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An Asymptotic Formula for a Trigonometric Sum of Vinogradov

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We obtain a representation formula for the trigonometric sum $f(m,n) := \sum_{a=1}^{m-1} \frac{|\sin(\pi a n/m)|}{\sin(\pi a/m)}$ and deduce from it the upper bound $f(m,n) < (4/\pi^2)\, m \log m + (4/\pi^2)(\gamma - \log(\pi/2) + 2C_G)\, m + O(m/\sqrt{\log m})$, where C_G is the supremum of the function $G(t) := \sum_{k=1}^{\infty} \log|2 \sin \pi k t|/(4k^2-1)$, over the set of irrationals. The coefficients on both the main term and the second term are shown to be best possible. This improves earlier bounds for $f(m,n)$. It is conjectured that $C_G = G(\sqrt{2}) \approx 0.236$. We also obtain the following asymptotic formula: If α is a real algebraic integer of degree 2 with $0 < \alpha < 1$, then for any rational approximation n/m of α with $0 < n < m$ we have $f(m,n) = (4/\pi^2)\, m \log m + (4/\pi^2)(\gamma - \log(\pi/2) + 2G(\alpha))\, m + O_\alpha(|\alpha - \frac{n}{m}|^{1/2}\, m \log m)$. © 2001 Elsevier Science

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1. INTRODUCTION

The sum

$$f(m,n) := \sum_{a=1}^{m-1} \frac{|\sin(\pi a n/m)|}{\sin(\pi a/m)}$$

arises in bounding incomplete exponential sums and character sums and is used for example in the study of the number of quadratic residues in a given interval of integers. Estimates for $f(m,n)$ have been given by several authors. Vinogradov [5] proved that for $m > 60$, $f(m,n) < m \log m - m$

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Niederreiter [2] improved that result to $f(m,n)<\frac{2}{\pi}m\log m+\frac{2}{5}m+n$. Cochrane [1] proved the more precise upper bound

$$f(m,n)<\frac{4}{\pi^2}m\log m+0.5m+O(1) \tag{1}$$

and the mean value estimate

(2)

$$m^{-1}\sum_{n=1}^{m} f(m,n)=\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}(\gamma-\log(\pi/2))\,m+O(\log m\log\log m),$$

where $\gamma=0.57721...$ is Euler's constant. It is plain from (2) that the constant $4/\pi^2$ in the main term in (1) is best possible. Peral [3] gave an improvement on the m term in (1) by reducing it from $0.50m$ to $0.32m$.[1] Peral [3] and Yu [6] also made improvements on the error term in (2).

In this paper we give a representation formula and an asymptotic formula for $f(m,n)$. We also determine the best possible constant on the m term in (1). It turns out to be about 0.24, a value substantially larger than the constant $(4/\pi^2)(\gamma-\log(\pi/2))\approx 0.05$ appearing in (2). Finally, we sharpen the mean value upper bound of Yu [6].

2. STATEMENT OF RESULTS

Let $n<m$ be positive integers. For any positive integer k let $\rho(kn)$ denote the least nonnegative residue of kn modulo m, and define

$$R(kn)=\min\{\rho(kn),m-\rho(kn)\}.$$

Let $C(t)$, $G(t)$ be the functions defined by

$$C(t)=\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{1}{4k^2-1}=\frac{1}{2}-\sum_{\substack{k=1\\ kt\in\mathbb{Z}}}^{\infty}\frac{1}{4k^2-1},$$

$$G(t):=\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{\log|2\sin\pi kt|}{4k^2-1}.$$

In Section 5 we show that the series $G(t)$ converges for any algebraic number t.

[1] Yu [6, Theorem 1] purported to obtain a further improvement in the m term, to about $0.1m$, but there was an error in Lemma 2 of his paper.

PROPOSITION (Representation formula). *The function $f(m,n)$ satisfies*

(3)

$$f(m,n)=\frac{8}{\pi^2}C\left(\frac{n}{m}\right)\left(m\log m+m\left(\gamma-\log\frac{\pi}{2}\right)\right)+\frac{8m}{\pi^2}G\left(\frac{n}{m}\right)+E(m,n),$$

where the error term $E(m,n)$ satisfies

$$-\frac{4}{\pi^2}\leqslant E(m,n)\leqslant\frac{m}{3\pi^2}\sum_{\substack{k=1\\ m\nmid nk}}^{\infty}\frac{1}{R(kn)^2\,(4k^2-1)}+\frac{4}{\pi^2}. \tag{4}$$

We see from (3) that the constant $C(\frac{n}{m})$, a value between 0 and 1/2, measures the reduction in the coefficient of the main term ($m\log m$) from its mean value of $4/\pi^2$. In particular, if n/m reduces to a fraction with small denominator then the main term is substantially reduced. On the other hand, if $(m,n)=1$, so that $C(n/m)=\frac{1}{2}-\sum_{m\mid k}\frac{1}{4k^2-1}=\frac{1}{2}+O(\frac{1}{m^2})$, we obtain

$$f(m,n)=\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}\left(\gamma-\log(\pi/2)+2G\left(\frac{n}{m}\right)\right)m \\ +E(m,n)+O\left(\frac{\log m}{m}\right), \tag{5}$$

and thus we see that the constant $G(\frac{n}{m})$ measures the deviation in the coefficient of the m term from the value appearing in the mean value estimate (2). Of course, one must bear in mind that the error term $E(m,n)$ can also contribute to the m term if $R(kn)$ is small for some small value of k.

