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# Some Results on the Spectra and Spectral Resolutions of a Class of Singular Integral Operators*

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## 1. Introduction

The operators in question are

$$T: f(s) \to m(s)f(s) + \frac{1}{2\pi i}\int_0^1 \frac{K(s,\sigma)}{s-\sigma}\,d\sigma\,, \tag{1}$$

$f$ denoting an arbitrary function in the space $L_2[0,1]$. The multiplier $m$ and the kernel $K$ are assumed to be continuously differentiable. The singular integral on the right of (1) is to be taken in the following sense:

$$\int_0^1 \frac{K(s,\sigma)f(\sigma)}{s-\sigma}\,d\sigma = \lim_{\varepsilon\to 0+}\int_0^1 \frac{K(s,\sigma)f(\sigma)}{s-\sigma+i\varepsilon}\,d\sigma\,, \tag{2}$$

the limit being in the mean square. (For a convenient account of such singular integrals, reference may be made to [4], Section 2.)

The present paper will be divided into three parts. In the first part, we locate the so-called *essential spectrum* of the general operator (1). In the second part, we study an operator $T_\infty$ on the infinite interval closely related to the operator (1) and, requiring that $m(s)-s$ and $K$ be small, and both $m$ and $K$ are real, establish the spectral resolution of the operator $T_\infty$ in an explicit form, using a modification of the Friedrichs perturbation method. In the third section, we use the form of the spectral resolution of $T_\infty$ to discuss the scattering theory of $T_\infty$. This in turn enables us to discuss the spectral theory of $T$, under the assumption that the kernel $K$ satisfies the end-point conditions

$$K(0,\sigma) \equiv K(1,\sigma) \equiv 0\,. \tag{3}$$

The results of the present paper, especially of its second section, are related to earlier work of Friedrichs (cf. his classical paper, [2]) on non-singular integral operators, and to work of Koppelman and Pincus [3] and Koppelman [1].

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## 2. The Essential Spectrum

It is convenient to introduce some notation. We write the integral (2) as

$$\int^{+} \frac{K(s, \sigma) f(\sigma)}{s - \sigma} d\sigma , \tag{4}$$

and the corresponding mean-square limit

$$\lim_{\varepsilon \to 0+} \int_0^1 \frac{K(s, \sigma) f(\sigma)}{s - \sigma - i\varepsilon} d\sigma \tag{5}$$

as

$$\int^{-} \frac{K(s, \sigma) f(\sigma)}{s - \sigma - i\varepsilon} d\sigma . \tag{6}$$

If $K \equiv 1$, we may write (4) as $I^+(f)$ and (6) as $I^-(f)$. The corresponding integrals

$$\lim_{\varepsilon \to 0+} \int_{-\infty}^{+\infty} \frac{f(\sigma)}{s - \sigma + i\varepsilon} d\sigma \tag{7}$$

and

$$\lim_{\varepsilon \to 0+} \int_{-\infty}^{+\infty} \frac{f(\sigma)}{s - \sigma - i\varepsilon} d\sigma , \tag{8}$$

extended over the infinite interval, will be written as $\int_\infty^+ (f)$ and $\int_\infty^- (f)$. Note that

$$\int_\infty^+ \left( \int_\infty^+ (f) \right) = -2\pi i \int_\infty^+ (f) , \qquad f \in L_2(-\infty, +\infty) , \tag{9}$$

$$\int_\infty^- \left( \int_\infty^- (f) \right) = +2\pi i \int_\infty^- (f) , \qquad f \in L_2(-\infty, +\infty) , \tag{10}$$

$$\int_\infty^- (f) - \int_\infty^+ (f) = 2\pi i f , \qquad f \in L_2(-\infty, +\infty) , \tag{11}$$

$$\int_\infty^- \left( \int_\infty^+ (f) \right) = \int_\infty^+ \left( \int_\infty^- (f) \right) = 0 , \qquad f \in L_2(-\infty, +\infty) . \tag{12}$$

These well-known results are most readily deduced by making use of the Fourier transform. We shall regard $L_2(0, 1)$ as the subspace of $L_2(-\infty, +\infty)$ consisting of all functions vanishing outside the interval $[0, 1]$. If $\chi$ denotes the operation of multiplying by the characteristic function of this interval, then

$$I^+(f) = \chi \int_\infty^+ (f) , \quad I^-(f) = \chi \int_\infty^- (f) , \qquad f \in L_2(0, 1) . \tag{13}$$

Since the kernel $K$ in (1) has a continuous first derivative, the two functions

$$k(s) = \frac{1}{2\pi i} K(s, s) , \quad k(s, \sigma) = \frac{1}{2\pi i} \frac{K(s, \sigma) - K(s, s)}{s - \sigma} \tag{14}$$

are continuous. If $k$ denotes the operation of multiplication by the function $k(s)$, then plainly the operator $T$ of (1) may be written

$$(Tf)(s) = ((m + kI^+)f) + \int_0^1 k(s, \sigma)d\sigma . \tag{15}$$

The commutator $mI^+ - I^+m$ maps the function $f$ into

$$\int_0^1 \left\{ \frac{m(s) - m(\sigma)}{s - \sigma} \right\} f(\sigma)d\sigma = \int_0^1 M(s, \sigma)f(\sigma)d\sigma . \tag{16}$$

Since $m$ has a continuous derivative, the kernel $M$ is continuous.

Suppose then that we let $\mathfrak{A}_0$ be the uniformly closed algebra of operators in the Hilbert space $L_2(0, 1)$ generated by all the operators $T$ of the form (1), $J$ be the ideal of $\mathfrak{A}_0$ consisting of all compact operators in $\mathfrak{A}_0$, and $\mathfrak{A}$ be the factor ring $\mathfrak{A}_0/J$. It follows at once from (16) that $\mathfrak{A}$ is a *commutative B-algebra*. Moreover, it follows at once from (15) that *the B-algebra* $\mathfrak{A}$ *has two generators*: *the* (class of the) *operator* $I^+$ *and the* (class of the) *operator* $S$ *of multiplication by* $s$.

