's inequality gives

$$\|x^n h_{n,\nu}^+(x; q)\|_+ \leq \exp\left(\sum_{k=n+1}^{\infty} kq^{\nu+k}\nu_{2k}\right).$$

Taking $\nu_n = a^n$ with $a < q^{-1/2}$ shows that

$$1 + \sum_{k=0}^{\infty} a^{2k} |K^+(n, k)| < \infty,$$

so that $x^n h_{n,\nu}^+(x; q) \in A^+$. This shows that the function $h_{n,\nu}^+(x; q)$ is in fact defined for $x > q^{1/2}$. A similar reasoning shows that $x^{-n} h_{n,\nu}^-(x; q) \in A^-$ and that $h_{n,\nu}^-(x; q)$ is defined for $x < q^{-1/2}$. From (7.8) we see that $h_{n,\nu}^+(x; q)$ has poles at the zeros of $(qx^{-2}; q)_{\infty}$ and that $x = \pm q^{1/2}$ are the poles of largest modulus. Similarly $h_{n,\nu}^-(x; q)$ has poles at the zeros of $(qx^2; q)_{\infty}$ and $x = \pm q^{-1/2}$ are the poles of smallest modulus.

Suppose now that ± 1 are not zeros of $h_{-1}^{\pm}(x; q)$. Evaluate the contour integral

$$I_+ = \frac{1}{2\pi i} \int_{|x|=1} \frac{h_{m,\nu}(x; q) h_{n,\nu}^+(x; q)}{h_{-1,\nu}^+(x; q)} dx.$$

If $m < n$ then near $x = \infty$ the integrand behaves as x^{m-n-1} and thus I_+ has no contribution from $x = \infty$. So when x_j^+ ($j \geq 1$) are the zeros of $h_{-1,\nu}^+(x; q)$, then

$$I_+ = - \sum_{j=1}^N \frac{h_{m,\nu}(x_j^+; q) h_{n,\nu}^+(x_j^+; q)}{[h_{-1,\nu}^+(x_j^+; q)]'},$$

where N is defined as in theorem 6.3 for $s = 1$. Similarly we compute the contour integral

$$I_- = \frac{1}{2\pi i} \int_{|x|=1} \frac{h_{m,\nu}(x; q) h_{n,\nu}^-(x; q)}{h_{-1,\nu}^-(x; q)} dx.$$

The integrand behaves as x^{n-m+1} near $x = 0$ and thus there is no pole at the origin when $m < n$ (even for $m \leq n + 1$). There are poles at the zeros x_j^- ($j \geq 1$) of $h_{-1,\nu}^-(x; q)$ and we thus have

$$I_- = \sum_{j=1}^M \frac{h_{m,\nu}(x_j^-; q) h_{n,\nu}^-(x_j^-; q)}{[h_{-1,\nu}^-(x_j^-; q)]'},$$

where M is defined as in theorem 6.3 for $s = 1$. Subtracting I_+ and I_- gives

$$I_- - I_+ = \frac{1}{2\pi i} \int_{|x|=1} h_{m,\nu}(x; q) \left(\frac{h_{n,\nu}^-(x; q) h_{-1,\nu}^+(x; q) - h_{n,\nu}^+(x; q) h_{-1,\nu}^-(x; q)}{h_{-1,\nu}^+(x; q) h_{-1,\nu}^-(x; q)} \right) dx.$$

The Laurent polynomial $h_{n,\nu}(x; q)$ is a solution of the three-term recurrence relation (7.1) and therefore a linear combination of the two special solutions $h_{n,\nu}^\pm(x; q)$. With the initial conditions $h_{0,\nu}(x; q) = 1$ and $h_{-1;\nu}(x; q) = 0$ and by combining (7.8) with lemma 6.2 we find

$$(x^{-1} - x)h_{n,\nu}(x; q) = h_{-1,\nu}^-(x; q)h_{n,\nu}^+(x; q) - h_{-1,\nu}^+(x; q)h_{n,\nu}^-(x; q),$$

so that

$$I_- - I_+ = \frac{1}{2\pi i} \int_{|x|=1} (x - x^{-1})h_{n,\nu}(x; q)h_{m,\nu}(x; q) \frac{dx}{h_{-1,\nu}^+(x; q)h_{-1,\nu}^-(x; q)}.$$

On the other hand, at a zero x_j^+ we see that $h_{n,\nu}^+(x_j^+; q)$ is a solution of (7.1) with initial value $h_{-1,\nu}^+(x_j^+; q) = 0$, so that $h_{n,\nu}^+(x_j^+; q) = h_{0,\nu}^+(x_j^+; q)h_{n,\nu}(x_j^+; q)$. Similarly at a zero x_j^- we have $h_{n,\nu}^-(x_j^-; q) = h_{0,\nu}^-(x_j^-; q)h_{n,\nu}(x_j^-; q)$. Therefore

$$\begin{aligned} I_- - I_+ &= \sum_{j=0}^{M(\nu,q)} h_{n,\nu}(x_j^-; q)h_{m,\nu}(x_j^-; q) \frac{h_{0,\nu}^+(x_j^-; q)}{[h_{-1,\nu}^+(x_j^-; q)]'} \\ &\quad + \sum_{j=0}^{M(\nu,q)} h_{n,\nu}(x_j^+; q)h_{m,\nu}(x_j^+; q) \frac{h_{0,\nu}^-(x_j^+; q)}{[h_{-1,\nu}^-(x_j^+; q)]'}. \end{aligned}$$

Combining both expressions for $I_- - I_+$ gives the orthogonality of the Laurent polynomials $h_{n,\nu}(x; q)$ and corresponds to the result given in theorem 6.3 for $s = 1$. In case $q^{1/2} < s < q^{-1/2}$ the orthogonality relations of theorem 6.3 can be derived in a similar way.

This approach can also be used to prove the orthogonality for the Laurent polynomials $x^{-1}h_{n,\nu}(x; q)$, cf. (2.3). Note also that the case $q = 0$ gives the orthogonality relations for the Chebyshev polynomials of the second kind.

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