To obtain the best possible constant on the m term in (1) our interest is in determining how large G can be over the set of rationals or more precisely in determining the constant

$$C_G=\lim_{m\to\infty}\ \sup_{(n,m)=1}\ G\left(\frac{n}{m}\right).$$

In Section 5 we show that C_G exists and that it is just the supremum of G over the set of irrationals. Numerical estimates have led us to conjecture that the supremum C_G is attained at the value $t=\sqrt{2}$ and so we would have

$$C_G=G(\sqrt{2})=0.2359\ldots.$$

In Section 8 we give another series representation for G, but it is not clear whether it will be of use in resolving this conjecture.

THEOREM 1. (a) *For any positive integers $n<m$ we have*

$$f(m,n)<\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}(\gamma-\log(\pi/2)+2C_G)\,m+O\left(\frac{m}{\sqrt{\log m}}\right).$$

(b) *The function $f(m,n)$ satisfies the following mean value estimate:*

$$\frac{1}{m}\sum_{n=1}^{m}f(m,n)<\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}m\left(\gamma-\log\frac{\pi}{2}\right)+\frac{2}{\pi}-\frac{2}{\pi^2}.$$

On account of the lower bound in (5), using $E(m,n)\geqslant -4/\pi^2$, it follows that the constant $(4/\pi^2)\,(\gamma-\log(\pi/2)+2C_G)\approx 0.24...$ on the m term in Theorem 1a is best possible. Yu [6] had the same upper bound as Theorem 1b, but with the constant $(4/\pi)-(2/\pi^2)-(\pi/6m)(1-(1/m))$, in place of $2/\pi-2/\pi^2$.

We also obtain the following asymptotic formula for $f(m,n)$.

THEOREM 2. *Suppose that α is a real algebraic integer of degree 2 with $0<\alpha<1$. Then for any rational approximation n/m of α with $0<n<m$ we have*

$$f(m,n)=\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}(\gamma-\log(\pi/2)+2G(\alpha))\,m$$
$$+O_\alpha\left(\left|\alpha-\frac{n}{m}\right|^{1/2}m\log m\right).$$

(The subscript α indicates a constant depending on α.)

In particular, if n/m is a best possible rational approximation of α, so that $|\frac{n}{m}-\alpha|<1/(2m^2)$, then

$$f(m,n)=\frac{4}{\pi^2}m\log m+\frac{4}{\pi^2}(\gamma-\log(\pi/2)+2G(\alpha))\,m+O_\alpha(\log m).$$

3. AUXILIARY LEMMAS

LEMMA 1. *For positive integers N we have*

$$\sum_{k=1}^{N}\frac{1}{2k-1}=\frac{\log N}{2}+\frac{\gamma}{2}+\log 2+O\left(\frac{1}{N^2}\right),$$

where the function in the big "Oh" satisfies

$$0\leqslant O\left(\frac{1}{N^2}\right)\leqslant\frac{1}{48N^2}.$$

Proof. By the Euler–MacLaurin expansion for the harmonic series we have

$$\sum_{n=1}^{N} \frac{1}{k} = \log N + \gamma + \frac{1}{2N} - \frac{B_2}{8N^2} - \frac{B_4}{64N^4} - \theta \frac{B_6}{384N^6},$$

where the B_i are the Bernoulli numbers $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$ and $0 < \theta < 1$; see for example Rademacher [4, (16.1)]. Thus

$$\sum_{k=1}^{N} \frac{1}{2k-1} = \sum_{k=1}^{2N} \frac{1}{k} - \frac{1}{2} \sum_{k=1}^{N} \frac{1}{k}$$
$$= \frac{1}{2} \log N + \log 2 + \frac{1}{2} \gamma + \frac{1}{48N^2} + \frac{B_4}{8} \left(\frac{1}{N^4} - \frac{1}{8N^4} \right)$$
$$+ \frac{B_6}{12} \left(\frac{\theta}{N^6} - \frac{\theta'}{32N^6} \right),$$

with $0 < \theta, \theta' < 1$. Since the last three terms are alternating in sign and decreasing in absolute value the lemma follows.

The following lemma sharpens Lemma 3.1 of Cochrane [1] and Lemma 1 of Yu [6].

LEMMA 2. *Let $m > 1$ be a natural number, then*

$$\sum_{a=1}^{m-1} \frac{1}{\sin\left(\frac{\pi a}{m}\right)} < \frac{2m}{\pi} \log m + \frac{2m}{\pi} \left(\gamma - \log \frac{\pi}{2} \right) + 1 - \frac{1}{\pi}.$$