In what follows, we shall, wherever this will lead to no essential confusion, omit to distinguish between an operator $T$ in $\mathfrak{A}_0$ and its class in the factor ring $\mathfrak{A}$.

Definition 1. The *essential spectrum* $\sigma_e(T)$ of an operator $T$ in Hilbert space is the spectrum of its natural homomorphic image in the factor algebra $B(H)/K$. Here, $B(H)$ denotes the algebra of all bounded operators in Hilbert space, and $K$ the two-sided ideal of compact operators.

Lemma 2. *A complex number* $\lambda$ *is in the complement of the essential spectrum of* $T$ *if and only if*

(a) *the range of* $\lambda I - T$ *is closed,*

(b) *the orthocomplement of the range of* $\lambda I - T$ *is finite dimensional,*

(c) *the nullspace of* $\lambda I - T$ *is finite dimensional.*

Proof. Let $V = \lambda I - T$. If $\lambda \notin \sigma_e(T)$, then $V$ has a left- and a right-hand inverse modulo the ideal of compact operators. Thus there exist operators $A$, $B$ and compact operators $C_1$, $C_2$ such that $AV = I + C_1$, $VB = I + C_2$. The nullspace of $V$ is consequently included in the nullspace of $I + C_1$. Hence, by the theory of compact operators, the nullspace of $V$ is finite dimensional. The range of $V$ plainly includes the range of $I + C_2$. Hence, by the theory of compact operators, the range of $V$ includes a closed space with a finite dimensional orthocomplement. Hence the range of $V$ is a closed space with a finite dimensional orthocomplement.

Conversely, suppose that (a), (b), and (c) are satisfied, so that $V$ is an operator having a finite dimensional nullspace $N$ and a closed range $R$ with finite dimensional orthocomplement $M$. Then $V$ is a 1–1 mapping of the orthocomplement $N^{\perp}$ onto $R$. By the closed graph theorem, there exists a bounded mapping $V_1$ of $R$ onto $N^{\perp}$ such that

$$(17)\qquad \begin{aligned} VV_1x &= x\,, & x \in R\,,\\ V_1Vy &= y\,, & y \in N^{\perp}\,. \end{aligned}$$

Let $P$ be the perpendicular projection onto $R$, and put $B = V_1P$. Put $A = V_{\perp}$. Since $(I - AV_1)y = 0$, $y \in N^{\perp}$, and since $N$ is finite dimensional, $I - AV_1$ has a finite dimensional range and is therefore compact. Since $VB = P$, while $I - P$ is compact, $VB - I$ is also compact. Thus $V$ has an inverse modulo the ideal of compact operators.

COROLLARY 3. *If the operator $T$ is self-adjoint or normal, then a complex number $\lambda$ is in the essential spectrum if and only if $\lambda$ is either a point eigenvalue of infinite multiplicity or a limit point of the spectrum.*

Proof: This statement follows immediately from the preceding lemma and the spectral theorem for normal operators. Details are left to the reader.

The spectrum of the multiplication operator $S$: $f(s) \to sf(s)$ is well-known and easily seen to be the closed unit interval $[0, 1]$, all of which is a continuous spectrum. Thus $\sigma_e(S) = [0, 1]$ by the preceding corollary. It is not much harder to discover the spectrum of the operator $I^+$. We may argue as follows. Let $U$ be the unitary transformation of $L_2(-\infty, +\infty)$ onto $L_2(0, 1)$ defined by the formula

$$(17a)\qquad U\colon f(\nu) \to \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\left(\frac{1}{1-x}\right)^{i\nu}\frac{1}{\sqrt{x(1-x)}}f(\nu)d\nu\,;$$

that this mapping is indeed a unitary mapping of the one space onto the other follows by an elementary change of variables from Plancherel's theorem. An easy computation then shows that

$$(18)\qquad (I^+Uf)(x) = -\frac{2\pi i}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{1}{[1+\exp(-2\pi\nu)]}f(\nu)\left(\frac{x}{1-x}\right)^{i\nu}\frac{1}{\sqrt{x(1-x)}}dx\,,$$
$$f \in L_2(-\infty, +\infty)\,.$$

Thus the transformation $U^{-1}I^+U$ is the operation of multiplying by $-2\pi i[1 + \exp(-2\pi\nu)]^{-1}$. It follows at once that $I^+$ is normal, and that its spectrum and essential spectrum coincide, both sets consisting of the closed interval of the negative imaginary axis from 0 to $-2\pi i$.

Let $\Omega$ denote Gelfand space of the commutative $B$-algebra $\mathfrak{A}$, i.e., the space of all homomorphisms $\omega$ of $\mathfrak{A}$ into the field of complex numbers, topologized in the usual way. If, for each $T \in \mathfrak{A}$, we write $\omega(T)$ as $T(\omega)$, it is well-known that $T(\omega)$ is a continuous function of $\omega$, and that the spectrum in $\mathfrak{A}$ of the element $T$, i.e., the set of $\lambda$ such that $\lambda I - T$ has no inverse in $\mathfrak{A}$, is the range of the continuous function $T(\omega)$.

