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A SERIES INVERSION FORMULA.

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1. *Introduction*. Suppose that the functions $f(x)$ and $e^{i\lambda(x-\pi)}f(x)$, where λ is not an integer, are expansible in Fourier series in the form

$$f(x)=\sum_{n=-\infty}^{\infty} a_n e^{inx}, \quad e^{i\lambda(x-\pi)}f(x)=\sum_{n=-\infty}^{\infty} b_n e^{-inx},$$

so that

$$\sum_{n=-\infty}^{\infty} a_n e^{inx}=\sum_{n=-\infty}^{\infty} b_n e^{-i(n+\lambda)x+i\lambda\pi}.$$

If we multiply by $e^{i(m+\lambda)x}$ and integrate term by term from 0 to 2π, we obtain

$$\frac{\sin\pi\lambda}{\pi}\sum_{n=-\infty}^{\infty}\frac{a_n}{m+n+\lambda}=b_m. \tag{1.1}$$

Similarly, if we multiply by e^{-imx} and integrate, we obtain the reciprocal formula

$$a_m=\frac{\sin\pi\lambda}{\pi}\sum_{n=-\infty}^{\infty}\frac{b_n}{m+n+\lambda}. \tag{1.2}$$

This can be verified directly by substituting (1.1) in (1.2), changing the order of summation, and using the formula

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n+r+\lambda)(n+s+\lambda)}=\frac{\pi^2}{\sin^2\pi\lambda}\quad (r=s), \qquad 0\quad (r\neq s).$$

In either case the work is so far purely formal, and the Fourier method could not be justified without introducing unnecessary restrictions. We have, for example,

$$a_n = e^{iny}, \quad b_n = e^{-i(n+\lambda)y+i\lambda\pi} \quad (0 < y < 2\pi),$$

and, in particular, if $a_n = (-1)^n$, then $b_n = (-1)^n$. Such a case would not be included in the Fourier theory.

It will, therefore, be assumed simply that the numbers a_n are given, and that the numbers b_n are defined by (1 . 1), and the object is to discover in what circumstances and in what sense the formula (1 . 2) holds. We shall consider the case $\lambda = \frac{1}{2}$, other cases being practically the same. In a recent paper* it was shown that, if $\Sigma |a_n|^p$, where $p > 1$, is convergent, then the formulae hold, and $\Sigma |b_n|^p$ is also convergent. But this condition is by no means necessary.

The series considered, of the form

$$\sum_{n=-\infty}^{\infty} u_n, \tag{1.3}$$

are convergent in the ordinary sense if the series

$$\sum_{n=1}^{\infty} u_n, \quad \sum_{n=1}^{\infty} u_{-n}$$

are both convergent, and they are summable $(C, 1)$ in the ordinary sense if these two series are both summable $(C, 1)$. The series (1 . 3) will be said to be convergent in the restricted sense if the series

$$\sum_{n=1}^{\infty} (u_n + u_{-n})$$

is convergent, and to be summable $(C, 1)$ in the restricted sense if this series is summable $(C, 1)$.

It is assumed throughout that the series (1 . 1) *are convergent in the ordinary sense.* To show the inadequacy of restricted convergence, let $a_n = 1$ for all values of n ($\lambda = \frac{1}{2}$). Then the series (1 . 1) converge in the restricted sense to the sum zero, and (1 . 2) is not satisfied.

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* E. C. Titchmarsh, *Math. Zeitschrift*, 25 (1926), 321–347. See also M. Riesz, *Math. Annalen* (to appear shortly).

The results are as follows :—

THEOREM I. *The series* (1 . 2) *are summable* $(C, 1)$ *in the restricted sense to the sums stated.*

The special interest of the theorem is that it shows that, if the series (1 . 2) converge at all, then they have the sums stated. The remaining theorems may be expressed in terms of the numbers

$$R_n = \sum_{m=n}^{\infty} \frac{a_m + a_{-m}}{m}, \quad R'_n = \sum_{m=n}^{\infty} \frac{a_m - a_{-m}}{m},$$

which may be any two sequences tending to zero.

THEOREM II. *The necessary and sufficient condition for the convergence of* (1 . 2) *in the restricted sense is*

$$\lim_{q\to\infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} = 0 \quad (0 < \theta < 1). \tag{1.4}$$

THEOREM III. *The necessary and sufficient condition for the summability* $(C, 1)$ *in the ordinary sense of* (1 . 2) *is the convergence of the series*

$$\sum_{n=1}^{\infty} \frac{R'_n}{n+\frac{1}{2}}. \tag{1.5}$$

THEOREM IV. *The necessary and sufficient conditions for convergence in the ordinary sense are the conditions of Theorems II and III, together with*

$$\lim_{q\to\infty} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{R'_n}{n-q-\frac{1}{2}} = 0. \tag{1.6}$$

2. LEMMA. *The series*

$$\frac{1}{\pi} \sum_{m=-\infty}^{\infty} \left(\frac{b_m}{m+n+\frac{1}{2}} - \frac{b_m}{m+\frac{1}{2}} \right)$$

converges in the ordinary sense to the sum $a_n - a_0$.