Proof. Assume first that m is even. Then

$$\sum_{a=1}^{m-1} \frac{1}{\sin\left(\frac{\pi a}{m}\right)} = 2 \sum_{a=1}^{m/2} \frac{1}{\sin\left(\frac{\pi a}{m}\right)} - 1$$
$$= -1 + 2 \sum_{a=1}^{m/2} \frac{m}{\pi a} + 2 \sum_{a=1}^{m/2} \left\{ \frac{1}{\sin\left(\frac{\pi a}{m}\right)} - \frac{m}{\pi a} \right\}$$
$$= -1 + \frac{2m}{\pi} \left(\log(m/2) + \gamma + \frac{1}{m} + O\left(\frac{1}{m^2} \right) \right)$$
$$+ \frac{2m}{\pi} \left[\frac{\pi}{m} \sum_{a=1}^{m/2} \left\{ \frac{1}{\sin\left(\frac{\pi a}{m}\right)} - \frac{m}{\pi a} \right\} \right],$$

where the O term is negative and so we will ignore it in what follows. Now the function

$$\frac{1}{\sin t}-\frac{1}{t}$$

is positive and increasing for $t \in [0, \frac{\pi}{2}]$ and so we have the following bound for the inner sum,

$$\frac{\pi}{m}\sum_{a=1}^{m/2}\left\{\frac{1}{\sin\left(\frac{\pi a}{m}\right)}-\frac{m}{\pi a}\right\}<\int_0^{\pi/2}\left(\frac{1}{\sin t}-\frac{1}{t}\right)dt+\left(1-\frac{2}{\pi}\right)\frac{\pi}{m}$$

$$=-\log \pi/4+\left(1-\frac{2}{\pi}\right)\frac{\pi}{m}.$$

Collecting all of the terms we get

$$\sum_{a=1}^{m-1}\frac{1}{\sin\left(\frac{\pi a}{m}\right)}<\frac{2m}{\pi}\left(\log\frac{m}{2}+\gamma+\frac{1}{m}-\log\frac{\pi}{4}\right)+2\left(1-\frac{2}{\pi}\right)-1$$

$$=\frac{2m}{\pi}\log m+\frac{2m}{\pi}\left(\gamma-\log\frac{\pi}{2}\right)+1-\frac{2}{\pi}.$$

The case of odd m is similar and so we leave the details to the reader.

LEMMA 3. *Let R be a natural number with $R \leqslant m/2$, and define*

$$S(R)=\sum_{h=1}^{R}\cot\left(\frac{(2h-1)\,\pi}{2m}\right).$$

Then

$$\text{(a)}\qquad S(R)=\frac{m\log m}{\pi}+\frac{m}{\pi}(\gamma-\log(\pi/4))+\frac{m}{\pi}\log\left|\sin\frac{R\pi}{m}\right|+E(R),$$

where

$$-\frac{1}{\pi}\leqslant E(R)\leqslant\frac{m}{24\pi R^2}+\frac{1}{\pi}.$$

In particular we have

$$\text{(b)}\quad \frac{m}{\pi}\log m+\frac{m}{\pi}(\gamma-\log(\pi/4))-\frac{1}{\pi} \leqslant S(m/2) \leqslant \frac{m}{\pi}\log m+\frac{m}{\pi}(\gamma-\log(\pi/4))+\frac{1}{\pi}+\frac{1}{6\pi m}.$$

Proof. Write

$$S(R) = \sum_{h=1}^{R}\left\{\cot\left(\frac{(2h-1)\pi}{2m}\right)-\frac{2m}{\pi(2h-1)}\right\}+\sum_{h=1}^{R}\frac{2m}{\pi(2h-1)} := S_1(R)+S_2(R),$$

say. According to Lemma 1 we have

$$S_2(R)=\frac{2m}{\pi}\left(\frac{1}{2}\log R+\frac{\gamma}{2}+\log 2+O(1/R^2)\right),$$

where $0\leqslant O(1/R^2)\leqslant 1/48R^2$, and so we get,

$$S_2(R)=\frac{m}{\pi}(\log R+\gamma+\log 4)+O\left(\frac{m}{R^2}\right)$$

with

$$0\leqslant O\left(\frac{m}{R^2}\right)\leqslant\frac{m}{24\pi R^2}.$$

Now write

$$S_1(R)=\frac{m}{\pi}\left[\frac{\pi}{m}\sum_{h=1}^{R}\left\{\cot\left(\frac{(2h-1)\pi}{2m}\right)-\frac{2m}{\pi(2h-1)}\right\}\right].$$

The latter sum is just a Riemann sum for the function $\cot x-\frac{1}{x}$ and thus since the function is negative, decreasing and bounded by $-\frac{2}{\pi}$ in $(0,\frac{\pi}{2})$ we get

$$\int_0^{\frac{(2R+1)\pi}{2m}}\left(\cot x-\frac{1}{x}\right)dx<\left[\frac{\pi}{m}\sum_{h=1}^{R}\left\{\cot\left(\frac{(2h-1)\pi}{2m}\right)-\frac{2m}{\pi(2h-1)}\right\}\right] <\int_0^{\frac{(2R-1)\pi}{2m}}\left(\cot x-\frac{1}{x}\right)dx.$$

We also have

$$\int_0^{\frac{(2R+1)\pi}{2m}}\left(\cot x-\frac{1}{x}\right)dx=\int_0^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx+\int_{\frac{R\pi}{m}}^{\frac{(2R+1)\pi}{2m}}\left(\cot x-\frac{1}{x}\right)dx$$
$$>\int_0^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx-\frac{2}{\pi}\frac{\pi}{2m}$$

and

$$\int_0^{\frac{(2R-1)\pi}{2m}}\left(\cot x-\frac{1}{x}\right)dx=\int_0^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx-\int_{\frac{(2R-1)\pi}{2m}}^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx$$
$$<\int_0^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx+\frac{2}{\pi}\frac{\pi}{2m}.$$