The space $\Omega$ is, of course, compact. Our next aim is to represent $\Omega$ in an explicit way. This may be done as follows. Since $S$ and $I^+$ generate the algebra $\mathfrak{A}$, the mapping $\omega \to [S(\omega), I^+(\omega)]$ is a continuous, 1-1 mapping of $\Omega$ into the two-dimensional plane, hence, a homeomorphism of $\Omega$ with a subset of the two-dimensional plane. Since the spectrum of $S$ is purely real while the spectrum of $I^+$ is pure imaginary, $S(\omega)$ is a real function and $I^+(\omega)$ a pure imaginary function; the map $\omega \to S(\omega) + I(\omega) = z(\omega)$ is a homeomorphism of $\Omega$ with a subset $\Omega_0$ of the complex plane. Since $\sigma_e(S) = [0, 1]$ and $\sigma_e(I) = [Oi, -2\pi i]$, $\mathscr{R}_e\, z(\omega)$ lies between zero and 1 and takes on all values between zero and 1, while $\mathscr{I}_m\, z(\omega)$ lies between zero and $-2\pi$ and takes on all values in this range. Thus $\Omega_0$ is a subset of the rectangle

$$\rho = \{z \mid 0 \leqq \mathscr{R}_e\, z \leqq 1; 0 \geqq \mathscr{I}_m\, z \geqq -2\pi\} ; \tag{19}$$

every horizontal and every vertical line in the rectangle $\rho$ intersects $\Omega_0$.

Let $k(s)$ be continuously differentiable, and vanish outside the interval $0 \leqq s \leqq 1$. Then plainly

$$kI^+ = kI^+_\infty . \tag{20}$$

Hence, using the commutativity of $\mathfrak{A}$, the following equations must hold as an equation in $\mathfrak{A}$, i.e., must hold modulo compact operators:

$$\begin{aligned} (kI^+)^2 = kI^+kI^+ &= kI^+kI^+_\infty \\ &= k(kI^+)I^+_\infty \\ &= k^2(I^+_\infty)^2 = k^2(-2\pi i I^+_\infty) \\ &= -2\pi i k^2 I^+ . \end{aligned} \tag{21}$$

Consequently we have

$$k^2(\omega)(I^+(\omega))^2 = -k^2(-2\pi i I^+(\omega)) \tag{22}$$

or

$$k^2(\omega)I^+(\omega)(I^+(\omega) \dotplus 2\pi i) = 0 . \tag{23}$$

In writing these last two equations, we have denoted the operation of multiplication by $k$ simply as the function $k$. Since $S(\omega) = \mathscr{R}_e\, z(\omega)$, we have

$$(P(S))(\omega) = P(\mathscr{R}_e\, z(\omega))$$

for each polynomial $P$ in the operator $S$. Thus, by the Weierstrass theorem, the same assertion must hold for every continuous function of the operator $s$. Hence equation (23) may be written

$$k^2(\mathscr{R}_e\, z)\mathscr{I}_m\,(z)(\mathscr{I}_m\,(z) + 2\pi i) = 0 \tag{24}$$

for each $z \in \Omega_0$ and for each continuously differentiable function $k(s)$ vanishing together with its first derivatives at $s = 0$ and $s = 1$. Thus it follows at once that if $z \in \Omega_0$, then either $\mathscr{R}_e\, z = 0$, $\mathscr{R}_e\, z = 1$, $\mathscr{I}_m\,(z) = 0$, or $\mathscr{I}_m\,(z) = -2\pi i$. The set $\Omega_0$ is therefore a subset of the periphery of the rectangle $\rho$ (cf. (19)). Let $V_1$ and $V_2$ be the transformation of $L_2(0, 1)$ into itself defined by $(V_1 f)(s) = f(1 - s)$ and $(V_2 f)(s) = f(s)$. Let $\tau_1$ and $\tau_2$ be, respectively, the automorphism and anti-automorphism of the ring of bounded operators in $L_2(0, 1)$ defined by

(25)
$$\tau_1:\ T \to V_1^{-1} T V_1\,,$$
$$\tau_2:\ T \to V_2^{-1} T^* V_2\,.$$

Then $\tau_1$ and $\tau_2$ map the ideal of compact operators into itself. Using (11) and (13), we have plainly

(26)
$$\tau_1 S = 1 - S\,,\quad \tau_1 I^+ = -I^- = -2\pi i - I^+$$

and

(27)
$$\tau_2 S = S\,,\quad \tau_2 I^+ = -I^- = -2\pi i - I^+\,.$$

By (26) and (27), $\tau_1$ and $\tau_2$ must map the algebra $\mathfrak{A}_0$ into itself. Thus $\tau_1$ and $\tau_2$ induce automorphisms of the factor-algebra $\mathfrak{A}$, which, for notational simplicity, we continue to denote with the same letters. By (26) and (27), we have

(28)
$$(\tau_1 S)(\omega) = 1 - \mathscr{R}_e\, z(\omega)\,,\quad (\tau_1 I^+)(\omega) = -2\pi i - \mathscr{I}_m\, z(\omega)$$

and

(29)
$$(\tau_2 S)(\omega) = \mathscr{R}_e\, z(\omega)\,,\quad (\tau_2 I^+)(\omega) = -2\pi i - \mathscr{I}_m\, z(\omega)\,.$$

Thus the range $\Omega_0$ of the function $z(\omega)$ is invariant under the two mappings

(30)
$$z \to 1 - 2\pi i - z$$

and

(31)
$$z \to \bar{z} - 2\pi i\,.$$

Since $\Omega_0$ is also known to be a subset of the periphery of the rectangle $\rho$ (cf. (19)) and to intersect every horizontal and every vertical line in the rectangle $\rho$, it follows that *$\Omega_0$ is the entire periphery of the rectangle $\rho$.*

Using the fact that, for each $T \in \mathfrak{A}_0$, $\sigma_e(T)$ is identical with the range of the function $T(\omega)$, this result may be formulated as follows:

THEOREM 4. *Let the singular integral operator $T$ of* (1) *be written in the form* (14)–(15). *Then the essential spectrum $\sigma_e(T)$ is the image under the mapping*

(32)
$$x + iy \to m(x) + k(x)iy$$

*of the periphery of the rectangle $\rho$ of* (19).