The series is

$$\frac{1}{\pi^2} \sum_{m=-\infty}^{\infty} \left(\frac{1}{m+n+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}} \right) \sum_{r=-\infty}^{\infty} \frac{a_r}{m+r+\frac{1}{2}},$$

and we have merely to prove that the order of the summation may be inverted. It will be sufficient to prove that

$$\sum_{m=-\infty}^{\infty} \sum_{r=s}^{\infty} = \sum_{r=s}^{\infty} \sum_{m=-\infty}^{\infty},$$

where $s > n$, $s > 0$, since the remaining part can be dealt with similarly; and for this it is sufficient to prove that

$$\lim_{p \to \infty} \sum_{r=s}^{\infty} \sum_{m=-\infty}^{-p} = 0, \quad \lim_{q \to \infty} \sum_{r=s}^{\infty} \sum_{m=q}^{\infty} = 0.$$

Now

$$\left(\frac{1}{m+n+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}\right) \frac{1}{m+r+\frac{1}{2}} = \frac{n}{r} \frac{1}{m+n+\frac{1}{2}} \left(\frac{1}{m+r+\frac{1}{2}} - \frac{1}{m+\frac{1}{2}}\right),$$

and obviously

$$\sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=-\infty}^{-p} \frac{1}{(m+n+\frac{1}{2})(m+\frac{1}{2})}, \quad \sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+\frac{1}{2})}$$

tend to zero. Also

$$\sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})}$$

is a steadily decreasing function of r, its maximum tending to zero as $q \to \infty$. Hence, by partial summation,

$$\sum_{r=s}^{\infty} \frac{a_r}{r} \sum_{m=q}^{\infty} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})} \to 0.$$

A similar partial summation gives

$$\sum^{\infty} \frac{a_r}{r} \sum_{m=-\infty}^{-p} \frac{1}{(m+n+\frac{1}{2})(m+r+\frac{1}{2})}$$

$$= \sum_{r=s}^{\infty} \frac{R_r''}{(r-n)(r-n+1)} \sum_{m=n-p+1}^{r-p} \frac{1}{m+\frac{1}{2}} - \sum_{r=s}^{\infty} \frac{R_r''}{(r-n+1)(r-p+\frac{3}{2})}$$

$$- \frac{R_{s-1}''}{s-n} \sum_{m=n-p+1}^{s-p} \frac{1}{m+\frac{1}{2}},$$

where $R''_n = \frac{1}{2}(R_n + R'_n)$. The last term is $O(1/p)$, the middle term is

$$O\left\{\int_s^{p-1} \frac{du}{u(p-u)} + \int_{p+1}^{\infty} \frac{du}{u(u-p)}\right\} = O\left(\frac{\log p}{p}\right),$$

and since

$$\left|\sum_{m=n-p+1}^{r-p} \frac{1}{m+\frac{1}{2}}\right| < \frac{K}{\sqrt{p}} \quad (r < \sqrt{p}), \qquad < K \log rp \quad (r \geqslant \sqrt{p}),$$

the first term is

$$\sum_{r<\sqrt{p}} o\left(\frac{1}{\sqrt{p}}\right) + \sum_{r \geqslant \sqrt{p}} O\left(\frac{\log r + \log p}{r^2}\right) = o(1).$$

This proves the lemma.

3. *Convergence in the restricted sense.* In view of the lemma it is sufficient to consider the series

$$\sum_{m=-\infty}^{\infty} \frac{b_m}{m+\frac{1}{2}}. \tag{3.1}$$

We have

$$\sum_{m=-q-1}^{q} \frac{b_m}{m+\frac{1}{2}} = \frac{1}{\pi} \sum_{m=-q-1}^{q} \frac{1}{m+\frac{1}{2}} \sum_{n=-\infty}^{\infty} \frac{a_n}{m+n+\frac{1}{2}}$$

$$= \frac{1}{\pi} \sum_{n=-\infty}^{\infty} a_n \sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})}.$$