Now

$$\int_0^{\frac{R\pi}{m}}\left(\cot x-\frac{1}{x}\right)dx=\left(\log\frac{\sin x}{x}\right]_0^{\frac{R\pi}{m}}$$
$$=\log\left(\sin\frac{R\pi}{m}\right)+\log(m)-\log(\pi)-\log(R),$$

and so we get

$$\frac{m}{\pi}\left[\log\left(\sin\frac{R\pi}{m}\right)+\log(m)-\log(\pi)-\log(R)\right]-\frac{1}{\pi}$$
$$<S_1(R)<\frac{m}{\pi}\left[\log\left(\sin\frac{R\pi}{m}\right)+\log(m)-\log(\pi)-\log(R)\right]+\frac{1}{\pi}.$$

Thus we obtain

$$S(R)=S_1(R)+S_2(R)$$
$$=\frac{m\log m}{\pi}+\frac{m}{\pi}\left[\log\left(\sin\frac{R\pi}{m}\right)-\log(\pi)-\log(R)\right]$$
$$+O(1)+\frac{m}{\pi}(\log R+\gamma+\log 4)+O\left(\frac{m}{R^2}\right)$$
$$=\frac{m\log(m)}{\pi}+\frac{m}{\pi}(\gamma-\log\pi/4)+\frac{m}{\pi}\log\left|\sin\frac{R\pi}{m}\right|+O(m/R^2)+O(1),$$

where $0\leqslant O(m/R^2)\leqslant m/(24\pi R^2)$ and $-\frac{1}{\pi}\leqslant O(1)\leqslant\frac{1}{\pi}$.

4. PROOF OF THE PROPOSITION

As in [1], [3], and [6], we start with the Fourier development of $|\sin x|$ given by

$$|\sin x| = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} \sin^2(kx).$$

We also use the following facts

$$\frac{\sin^2(rx)}{\sin x} = \sin x + \sin 3x + \cdots + \sin(2r-1)\, x$$

and

$$\sum_{s=1}^{n} \sin sx = \frac{\cos \frac{1}{2} x - \cos(n+\frac{1}{2})\, x}{2 \sin \frac{1}{2} x}.$$

It follows readily that

$$f(m, n) = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1} S(m, n, k), \tag{7}$$

where

$$S(m, n, k) := \sum_{h=1}^{kn} \cot\left(\frac{(2h-1)\,\pi}{2m}\right). \tag{8}$$

Due to the symmetry of the cotangent function with respect to $\frac{\pi}{2}$ one has

$$\sum_{h=1}^{m} \cot\left(\frac{(2h-1)\,\pi}{2m}\right) = 0,$$

and so

$$S(m, n, k) = \sum_{h=1}^{R(kn)} \cot\left(\frac{(2h-1)\,\pi}{2m}\right) = S(R(kn)), \tag{9}$$

where $R(kn)$ is as defined in Section 2 and $S(R(kn))$ is as defined in Lemma 3. We define $S(kn) := S(R(kn))$ and put $S(0) = 0$. It follows that

$$f(m, n) = \frac{8}{\pi} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{S(kn)}{4k^2-1}. \tag{10}$$

Now, by Lemma 3, if $m \nmid kn$ then

$$S(kn) = \frac{m\ln(m)}{\pi} + \frac{m}{\pi}(\gamma - \ln \pi/4) + \frac{m}{\pi}\ln\left|\sin\frac{kn\pi}{m}\right| + E(kn),$$

where $E(kn) := E(R(kn))$ satisfies

$$-\frac{1}{\pi} \leqslant E(kn) \leqslant \frac{m}{24\pi R(kn)^2} + \frac{1}{\pi}. \tag{11}$$

Thus we obtain

(12)

$$\begin{aligned} f(m,n) &= \frac{8}{\pi^2} C(n/m)\left(m\log m + m\left(\gamma - \log\frac{\pi}{4}\right)\right) \\ &\quad + \frac{8m}{\pi^2} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{\ln|\sin(kn\pi/m)|}{4k^2-1} + E(m,n) \\ &= \frac{8}{\pi^2} C(n/m)\left(m\log m + m\left(\gamma - \log\frac{\pi}{2}\right)\right) + \frac{8m}{\pi^2} G(n/m) + E(m,n), \end{aligned}$$

where G and C are as defined in Section 2 and where

$$E(m,n) = \frac{8}{\pi} \sum_{\substack{k=1 \\ m \nmid kn}}^{\infty} \frac{E(kn)}{4k^2-1}.$$

The Proposition follows from (11) and (12).

5. THE FUNCTIONS $F(t)$ AND $G(t)$

Our interest in this section is the study of the behavior of the function $G(t)$ defined in Section 2, but we shall do this by studying the related function

$$F(t) := \sum_{kt \notin \mathbb{Z}} \frac{\log|\sin kt\pi|}{4k^2-1},$$

which is a slightly easier task because all of the terms in the latter sum are nonpositive. We have

$$G(t) = F(t) + (\log 2)\, C(t), \tag{13}$$

where $C(t)$ is as defined in Section 2. In particular if we let C_F be defined by

$$C_F = \lim_{m \to \infty} \sup_{(n,m)=1} F\left(\frac{n}{m}\right)$$

and let C_G be the analagous supremum for G then it is easy to see that

$$C_G = C_F + \frac{\log 2}{2}.$$

Now, F is periodic modulo 1 and $F(1-t) = F(t)$ and so we may restrict our attention to the interval $(0, \frac{1}{2})$. We note that for any algebraic number t the series $F(t)$ converges to a finite value. This is obvious if t is rational. If t is algebraic of degree $d \geqslant 2$ then by Liouville's theorem there exists a constant $c = c(t)$ such that

$$\left| t - \frac{a}{k} \right| > \frac{c}{k^d}$$

for any rational number a/k. For any positive integer k let a be the positive integer such that $|kt-a| < 1/2$. Then $|kt-a| = k\,|t-\frac{a}{k}| \geqslant c/k^{d-1}$, and so

$$|\sin(kt\pi)| = |\sin(a\pi + (kt-a)\,\pi)| = |\sin(kt-a)\,\pi| \geqslant \sin\left(\frac{c\pi}{k^{d-1}}\right) \geqslant \frac{c}{k^{d-1}}.$$