### 3. Spectral Theory of a Slightly Perturbed Operator

We return to the consideration of the operator (1). If $m(s)$ and $K(s, s)$ are real, then Theorem 4 assures us that the essential spectrum $\sigma_e(T)$ is real: the spectrum of $T$ consists of the closed interval from

(33)
$$a = \min_{0 \le s \le 1} \min\,(m(s), m(s) - K(s, s))$$

to

$$b = \max_{0 \leq s \leq 1} \max (m(s), m(s) - K(s, s)) , \tag{34}$$

together with a possible sequence of isolated points having no limit points outside this closed interval. In the present section, we shall assume not only that $m(s)$ and $K(s, s)$ are real, but that $K$ satisfies the end-point conditions (3), and that $m(s) - s$ and $K(s, s)$ are small together with their first four derivatives. This assumption will allow us to show by the methods of perturbation theory that $I$ is similar to the operation $M$ of multiplication by $m(s)$. Note that $m$ and $K$ are assumed to be four times continuously differentiable.

We begin by showing that, without loss of generality, we may assume that $m(s) \equiv s$. To this end, we argue as follows. First note that $m'(s) > 0$ so that $m$ is monotone increasing. Since instead of considering $T$ we may consider a linear function $cT + d$ of it, it can be assumed without loss of generality that $m(0) = 0$, $m(1) = 1$. Let $U_0$ be the unitary transformation

$$f(s) \to f(m^{-1}(s))\left\{\frac{d}{ds} m^{-1}(s)\right\}^{1/2} . \tag{35}$$

Then it is plain that $T_1 = U_0^{-1} T U_0$ is the operator

$$T_1 \colon f(s) \to sf(s) + \frac{1}{2\pi i}\int_0^1 \frac{\left[ K(m(s), m(\sigma))\left\{\frac{d}{ds} m(s)\right\}^{1/2}\left\{\frac{s - \sigma}{m(s) - m(\sigma)}\right\}\right]}{s - \sigma} f(\sigma)d\sigma . \tag{36}$$

If we write the kernel in the denominator of (36) as $K_1(s, \sigma)$, then it is clear from the assumed properties of $m$ that $K_1(s, \sigma)$ is small together with its first three derivatives, and satisfies the end-conditions (3). Since $T$ is unitary-equivalent to $T_1$, it follows without loss of generality that we may confine our attention to the operator $T_1$, i.e., we may assume that $m(s) \equiv s$. This assumption will be made in all that follows. It is true that in making the passage from $K_1$ to $K$ we lose one derivative; since in the following analysis three continuous derivatives will suffice for all purposes, this is no real loss.

We now extend $K$ to a small three-times continuously differentiable kernel defined for all values of the variables $s, \sigma$, which we continue to denote by the letter $K$. We assume the extension to be made in such a way that $K(s, s)$ is real for all $s$, and such that $K$ vanishes outside a compact subset of $E' \times E'$. Instead of attempting the spectral reduction of the operator $T$ of (1) directly, we shall first analyze the corresponding operator

$$T_\infty \colon f(s) \to sf(s) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{K(s, \sigma)}{s - \sigma} f(\sigma)d\sigma , \quad f \in L_2(-\infty, +\infty) . \tag{37}$$

The domain of $T_\infty$ is to consist of all functions $f \in L_2(-\infty, +\infty)$ such that $sf(s)$ is also square-integrable on the infinite axis. The operator $T$ is evidently the restriction of $T_\infty$ to its invariant subspace $L_2(0, 1)$.

Set

$$\alpha(s) = s - K(s, s)\ ; \tag{38}$$

by our assumption on the smallness of $K$, $\alpha'(s) > 0$ and $\alpha(s) = s$ if $|s|$ is sufficiently large. Thus $\alpha$ increases monotonically. Set

$$(U_1 f)(s) = \frac{1}{2\pi i}\int_\infty^+ \frac{\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma}\,d\sigma - \frac{1}{2\pi i}\int_\infty^- \frac{f(\sigma)}{s-\sigma}\,d\sigma\ . \tag{39}$$

It is plain from the properties of $\alpha$ and the boundedness of the maps $\int_\infty^+$ and $\int_\infty^-$ that $U_1$ is a bounded mapping of $L_2(-\infty, +\infty)$ into itself.

We continue to write

$$k(s, \sigma) = \frac{1}{2\pi i}\,\frac{K(s,\sigma) - K(s,s)}{s-\sigma}\ , \tag{40}$$

and note that just as in (15) we may write

$$(T_\infty f)(s) = sf(s) + \frac{K(s,s)}{2\pi i}\int_\infty^+ \frac{f(\sigma)}{s-\sigma}\,d\sigma + \int_{-\infty}^{+\infty} k(s,\sigma)f(\sigma)d\sigma\ . \tag{41}$$

Using (9)–(12) of the preceding section, it follows that

$$\begin{aligned}(T_\infty U_1 f)(s) = {} & \frac{1}{2\pi i}\int_\infty^+ \frac{s\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma}\,d\sigma + \frac{1}{2\pi i}\int_\infty^- \frac{sf(\sigma)}{s-\sigma}\,d\sigma \\ & - \frac{K(s,s)}{2\pi i}\int_\infty^+ \frac{\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma}\,d\sigma \\ & - \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left\{\int_\infty^- \frac{k(s,\sigma_1)}{\sigma-\sigma_1}\,d\sigma_1\right\} f(\sigma)d\sigma \\ & + \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\left\{\int_\infty^+ \frac{k(s,\sigma_1)}{\sigma-\sigma_1}\,d\sigma_1\right\} \alpha'(\sigma)f(\alpha(\sigma))d\sigma\ .\end{aligned} \tag{42}$$