Now, if $n \neq 0$,

$$\sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})} = \frac{1}{n} \sum_{m=-q-1}^{q} \left(\frac{1}{m+\frac{1}{2}} - \frac{1}{m+n+\frac{1}{2}}\right)$$

$$= -\frac{1}{n} \sum_{m=n-q-1}^{n+q} \frac{1}{m+\frac{1}{2}}$$

$$= -u_{n,q}/n,$$

say, while, if $n = 0$, we obtain

$$\sum_{m=-q-1}^{q} \frac{1}{(m+\frac{1}{2})^2} = \pi^2 + O\left(\frac{1}{q}\right).$$

Hence

$$\sum_{m=-q-1}^{q} \frac{b_m}{m+\frac{1}{2}} = \pi a_0 - \frac{1}{\pi} \sum_{n=-\infty}^{\infty}{}' \frac{a_n}{n} u_{n,q} + O\left(\frac{1}{q}\right)$$

$$= \pi a_0 - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} + O\left(\frac{1}{q}\right),$$

since $u_{n,q} = -u_{-n,q}$. Now

$$\sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} = \sum_{n=1}^{\infty} R_n (u_{n,q} - u_{n-1,q})$$

$$= \sum_{n=1}^{k} + \sum_{n=k+1}^{(1-\theta)q} + \sum_{(1-\theta)q}^{(1+\theta)q} + \sum_{(1+\theta)q}^{\infty},$$

where $k = [\sqrt{q}]$, and $0 < \theta < 1$.* Now $|R_n| \leqslant R$ (all n), $|R_n| < \epsilon$ $(n > \sqrt{q_0})$; also $u_{n,q}$ increases steadily for $n < q$, and then decreases steadily to zero, while

$$u_{n,q} < \frac{2n}{q-n+\frac{3}{2}} \quad (n < q); \quad u_{n,q} < \frac{2q+2}{n-q-\frac{1}{2}} \quad (n > q).$$

Hence, for $q > q_1$,

$$\left|\sum_{n=1}^{k}\right| < R\frac{2\sqrt{q}}{q-\sqrt{q}+\frac{3}{2}}, \quad \left|\sum_{k+1}^{(1-\theta)q}\right| < \epsilon\frac{2(1-\theta)q}{\theta q+\frac{3}{2}}, \quad \left|\sum_{(1+\theta)q}^{\infty}\right| < \epsilon\frac{2q+2}{\theta q-\frac{1}{2}}.$$

It follows that

$$\sum_{n=1}^{\infty} \frac{a_n + a_{-n}}{n} u_{n,q} = \sum_{(1-\theta)q}^{(1+\theta)q} R_n (u_{n,q} - u_{n+1,q}) + o(1)$$

$$= \sum_{(1-\theta)q}^{(1+\theta)q} R_n \left(\frac{1}{n+q+\frac{1}{2}} - \frac{1}{n-q-\frac{3}{2}}\right) + o(1)$$

$$= -\sum_{(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} + o(1),$$

and this proves Theorem II.

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* I write $\sum_{n=a}^{b} = \sum_{a \leqslant n \leqslant b}$, whether a and b are integers or not. To avoid trivial difficulties with the limits, let θ be irrational.

4. *Summability in the restricted sense.* In view of what has been proved already, Theorem I is true if

$$(4.1)\qquad \sum_{q=1}^{Q} \sum_{n=(1-\theta)q}^{(1+\theta)q} \frac{R_n}{n-q-\frac{1}{2}} = o(Q).$$

The left-hand side is equal to

$$-\sum_{n=1}^{(1-\theta)Q} R_n \sum_{q=n/(1+\theta)}^{n/(1-\theta)} \frac{1}{q-n+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)Q} R_n \sum_{q=n/(1+\theta)}^{Q} \frac{1}{q-n+\frac{1}{2}}$$

$$= -\sum_{n=1}^{(1-\theta)Q} R_n \sum_{p=-\theta n/(1+\theta)}^{\theta n/(1-\theta)} \frac{1}{p+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)Q} R_n \sum_{p=-\theta n/(1+\theta)}^{Q-n} \frac{1}{p+\frac{1}{2}}$$

$$= -\sum_{n=1}^{(1-\theta)Q} R_n \sum_{\theta n/(1+\theta)-1}^{\theta n/(1-\theta)} \frac{1}{p+\frac{1}{2}} - \sum_{n=(1-\theta)Q}^{(1+\theta)(Q+1)/(1+2\theta)} R_n \sum_{\theta n/(1+\theta)-1}^{Q-n} \frac{1}{p+\frac{1}{2}}$$

$$+ \sum_{(1+\theta)(Q+2)/(1+2\theta)}^{Q} R_n \sum_{Q-n+1}^{\theta n/(1+\theta)-1} \frac{1}{p+\frac{1}{2}} + \sum_{n=Q+1}^{(1+\theta)Q} R_n \sum_{n-Q-1}^{\theta n/(1+\theta)-1} \frac{1}{p+\frac{1}{2}}.$$