Thus

$$\ln|\sin(kt\pi)| \geqslant \ln c - (d-1)\ln k,$$

and so the series $F(t)$ converges. There are, however, transcendental values of t where the series $F(t)$ diverges.

The behavior of $F(t)$ is very wild. If t is a rational number and $\{t_n\}$ is any sequence of real numbers converging to t with the $t_n \neq t$ for all n, then $F(t_n) \to -\infty$. Indeed, if $t = \frac{a}{m}$, then the term $k = m$ in the definition of $F(t_n)$ tends to $-\infty$. However, if $\{t_n\}$ is a sequence of rationals converging to an algebraic number t of degree at least 2 then, provided it converges "sufficiently fast", it can be shown that $F(t_n) \to F(t)$. Here we prove

Lemma 4. *Suppose that α is a real algebraic integer of degree 2 with conjugate $\bar{\alpha}$ and that n/m is any rational number. Then*

$$\left| F(\alpha) - F\left(\frac{n}{m}\right) \right| \ll \sqrt{|\alpha| + |\bar{\alpha}|}\,\left| \alpha - \frac{n}{m} \right|^{1/2} (\log m + \log(|\alpha| + |\bar{\alpha}|)).$$

Proof. We may assume that $(m, n) = 1$. Let $f(x) = x^2 + ax + b = (x-\alpha)(x-\bar{\alpha})$ be the minimal polynomial for α. For any rational number l/k we have

$$\left|\alpha - \frac{l}{k}\right| > \frac{c(\alpha)}{k^2}, \tag{14}$$

where the constant $c(\alpha)$ may be taken as $1/(1+|\alpha|+|\bar{\alpha}|)$. Set $\delta = |\alpha - \frac{n}{m}|$. If $k < (c(\alpha)/2\delta)^{1/2}$ and l is the nearest integer to $k\alpha$ then we have

$$|k\alpha - l| = k\left|\alpha - \frac{l}{k}\right| > \frac{c(\alpha)}{k} > 2k\delta = 2\left|k\alpha - k\frac{n}{m}\right|.$$

In particular for such k there is no integer between $k\alpha$ and $k\frac{n}{m}$, and so by the mean value theorem,

$$|\log|\sin(k\alpha\pi)| - \log|\sin(kn\pi/m)|\,| \leqslant \cot\left(\frac{c(\alpha)\,\pi}{2k}\right)\left|k\alpha\pi - k\frac{n}{m}\pi\right| \leqslant 2k^2\delta/c(\alpha).$$

It follows that

$$\begin{aligned}|F(n/m) - F(\alpha)| &\ll \frac{\delta}{c(\alpha)} \sum_{k<(c(\alpha)/2\delta)^{1/2}} \frac{k^2}{4k^2-1} \\ &\quad + \sum_{k \geqslant (c(\alpha)/2\delta)^{1/2}} \frac{|\log|\sin(kn\pi/m)|\,|}{4k^2-1} \\ &\quad + \sum_{k \geqslant (c(\alpha)/2\delta)^{1/2}} \frac{|\log|\sin(k\alpha\pi)|\,|}{4k^2-1} \\ &\ll \frac{\delta}{c(\alpha)}(c(\alpha)/\delta)^{\frac{1}{2}} + \sum_{k \geqslant (c(\alpha)/2\delta)^{1/2}} \frac{\log m + \log k - \log c(\alpha)}{4k^2-1} \\ &\ll \left(\frac{\delta}{c(\alpha)}\right)^{1/2} \log\frac{m}{c(\alpha)}.\end{aligned}$$

LEMMA 5. *Let n/m be a rational number with $0 < n/m < 1/2$ and $(n, m) = 1$. Then there exists a real algebraic integer α of degree 2 with $0 < \alpha < 1/2$, $-(m+\frac{1}{2}) < \bar{\alpha} < 0$ and*

$$|F(\alpha) - F(n/m)| \ll \frac{\log m}{\sqrt{m}}.$$

Proof. Since $(nm, m^2) = m$ there exist positive integers a, b such that

$$0 < |n^2 + anm - bm^2| < m/2. \tag{15}$$

We let a, b be the solution of (15) with a, b both positive and a minimal. Let $p(x) = x^2 + ax - b = (x - \alpha)(x - \bar{\alpha})$, say. Since the discriminant of $p(x)$ is positive the roots α, $\bar{\alpha}$ are real. We let α be the positive root and $\bar{\alpha}$ the negative root. Now by (15) we have

$$\left|\frac{n}{m} - \alpha\right| \left|\frac{n}{m} - \bar{\alpha}\right| = \left|p\left(\frac{n}{m}\right)\right| < \frac{1}{2m}, \tag{16}$$