It follows from (40) that the function $k(s, \sigma)$ has two continuous derivatives, vanishes if $|s|$ is sufficiently large, and vanishes as $|\sigma| \to \infty$ at least as fast as $|\sigma|^{-1}$. Consequently, using standard lemmas on singular integrals (cf. J. Schwartz [4], Section 3 for statement and proof of these lemmas) we conclude that each of the integrals

$$\int_\infty^\pm \frac{k(s,\sigma_1)}{\sigma-\sigma_1}\,d\sigma_1 = \phi_\pm(s,\sigma) \tag{43}$$

has a Hölder continuous derivative of any order less than 1, vanishes if $|s|$ is sufficiently large, and vanishes as $|\sigma| \to \infty$ at least as fast as $|\sigma|^{-1}$. Thus, if we put

$$l(s,\sigma) = \frac{1}{2\pi i}\,\phi_+(s, \alpha^{-1}(\alpha)) - \frac{1}{2\pi i}\,\phi_-(s,\sigma)\ , \tag{44}$$

the function $l$ has the same properties. Since

$$\int_{-\infty}^{+\infty} \alpha'(\sigma)f(\alpha(\sigma))d\sigma - \int_{-\infty}^{+\infty} f(\sigma)d\sigma = 0 \tag{45}$$

for each $f \in L_2(-\infty, +\infty)$ such that $sf(s)$ is also square-integrable, we may conclude from (42) that

$$(46)\quad \begin{aligned}(T_\infty U_1 f)(s) = \frac{1}{2\pi i}\int_\infty^{+}\frac{\sigma\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma}d\sigma - \frac{1}{2\pi i}\int_\infty^{-}\frac{\sigma f(\sigma)}{s-\sigma}d\sigma \\ - \frac{K(s,s)}{2\pi i}\int_\infty^{+}\frac{\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma}d\sigma + \int_{-\infty}^{+\infty} l(s,\sigma)f(\sigma)d\sigma\end{aligned}$$

for each $f \in L_2(-\infty, +\infty)$ such that $sf(s)$ is also square-integrable. Thus, if we put

$$(47)\quad L(s,\sigma) = -\frac{K(s,s) - K(\alpha^{-1}(\sigma), \alpha^{-1}(\sigma))}{s - \alpha^{-1}(\sigma)} + l(s,\sigma)\,,$$

we may write

$$(48)\quad \begin{aligned}(T_\infty U_1 f)(s) = \frac{1}{2\pi i}\int_\infty^{+}\frac{\alpha(\sigma)\alpha'(\sigma)f(\alpha(\sigma))}{s-\sigma} - \frac{1}{2\pi i}\int_\infty^{-}\frac{\sigma f(\sigma)}{s-\sigma}d\sigma \\ + \int_{-\infty}^{+\infty} L(s,\sigma)f(\sigma)d\sigma\end{aligned}$$

for each function $f \in L_2(-\infty, +\infty)$ such that $sf(s)$ is also square-integrable. If we define the closed operator $S_\infty$ by setting

$$(49)\quad \begin{aligned}\text{Domain}(S_\infty) = \left\{f \in L_2(-\infty, +\infty) \,\middle|\, \int_{-\infty}^{+\infty} s^2 |f(s)|^2 ds < \infty\right\}, \\ (S_\infty f)(s) = sf(s)\,, \qquad f \in \text{Domain}(S_\infty)\,,\end{aligned}$$

we may write (48) as

$$(50)\quad (T_\infty U_1 f)(s) = (U_1 S_\infty f)(s) + \int_{-\infty}^{+\infty} L(s,\sigma)f(\sigma)d\sigma\,.$$

The proper course of argument is now obvious. The integral operator $L$ defined by the final term of the sum (50) is a regular integral operator with a Hölder continuous kernel vanishing rapidly at infinity. If $U_1$ is known to be invertible, we can write (50) as $U_1^{-1}T_\infty U_1 = S_\infty + U_1^{-1}L$. If $U_1^{-1}L$ has a kernel which is smooth, small and vanishes rapidly at infinity, then $T_\infty$ is similar to the sum of $S_\infty$ and a *regular* integral operator. By a result of Friedrichs [2] (cf. also Schwartz, [3], Section 3), $T_\infty$ is then similar to $S_\infty$.

To translate this outline into a valid proof, we have only to establish the following lemma.

LEMMA 5. *If the function* $K(s,s)$ *in* (38) *and* (39) *is sufficiently small and has derivatives of the first three orders bounded by a sufficiently small constant* $\varepsilon$, *then the mapping* $U_1$ *defined by* (39) *is bounded and has a bounded inverse*

(a) *as a map of* $L_2(-\infty, +\infty)$ *into itself,*

(b) *as a map of the space* $H_\gamma$ *of all functions with finite norm*

$$(51) \qquad \|f\|_\gamma = \max_{h,s} \left| \frac{f(s+h)-f(s)}{h^\gamma} \right| + \max_s |sf(s)|$$

*into itself.*

*Moreover,*

(a′) *the inverse of* $U_1$*, regarded as a map of* $L_2$ *into itself, has a bound depending only on* $\varepsilon$*, and remaining finite as* $\varepsilon \to 0$;

(b′) *the inverse of* $U_1$*, regarded as a map of* $H_\gamma$ *into itself, has a bound depending only on* $\varepsilon$ *and* $\gamma$*, and remaining finite as* $\varepsilon \to 0$.

(Note: in (51) we take $1 > \gamma > 0$.)