For $Q > Q_0(\epsilon)$, the first two terms give

$$\sum_{n\leqslant\sqrt{Q}} O(1) + \epsilon \sum_{n>\sqrt{Q}}^{(1-\theta)Q} O(1) + \epsilon \sum_{(1-\theta)Q}^{(1+\theta)(Q+1)/(1+2\theta)} O\left(\frac{Q}{n}\right) = O(\sqrt{Q}) + \epsilon O(Q).$$

Also replacing R_n by ϵ in the last two terms is equivalent to replacing R_n by ϵ in (4.1), with error $O(\sqrt{Q})+\epsilon O(Q)$, as has just been proved; and

$$\epsilon \sum_{q=1}^{Q} \sum_{(1-\theta)q}^{(1+\theta)q} \frac{1}{n-q-\frac{1}{2}} = \epsilon \sum_{q=1}^{Q} O(1) = \epsilon O(Q).$$

This proves Theorem I.

5. *Summability in the ordinary sense.* Suppose that the series (3.1) is summable $(C, 1)$ in the ordinary sense. Then

$$(5.1)\qquad \sum_{m=1}^{q} \left(1-\frac{m-1}{q}\right) \frac{b_m}{m+\frac{1}{2}} = A + o(1).$$

Let
$$b'_m = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{a_n}{m+n+\frac{1}{2}}, \quad b''_m = \frac{1}{\pi} \sum_{n=-\infty}^{-1} \frac{a_n}{m+n+\frac{1}{2}}.$$

Then
$$\sum_{m=1}^{q} \left(1-\frac{m-1}{q}\right) \frac{b'_m+b''_m}{m+\frac{1}{2}} = A' + o(1).$$

But, by Theorem I,

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b''_m-b''_{-m-1}}{m+\frac{1}{2}}=o(1).$$

Hence
$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}=A'+o(1).$$

Now
$$b'_m+b''_{-m-1}=\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{a_n-a_{-n}}{m+n+\frac{1}{2}}=o(1).$$

Hence
$$\sum_{m=1}^{q}\frac{m-1}{q}\,\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}=\frac{1}{q}\sum_{m=1}^{q}o(1)=o(1).$$

Hence

$$\begin{aligned}
\pi A'+o(1) &= \pi\sum_{m=1}^{q}\frac{b'_m+b''_{-m-1}}{m+\frac{1}{2}}\\
&=\sum_{m+1}^{q}\frac{1}{m+\frac{1}{2}}\sum_{n-1}^{\infty}\frac{a_n-a_{-n}}{m+n+\frac{1}{2}}\\
&=\sum_{m=1}^{q}\sum_{n=1}^{\infty}\left(\frac{1}{m+\frac{1}{2}}-\frac{1}{m+n+\frac{1}{2}}\right)(R'_n-R'_{n+1})\\
&=\sum_{n=1}^{\infty}\left(\frac{1}{n+\frac{1}{2}}-\frac{1}{n+q+\frac{1}{2}}\right)R'_n\\
&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}-\sum_{n=1}^{q}\frac{R'_n}{n+q+\frac{1}{2}}+q\sum_{n=q+1}^{\infty}\frac{R'_n}{(n+\frac{1}{2})(n+q+\frac{1}{2})}\\
&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}+\sum_{n=1}^{q}o\left(\frac{1}{q}\right)+q\sum_{n=q+1}^{\infty}o\left(\frac{1}{n^2}\right)\\
&=\sum_{n=1}^{q}\frac{R'_n}{n+\frac{1}{2}}+o(1).
\end{aligned}$$

This proves that the condition stated is necessary. Conversely, if the condition is satisfied, we find, on reversing the above steps, that (5 . 1) holds. But, by Theorem I,

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b_m-b_{-m-1}}{m+\frac{1}{2}}$$

tends to a limit. Hence

$$\sum_{m=1}^{q}\left(1-\frac{m-1}{q}\right)\frac{b_{-m-1}}{m+1}$$

also tends to a limit. Hence the series is summable $(C, 1)$ in the ordinary sense.