whence

$$\left|\frac{n}{m} - \alpha\right| < \frac{1}{2m\,|\bar{\alpha}|}. \tag{17}$$

We claim that $0 < \alpha < \frac{1}{2}$. Indeed, if $\alpha \geqslant 1$ then since $0 < \frac{n}{m} < \frac{1}{2}$ it follows that $|\frac{n}{m} - \alpha|\,|\frac{n}{m} - \bar{\alpha}| \geqslant \frac{1}{2} \geqslant \frac{1}{2m}$, contradicting (16). If $\frac{1}{2} \leqslant \alpha < 1$ then $\bar{\alpha} = -b/\alpha < -1$ and thus since $0 < \frac{n}{m} \leqslant \frac{1}{2} - \frac{1}{2m}$, we have $|\frac{n}{m} - \alpha|\,|\frac{n}{m} - \bar{\alpha}| \geqslant \frac{1}{2m}$, again in violation of (16). In particular, it follows that α is not in $\mathbb{Z}$ and therefore is an algebraic integer of degree 2.

In order to determine the size of $\bar{\alpha}$ we observe that since a is a minimal solution of (15) we have either $a < m$ or $b < n$. If $a < m$ then since $\alpha + \bar{\alpha} = -a$ we have $|\bar{\alpha}| = a + \alpha < m + \frac{1}{2}$. Suppose now that $b < n$. Then since $|\alpha\bar{\alpha}| = b < n$ and, by (17), $\alpha > \frac{n}{m} - \frac{1}{2m|\bar{\alpha}|}$ we have

$$|\bar{\alpha}| < \frac{n}{\alpha} < \frac{n}{\dfrac{n}{m} - \dfrac{1}{2m\,|\bar{\alpha}|}}.$$

It follows that $|\bar{\alpha}| < m + \frac{1}{2n} \leqslant m + \frac{1}{2}$. Applying Lemma 4 to this choice of α and using (17) yields Lemma 5.

It follows from Lemmas 4 and 5 that the constant

$$C_F := \lim_{m \to \infty} \sup_{(n,m)=1} F(n/m),$$

exists and is equal to the supremum of F over the set $\mathscr{R}$ of real algebraic integers of degree 2. Consider now the truncated series $F_K(t) = \sum_{k=1}^{K} \log|\sin kt\pi|/(4k^2 - 1)$. For any value of K, F_K is continuous on the set of irrationals $\mathscr{I}$ and thus since $\mathscr{R}$ is a dense subset of $\mathscr{I}$ we have

$$C_F = \sup_{t \in \mathscr{R}} F(t) = \lim_{K \to \infty} \sup_{t \in \mathscr{R}} F_K(t) = \lim_{K \to \infty} \sup_{t \in \mathscr{I}} F_K(t) = \sup_{t \in \mathscr{I}} F(t),$$

that is, C_F is just the supremum of F over the set of all irrationals.

6. PROOF OF THEOREM 1

LEMMA 6. *For any positive integers $n < m$ we have either*

(a)
$$f(m, n) \leqslant \frac{4}{\pi^2} m \log m + O(1)$$

or

(b)
$$f(m, n) = \frac{8}{\pi^2} C(n/m) \left(m \log m + m \left(\gamma - \log \frac{\pi}{4} \right) \right) + \frac{8m}{\pi^2} F\left(\frac{n}{m}\right) + O\left(\frac{m}{\sqrt{\log m}}\right).$$

Proof. Define the sets

$$A = \left\{ n \in \{ 1, 2, 3 \cdots, m \} \;\middle|\; \exists k \leqslant \frac{\sqrt{\log m}}{2} \text{ such that } R(kn) \leqslant \frac{\sqrt{m}}{\pi} \right\}$$

and

$$B = \{ 1, 2, 3 \cdots, m \} - \{A\}.$$

Suppose first that $n \in A$. Then there exists $k_0 \leqslant \sqrt{\log m}/2$ such that $R(k_0 n) \leqslant \sqrt{m}/\pi$. Using the inequality $\cot x < 1/x$ and then Lemma 1, we have that if $m \nmid k_0 n$ then

$$\begin{aligned} S(k_0 n) &= \sum_{h=1}^{R(k_0 n)} \cot\left(\frac{(2h-1)\pi}{2m}\right) \leqslant \sum_{h=1}^{[\frac{\sqrt{m}}{\pi}]} \cot\left(\frac{(2h-1)\pi}{2m}\right) \\ &< \frac{2m}{\pi} \sum_{h=1}^{[\frac{\sqrt{m}}{\pi}]} \frac{1}{(2h-1)} \\ &< \frac{2m}{\pi} \left[\frac{1}{2} \log\left(\frac{\sqrt{m}}{\pi}\right) + \frac{\gamma}{2} + \log 2 + \frac{\pi^2}{48m} \right] \\ &= \frac{m \log m}{\pi} + \frac{m}{\pi} \left(\gamma - \log\left(\frac{\pi}{4}\right) \right) + \frac{\pi}{24} - \frac{m \log m}{2\pi}, \end{aligned}$$

and thus by Lemma 3b we obtain

(18)
$$S(k_0 n) < S(m/2) + \frac{1}{\pi} + \frac{\pi}{24} - \frac{m}{2\pi} \log m.$$