Proof: Let $\beta(s)$ be the inverse function of $\alpha(s)$; let $B$ be defined by $(Bf)(s) = \beta'(s)f(\beta(s))$. Set

$$(52) \qquad V_1 = B\frac{1}{2\pi i}\int_\infty^+ - \frac{1}{2\pi i}\int_\infty^- .$$

Using (9)–(12) and making an elementary change of variables, it follows that

$$(53) \qquad (V_1U_1f)(s) = \frac{1}{2\pi i}\int_\infty^- \frac{f(\sigma)}{s-\sigma}\,d\sigma - \frac{\beta'(s)}{2\pi i}\int_\infty^+ \frac{f(\sigma)}{\beta(s)-\beta(\sigma)}\,d\sigma .$$

The second integral on the right of (53) is, of course, to be defined as

$$(54) \qquad \lim_{\varepsilon\to 0+}\int_{-\infty}^{+\infty} \frac{f(\sigma)}{\beta(s)-\beta(\sigma)+i\varepsilon}\,d\sigma ,$$

the limit being in the mean square. Set

$$(54a) \qquad D(s,\sigma) = \left\{\frac{\beta'(s)(s-\sigma)}{(\beta(s)-\beta(\sigma))} - 1\right\}(s-\sigma)^{-1} ;$$

then (53) may be written, using (11), as

$$(55) \qquad (V_1U_1f)(s) = f(s) + \int_{-\infty}^{+\infty} D(s,\sigma)f(\sigma)d\sigma .$$

Let $\varepsilon$ be a bound for the function $K$ and its first two derivatives; by the symbol $\eta$ we shall denote any function of $\varepsilon$ which goes to zero with $\varepsilon$. It is clear that the function $D$ vanishes if both $|s|$ and $|\sigma|$ are sufficiently large; moreover, the quantity

$$(56) \qquad \sup_{-\infty<\sigma<\infty} (1+|\sigma|)\,\|D(\cdot,\sigma)\|_\gamma$$

is readily seen to be finite and to have the bound $\eta$. Thus the integral operator on the right of (55) has the bound $\eta$, whether it be considered as an integral operator in the space $L_2(-\infty,\infty)$ or in the space $H_\gamma$ defined under (b) of Lemma 5. It follows that $V_1U_1$ has an inverse $J$, with norm bounded by $1+\eta$. Set $V_2 = JV_1$; then $V_2$ is a left inverse for $U_1$.

Our lemma will be proved if only we can show that $V_2$ is also a right inverse for $U$ and estimate the norm of $V_2$. For this purpose, it is sufficient to show that $V_2 g = 0$ implies $g = 0$. Since the operator $J$ is plainly invertible, we have only to show that $V_1 g = 0$ implies $g = 0$. This we do as follows. Suppose that $V_1 g = 0$. Then, by (12), we have

$$\int_{\infty}^{+} \frac{d\sigma_1}{s - \sigma_1} \beta'(\sigma_1) \int_{\infty}^{+} \frac{g(\sigma)}{\beta(\sigma_1) - \sigma} d\sigma = 0 . \tag{57}$$

Making a change of variable, and replacing $s$ by $\alpha(s)$, we have consequently

$$\alpha'(s) \int_{\infty}^{+} \frac{d\sigma_1}{\alpha(s) - \alpha(\sigma_1)} \left\{ \int_{\infty}^{+} \frac{g(\sigma)}{\sigma_1 - \sigma} \right\} d\sigma = 0 . \tag{58}$$

Set

$$E(s, \sigma) = \left\{ \alpha'(s) \frac{s - \sigma}{\alpha(s) - \alpha(\sigma)} - 1 \right\} (s - \sigma)^{-1} . \tag{59}$$

Then, as above, we may conclude that

$$\sup_{-\infty < \sigma < +\infty} (1 + |\sigma|) \| E(\cdot, \sigma) \|_\gamma \tag{60}$$

has the bound $\eta$, and that (58) implies

$$\int_{\infty}^{+} \frac{g(\sigma)}{s - \sigma} + \int_{-\infty}^{+\infty} E(s, \sigma) \left\{ \int_{\infty}^{+} \frac{g(\sigma_1)}{\sigma - \sigma_1} \right\} d\sigma = 0 . \tag{61}$$

Since the norm (60) is small, it follows at once from (61) that

$$\int_{\infty}^{+} \frac{g(\sigma)}{s - \sigma} d\sigma = 0 . \tag{62}$$

On the other hand, by (52) and (12) we see that $V_1 g = 0$ implies

$$\int_{\infty}^{-} \frac{g(\sigma)}{s - \sigma} d\sigma = 0 . \tag{63}$$

Thus, by (11), $V_1 g = 0$ implies $g = 0$, and $V_2$ is therefore an inverse for $U_1$.

It only remains to estimate the norm of the transformation $V_2$. Since $V_2 = J V_1$ and $J$ is known to have a norm bounded by $1 + \eta$, it is sufficient to estimate the norm of $V_1$. Since the map $B$ in either of the spaces $L_2$ or $H_\gamma$ has plainly a norm dependent only on the bound $\varepsilon$ for the derivatives of $\beta(s) - s$, it follows from (52) that it is sufficient to show that $\int_{\infty}^{-}$ and $\int_{\infty}^{+}$ are bounded mappings in these spaces. This, however, is a well-known property of singular integrals, proved, for example, in [4], Section 3.