6. *Convergence in the ordinary sense.* If the series (3 . 1) is convergent in the ordinary sense, then

$$\sum_{m=0}^{q} \frac{b'_m}{m+\frac{1}{2}} + \sum_{m=0}^{q} \frac{b''_m}{m+\frac{1}{2}} \tag{6.1}$$

and

$$\sum_{m=-q-1}^{-1} \frac{b'_m}{m+\frac{1}{2}} + \sum_{m=-q-1}^{-1} \frac{b''_m}{m+\frac{1}{2}} \tag{6.2}$$

both tend to limits when $q \to \infty$. Also it follows from the working of § 5 that the existence of the limit of

$$\sum_{m=0}^{q} \frac{b'_m}{m+\frac{1}{2}} - \sum_{m=-q-1}^{-1} \frac{b''_m}{m+\frac{1}{2}} \tag{6.3}$$

is a necessary condition. Adding (6 . 2) and (6 . 3), we see that the restricted convergence of

$$\sum_{m=-\infty}^{\infty} \frac{b'_m}{m+\frac{1}{2}}$$

is a necessary condition, and similarly for b''_m. Applying Theorem II to each of these series, we obtain the conditions (1 . 5) and (1 . 6); and, as we have seen, the existence of the limit of (6 . 3) is equivalent to the convergence of (1 . 5). Thus the conditions are necessary, and it is easily seen that they are also sufficient.

It may be noted that a simple sufficient condition for convergence in the ordinary sense is the convergence in the ordinary sense of

$$\sum_{-\infty}^{\infty} \frac{a_m}{m+\frac{1}{2}} \log|m+\tfrac{1}{2}|.$$

But the condition is not necessary; *e.g.* let

$$a_n = (-1)^n n (\log n)^{-\frac{1}{2}} \quad (n > 1), \quad a_n = 0 \quad (n < 1).$$

7. *Example.* Let

$$a_m = \frac{1}{(m+\frac{1}{2})^{2k}},$$

where k is a positive integer. Then

$$\pi b_n = \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}(m+n+\frac{1}{2})}$$

$$= \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}} \frac{1}{n} \left\{ \frac{1}{(m+\frac{1}{2})} - \frac{1}{(m+n+\frac{1}{2})} \right\}$$

$$= \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}} - \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}(m+n+\frac{1}{2})}.$$

This process may be repeated until we arrive at the series

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})(m+n+\frac{1}{2})},$$

which is zero if $n \neq 0$. Hence, if $n \neq 0$,

$$\pi b_n = \frac{1}{n} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k}} - \frac{1}{n^2} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^{2k-1}} + \ldots + \frac{1}{n^{2k-1}} \sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^2}.$$

Now, if r is even,

$$\sum_{m=-\infty}^{\infty} \frac{1}{(m+\frac{1}{2})^r} = 2(2^r-1)\,\zeta(r),$$

while the sum is zero if r is odd. Hence, if $n \neq 0$,

$$b_n = \frac{2}{\pi} \sum_{p=1}^{k} \frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+1}}.$$

Also, clearly $b_0 = 0$. Hence the relation

$$\sum_{n=-\infty}^{\infty} a_n^2 = \sum_{n=-\infty}^{\infty} b_n^2$$

gives
$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\frac{1}{2})^{4k}} = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty}{}' \left\{ \sum_{p=1}^{k} \frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+1}} \right\}^2$$

$$= \frac{8}{\pi^2} \sum_{n=1}^{\infty} \sum_{p=1}^{k} \sum_{q=1}^{k} \frac{(2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)}{n^{4k-2p-2q+2}},$$

that is to say

(8.51) $(2^{4k}-1)\,\zeta(4k)$

$$= \frac{4}{\pi^2} \sum_{p=1}^{k} \sum_{q=1}^{k} (2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)\,\zeta(4k-2p-2q+2).$$

Similarly if we take

$$a_m = \frac{1}{(m+\frac{1}{2})^{2k+1}}$$

we obtain

$$b_0 = \frac{1}{\pi}(2^{2k+2}-1)\,\zeta(2k+2), \quad b_n = -\frac{2}{\pi}\sum_{p=1}^{k}\frac{(2^{2p}-1)\,\zeta(2p)}{n^{2k-2p+2}} \quad (n \neq 0).$$

This leads to the relation

$$(8.52)\quad (2^{4k+2}-1)\,\zeta(4k+2) = \frac{2}{\pi^2}(2^{2k+2}-1)^2\zeta^2(2k+2)$$
$$+\frac{4}{\pi^2}\sum_{p=1}^{k}\sum_{q=1}^{k}(2^{2p}-1)(2^{2q}-1)\,\zeta(2p)\,\zeta(2q)\,\zeta(4k-2p-2q+4).$$

We have thus proved quite independently of the circular functions that the numbers $\zeta(2n)$ are all rational multiples of integral powers of $\zeta(2)$ [since the constant π^2 enters the above formulae as the value of $6\,\zeta(2)$].