Now, since $S(kn) \leqslant S(m/2)$ for all values of k we obtain from (18) and (10) that

$$f(m, n) = \frac{8}{\pi} \sum_{m \nmid kn} \frac{S(kn)}{4k^2-1} \leqslant \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{S(m/2)}{4k^2-1} - \frac{4m}{\pi^2} \frac{\log m}{4k_0^2-1} + O(1).$$

We note that the latter inequality holds as well if $m \mid k_0 n$. Using Lemma 3b and the fact that $k_0 \leqslant \sqrt{\log m}/2$ we get

$$f(m, n) \leqslant \frac{4}{\pi^2} m \log m + \frac{4}{\pi^2}\left(\gamma - \log\left(\frac{\pi}{4}\right) - 1\right) + O(1) < \frac{4}{\pi^2} m \log m + O(1).$$

Suppose now that $n \in B$. Then, for any $k \leqslant \sqrt{\log m}/2$ we have $R(kn) \geqslant \sqrt{m}/\pi$. Now, by the proposition,

$$f(m, n) = \frac{8}{\pi^2} C(n/m)\left(m \log m + m\left(\gamma - \log \frac{\pi}{4}\right)\right) + \frac{8m}{\pi^2} F\left(\frac{n}{m}\right) + E(m, n),$$

where

$$E(m, n) \leqslant \frac{m}{3\pi^2} \sum_{\substack{k=1 \\ m \nmid kn}} \frac{1}{R(kn)^2\,(4k^2-1)} + 1.$$

It follows that

$$\begin{aligned} E(m, n) &\leqslant \frac{m}{3\pi^2} \sum_{k \leqslant \frac{\sqrt{\log m}}{2}} \frac{\pi^2}{m(4k^2-1)} + \frac{m}{3\pi^2} \sum_{k > \frac{\sqrt{\log m}}{2}} \frac{1}{4k^2-1} + 1 \\ &\leqslant \frac{1}{6} + \frac{m}{3\pi^2 \sqrt{\log m}} + 1. \end{aligned}$$

Lemma 7. *For any positive integers $n < m$ we have*

$$\frac{1}{m} \sum_{h=1}^{m} |\sin(\pi h n/m)| = \frac{4}{\pi} C\left(\frac{n}{m}\right) < \frac{2}{\pi}.$$

Proof. Using the Fourier exansion for $|\sin t|$ given in Section 4 we have

$$\sum_{h=1}^{m} |\sin(\pi h n/m)| = \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{4k^2-1}\left(\sum_{h=1}^{m} \sin^2(\pi h k n/m)\right).$$

The lemma now follows from the identity,

$$\sum_{h=1}^{m} \sin^2(\pi hkn/m) = \begin{cases} \dfrac{m}{2} & \text{if } \ k\dfrac{n}{m} \notin \mathbb{Z}, \\ 0 & \text{if } \ k\dfrac{n}{m} \in \mathbb{Z}. \end{cases}$$

Proof of Theorem 1a. If n/m satisfies alternative (a) in Lemma 6 then we are done and so we may assume that alternative (b) holds, that is

$$f(m, n) = \frac{8}{\pi^2} C(n/m)\left(m \log m + m\left(\gamma - \log\frac{\pi}{4}\right)\right) + \frac{8m}{\pi^2} F\left(\frac{n}{m}\right) + O\left(\frac{m}{\sqrt{\log m}}\right).$$

Let α be an algebraic integer of degree 2 satisfying Lemma 5. Then we have

$$\begin{aligned} f(m, n) &= \frac{8}{\pi^2} C(n/m)\left(m \log m + m\left(\gamma - \log\frac{\pi}{4}\right)\right) \\ &\quad + \frac{8m}{\pi^2} F(\alpha) + O(\sqrt{m} \log m) + O\left(\frac{m}{\sqrt{\log m}}\right) \\ &\leqslant \frac{4}{\pi^2}\left(m \log m + m\left(\gamma - \log\frac{\pi}{4}\right)\right) + \frac{8m}{\pi^2} C_F + O\left(\frac{m}{\sqrt{\log m}}\right). \end{aligned}$$

The theorem follows from the fact that $C_F = C_G - \frac{\log 2}{2}$.

Proof of Theorem 1b. We have

$$\sum_{n=1}^{m-1} f(m, n) = \sum_{n=1}^{m-1} \sum_{a=1}^{m-1} \frac{|\sin(\pi an/m)|}{\sin(\pi a/m)},$$

and so by Lemma 7,

$$\sum_{n=1}^{m-1} f(m, n) < \frac{2m}{\pi} \sum_{a=1}^{m-1} \frac{1}{\sin(\pi a/m)}.$$

Using Lemma 2 we get

$$\begin{aligned} \sum_{n=1}^{m-1} f(m, n) &< \frac{2m}{\pi}\left(\frac{2m}{\pi} \log m + \frac{2m}{\pi}\left(\gamma - \log\frac{\pi}{2}\right) + 1 - \frac{1}{\pi}\right) \\ &= \frac{4m^2 \log m}{\pi^2} + \frac{4m^2}{\pi^2}\left(\gamma - \log\frac{\pi}{2}\right) + \frac{2m}{\pi} - \frac{2m}{\pi^2}. \end{aligned}$$