It is plain from (47), (44), and (43) and the lemmas on singular integrals of [4], Section 3, that the kernel $L$ of (50) has a norm

$$
\begin{aligned}
\| L \|_{\gamma,\gamma} = \max_{h,s} \max_{\mu,\sigma} & \left| \frac{L(s+h, \sigma+\eta) - L(s+h, \sigma) - L(s, \sigma+\eta) + L(s,\sigma)}{h^\gamma \mu^\gamma} \right| \\
& + \max_{h,s} \max_{\sigma} (1 + |\sigma|) \left| \frac{L(s+h,\sigma) - L(s,\sigma)}{h^\gamma} \right| \\
& + \max_{s} \max_{\mu,\sigma} (1 + |s|) \left| \frac{L(s,\sigma+\eta) - L(s,\sigma)}{\mu^\gamma} \right| \\
& + \max_{s} \max_{\sigma} (1 + |s|)(1 + |\sigma|) \, |L(s,\sigma)|
\end{aligned} \tag{64}
$$

bounded by $\eta$. By Lemma 5(b), the operator $U_1^{-1}L = L_1$ is also an integral operator with a norm $\| L_1 \|_{\gamma,\gamma}$ bounded by $\eta$. We have observed that

$$U_1^{-1} T_\infty U_1 = S_\infty + L_1 .$$

Thus $T_\infty^{(1)} = U_1^{-1} T_\infty U_1$ satisfies the hypotheses needed for the application of the Friedrichs theorem on perturbation of a multiplication operator by a regular integral operator.

Applying the theorem of Friedrichs in the form set forth in [4], Section 3, we obtain the following result.

THEOREM 6. *Let $K(s, \sigma)$ be a function with three continuous derivatives, vanishing if $|s| + |\sigma|$ is sufficiently large. Let $K(s, s)$ be real. Then, if $K(s, \sigma)$ and its first three derivatives have a sufficiently small upper bound $\varepsilon$, the operators $S_\infty$ and $T_\infty$ defined by*

$$
\begin{aligned}
&\text{domain}\,(S_\infty) = \left\{ f \in L_2(-\infty, +\infty) \,\middle|\, \int_{-\infty}^{+\infty} s^2 \, |f(s)|^2 ds < \infty \right\}, \\
&(S_\infty f)(s) = s f(s)
\end{aligned} \tag{65}
$$

*and*

$$
\begin{aligned}
&\text{domain}\,(T_\infty) = \text{domain}\,(S_\infty)\,, \\
&(T_\infty f)(s) = s f(s) + \frac{1}{2\pi i} \int_\infty^+ \frac{K(s,\sigma)}{s - \sigma} f(\sigma) d\sigma
\end{aligned} \tag{66}
$$

*are equivalent. Indeed, we may write $S_\infty = U_1^{-1} T_\infty U_2$, where $U = U_1 U_2$, and where*

$$(U_1 f)(s) = \frac{1}{2\pi i} \int_\infty^+ \frac{\alpha'(\sigma) f(\alpha(\sigma))}{s - \sigma} d\sigma - \frac{1}{2\pi i} \int_\infty^- \frac{f(\sigma)}{s - \sigma} d\sigma \,, \tag{67}$$

$$\alpha(s) = s - K(s, s)\,, \tag{68}$$

$$(U_2 f)(s) = f(s) + \int_\infty^+ \frac{H(s,\sigma)}{s - \sigma} f(\sigma) d\sigma \,, \tag{69}$$

*$H$ being a kernel whose norm $\| H \|_{\gamma,\gamma}$ (cf. (64)) is bounded by a function $\eta(\varepsilon)$ which approaches zero with $\varepsilon$.*

## 4. Scattering Theory

Theorem 6 enables us to discuss the scattering theory of the operator $T_\infty$ quite readily. This theory will in turn yield the small amount of additional information about the operator $U$ of that theorem which we need to be able to deduce the spectral representation of the operator $T$ of (1) from that of the closely related operator $T_\infty$.

First note that because $T_\infty$ is similar to the self-adjoint operator $S_\infty$, the family of operators $\exp\{itT_\infty\}$ are uniformly bounded in the parameter $t$. We shall show that the limit

$$\lim_{t\to\infty} \exp\{-itS_\infty\}\exp\{itT_\infty\} \tag{70}$$

exists in the strong topology of operators, and equals $U^{-1}$. We proceed as follows. By Theorem 6, this limit may be written as

$$\exp\{-itS_\infty\}U\exp\{itS_\infty\}U^{-1}\,. \tag{71}$$

Thus, we have only to show that

$$\lim_{t\to\infty} \exp\{-itS_\infty\}U\exp\{itS_\infty\} = I \tag{72}$$

in the strong topology of operators. But (72) is in turn a consequence of the two limiting relationships

$$\lim_{t\to\infty} \exp\{-itS_\infty\}U_1\exp\{itS_\infty\} = I \tag{73}$$

and

$$\lim_{t\to\infty} \exp\{-itS_\infty\}U_2\exp\{itS_\infty\} = I\,. \tag{74}$$

A strong limiting relationship among uniformly bounded operators holds everywhere if it holds on a dense set. Thus, to prove (73) we have only to show that

$$\lim_{t\to\infty} \frac{1}{2\pi i}\int_\infty^- \frac{\exp\{it(s-\sigma)\}}{s-\sigma} f(\sigma)d\sigma = f(s) \tag{75}$$

and

$$\lim_{t\to\infty} \frac{1}{2\pi i}\int_\infty^+ \frac{\exp\{it(s-\sigma)\}}{s-\sigma} f(\sigma)d\sigma = 0 \tag{76}$$

in the topology of $L_2$ for each $f$ in $C^\infty$ vanishing outside a finite interval (cf. (67)). Similarly, to prove (74) we have only to show that

$$\lim_{t\to\infty} \frac{1}{2\pi i}\int_\infty^+ \frac{H(s,\sigma)}{s-\sigma} \exp\{it(s-\sigma)\}f(\sigma)d\sigma = 0 \tag{77}$$

in the topology of $L_2$ for each $f\in C^\infty$ vanishing outside a finite interval (cf.