7. PROOF OF THEOREM 2

Let α be a real algebraic integer of degree 2 with $0<\alpha<1$ and let n, m be positive integers with $0<n<m$. As in the proof of Lemma 4 we set $\delta=|\alpha-\frac{n}{m}|$ and let $c(\alpha)$ be the constant in (14). Suppose that k is a positive integer with $k<(c(\alpha)/2\delta)^{1/2}$ and that $kn\equiv j \pmod m$ with $|j|\leqslant m/2$, say $kn=j+lm$. Then

$$\frac{c(\alpha)}{k^2}<\left|\alpha-\frac{l}{k}\right|\leqslant\left|\alpha-\frac{n}{m}\right|+\left|\frac{n}{m}-\frac{l}{k}\right|=\delta+\frac{|j|}{mk},$$

and so

$$\frac{|j|}{mk}>\frac{c(\alpha)}{k^2}-\delta>\frac{c(\alpha)}{2k^2},$$

that is $|j|>mc(\alpha)/(2k)$. It follows from (11) that for $k<(c(\alpha)/2\delta)^{1/2}$, we have

$$E(kn)=E(|j|)\ll\frac{m}{j^2}\ll_\alpha\frac{k^2}{m},$$

and hence

$$E(m,n):=\frac{8}{\pi}\sum_{m\nmid kn}\frac{E(kn)}{4k^2-1}\ll_\alpha\frac{1}{m}\sum_{k<(c(\alpha)/2\delta)^{1/2}}1+m\sum_{k>(c(\alpha)/2\delta)^{1/2}}\frac{1}{k^2},$$

that is,

$$E(m,n)\ll_\alpha\frac{1}{m\delta^{1/2}}+m\delta^{1/2}\ll_\alpha m\delta^{1/2}, \tag{19}$$

the last inequality following from $\delta=|\alpha-\frac{n}{m}|\geqslant c(\alpha)/m^2$. Thus by (12), (19), and Lemma 4,

$$f(m,n)=\frac{8}{\pi^2}C\left(\frac{n}{m}\right)(m\log m+(\gamma-\log(\pi/4))\,m)+\frac{8m}{\pi^2}F(\alpha)$$
$$+O_\alpha(m\delta^{1/2}\log m).$$

Suppose that $n/m=a/b$ where a, b are positive integers with $(a,b)=1$. Then since

$$\frac{c(\alpha)}{b^2}<\left|\alpha-\frac{a}{b}\right|=\delta,$$

we have

$$C\left(\frac{n}{m}\right)=\frac{1}{2}-\sum_{b\mid k}\frac{1}{4k^2}-1=\frac{1}{2}+O_\alpha(\delta),$$

and thus since $\delta<1$ we get

$$f(m,n)=\frac{4}{\pi^2}\,(m\log m+(\gamma-\log(\pi/4))\,m)+\frac{8m}{\pi^2}\,F(\alpha)+O_\alpha(m\delta^{1/2}\log m).$$

The theorem follows from the fact that $F(\alpha)=G(\alpha)+\frac{\log 2}{2}$.

8. ANOTHER FORMULA FOR $G(t)$

We start by noting that the function $-\log|2\sin(\pi t)|$ has the following Fourier series (convergent in (0, 1)),

$$-\log|2\sin(\pi x)|=\sum_{h=1}^{\infty}\frac{\cos 2\pi hx}{h}.$$

Thus we have the expansion

$$\begin{aligned}
-G(t)&=\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{1}{4k^2-1}\left(\sum_{h=1}^{\infty}\frac{\cos(2\pi kht)}{h}\right)\\
&=\sum_{h=1}^{\infty}\frac{1}{h}\left(\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{1}{4k^2-1}\cos(2\pi kht)\right)\\
&=\sum_{h=1}^{\infty}\frac{1}{h}\left(\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}\frac{1}{2}\left(\frac{1}{2k-1}-\frac{1}{2k+1}\right)\cos(2\pi kht)\right)\\
&=\sum_{h=1}^{\infty}\frac{1}{h}\left(\frac{1}{2}+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{2k-1}\,(\cos(2\pi kht)\right.\\
&\qquad\left.-\cos(2\pi(k-1)\,ht))-\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{1}{4k^2-1}\right).
\end{aligned}$$

From the identity

$$\cos(2\pi kht)-\cos(2\pi(k-1)\,ht)=-2\sin(\pi ht)\sin(\pi h(2k-1)\,t),$$

we get

$$-G(t)=\sum_{h=1}^{\infty}\frac{1}{h}\left(C(t)-\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(\pi ht)\sin(\pi h(2k-1)t)\right)$$

$$=\sum_{h=1}^{\infty}\frac{1}{h}\left(C(t)-\sin(\pi ht)\sum_{k=1}^{\infty}\frac{1}{2k-1}\sin(\pi h(2k-1)t)\right).$$

Now the inner series is just the Fourier expansion of the step function $\frac{\pi}{4}\chi(t)$ where

$$\chi(t)=\begin{cases}-1 & \text{if } t\in(-1,0)\\ 1 & \text{if } t\in(0,1).\end{cases}$$

Thus, wherever the series $G(t)$ converges, we obtain the identity

$$G(t):=\sum_{\substack{k=1\\ kt\notin\mathbb{Z}}}^{\infty}\frac{\log|2\sin(\pi kt)|}{4k^2-1}=\frac{8}{\pi^2}\sum_{h=1}^{\infty}\frac{1}{h}\left(\frac{\pi}{4}|\sin(\pi ht)|-C(t)\right). \tag{20}$$

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