(69)). Statements (75) and (76) follow readily by use of the Fourier transformation. If we introduce

$$h(s, \sigma) = \frac{H(s, \sigma) - H(s, s)}{s - \sigma}, \tag{78}$$

then the integral in (77) may be written as

$$H(s, s)\int_{\infty}^{+}\frac{\exp\{it(s-\sigma)\}}{s-\sigma}f(\sigma)d\sigma + \exp\{its\}\int_{-\infty}^{+\infty}h(s, \sigma)f(\sigma)\exp\{-it\sigma\}d\sigma\,. \tag{79}$$

The first term in (79) approaches zero as $t \to \infty$ by (76). Since $\| H \|_{\gamma,\gamma} < \infty$, the kernel $h(s, \sigma)$ is of Hilbert-Schmidt class. Thus the function $h(\sigma, \sigma)f(\sigma)$, regarded as a function of $\sigma$ with values in the space $L_2(-\infty, +\infty)$ is integrable. The second term in (79) consequently approaches zero as $t \to \infty$ by the Riemann-Lebesgue lemma, generalized to vector-valued functions. (We remark that the conventional proof of this lemma via approximation of an arbitrary integrable function by step-functions holds as well for vector-valued as for scalar-valued functions.)

Thus we have verified that the limit (70) exists and equals $U^{-1}$.

It is even easier to show that the limit

$$\lim_{t\to\infty} \exp\{itT_\infty\}\exp\{-itS_\infty\} \tag{80}$$

exists as a strong limit. Since the parameter family of operators in (80) is uniformly bounded it is sufficient to show that the limit exists on a dense set of functions in $L_2(-\infty, +\infty)$. Suppose then that $f \in C^\infty$ and vanishes outside a bounded interval. Obviously, we only need to show that

$$\int_1^\infty \left|\frac{d}{dt}\exp\{itT_\infty\}\exp\{-itS_\infty\}f\right| dt < \infty\,; \tag{81}$$

i.e., to show that

$$\int_1^\infty |(T_\infty - S_\infty)\exp\{-itS_\infty\}f|\,dt < \infty\,. \tag{82}$$

For this, we need only a sufficiently exact bound for the norm of the expression

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{K(s, \sigma)\exp\{-it\sigma\}}{s-\sigma}f(\sigma)d\sigma \\ &= K(s, s)\int_{-\infty}^{+\infty}\frac{\exp\{-it\sigma\}f(\sigma)}{s-\sigma}d\sigma + \int_{-\infty}^{+\infty}k(s, \sigma)\exp\{-it\sigma\}f(\sigma)d\sigma\,.\end{aligned} \tag{83}$$

Since $k(s, \sigma)f(\sigma)$ vanishes outside a finite interval and is twice continuously differentiable, the maximum of the second integral on the right of (83) goes to zero as fast as $t^{-2}$, as $t \to \infty$; since the integral vanishes outside a fixed finite interval, the $L_2$ norm of the function which it defines goes to zero as fast as $t^{-2}$, as $t \to \infty$. The Fourier-transform $\hat{g}$ of the function defined by the first integral on the right of (83) is

$$\hat{g}(\xi) = \begin{cases} \hat{f}(\xi) & \text{if } \xi < -t\,, \\ 0 & \text{if } \xi > -t\,. \end{cases} \tag{84}$$

Since $\hat{f}(\xi)$ goes to zero faster than any power $|\xi|^{-N}$, as $|\xi| \to \infty$, it is clear that the $L_2$ norm of the first term on the right of (83) goes to zero equally rapidly. Thus (82) holds, and hence the limit (80) exists.

Since the product of two strong limits is the strong limit of the product, and since the limit (70) is known to equal $U^{-1}$, the limit (80) must equal $U$.

The limiting formulae (70) and (80) for the transformations $U^{-1}$ and $U$ which we have derived show that the common invariant subspace $L_2(0, 1)$ of the operators $T_\infty$ and $S_\infty$ must also be invariant under $U^{-1}$ and $U$. Let $V$ denote the restriction of $U$ to this subspace; plainly $V$ is a 1-1 map of $L_2(0, 1)$ onto itself, and equally plainly $V^{-1}TV = S$. This gives the spectral analysis of $T$.

We append a final remark on scattering theory. Our argument shows that, if the operator $T$ is defined by

$$\begin{aligned} T\colon f(s) \to m(s)f(s) &+ \frac{K(s,s)}{2\pi i} \lim_{\varepsilon \to 0+} \int_0^1 \frac{f(\sigma)}{s-\sigma}\, d\sigma \\ &+ \int_0^1 k(s,\sigma) f(\sigma)\, d\sigma\,, \end{aligned} \tag{85}$$

$m(s)$ being close to $s$, $K(s, s)$ being real, small and vanishing at $s = 0$ and $s = 1$, and the kernel $k$ being regular, then the limit

$$\lim_{t\to\infty} \exp\{itT\} \exp\{-itm(S)\} \tag{86}$$

exists in the strong topology of operators. As our argument shows (cf. (81)-(84) and the associated text), it is essential here that we let $t \to \infty$ and not $t \to -\infty$. Formulae (81)-(84) and the associated text make it plain that to obtain the corresponding limit as $t \to -\infty$, we must rewrite formula (85) so that the integral in the second term on the right appears as $I^-$ and not as $I^+$. This may readily be accomplished, using formula (11); we have

$$\begin{aligned} T\colon f(s) \to (m(s) - K(s,s))f(s) &+ \frac{K(s,s)}{2\pi i} \lim_{\varepsilon \to 0-} \int_0^1 \frac{f(\sigma)}{s-\sigma}\, d\sigma \\ &+ \int_0^1 k(s,\sigma) f(\sigma)\, d\sigma\,. \end{aligned} \tag{87}$$

Thus, as $t \to -\infty$, it is the limit

$$\lim_{t\to-\infty} \exp\{itT\} \exp\{-it(m(S) - K(S,S))\} \tag{88}$$

which exists in the strong topology, and not the limit (86). The scattering theory of the operator $T$ thus exhibits a curious asymmetry.

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