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JACOBIANS OF RIEMANN SURFACES AND THE SOBOLEV SPACE $H^{1/2}$ ON THE CIRCLE

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Abstract. We solve the problem of embedding the universal cover of the Jacobian of any Riemann surface, X, into the "universal Jacobian" $H^{1/2}(S^1)$, (see [7]). The latter is the Sobolev space of half-order differentiable functions on the unit circle. As the complex structure on X is deformed, we show how to define a ($\mathcal{T}(X)$-parametrized) family of these natural embeddings, such that they provide a holomorphic homomorphism of a vector bundle over the Teichmüller space $\mathcal{T}(X)$, into the tautological vector bundle over the universal Siegel space which commutes with the universal period map.

In sections 6 and 7 we study the above embeddings with reference to towers of Jacobians arising from towers of finite coverings of Riemann surfaces. We can pass to inductive limits and see the role of the commensurability automorphism groups acting on the limit objects.

1. Introduction

The half-order Sobolev space on the unit circle, $H^{1/2}(S^1)$, plays an elegant role in the theory of Teichmüller spaces, as shown in earlier joint work of the second author with Dennis Sullivan, (see [7]). We recall that $H^{1/2}(S^1)$ stands for the real Hilbert space of real functions f on S^1 (modulo the constant functions), whose Fourier expansions: $f(e^{i\theta}) = \sum_{n=-\infty}^{\infty} u_n e^{in\theta}$, $u_{-n} = \bar{u}_n$, have the property that the sequence $\{\sqrt{n}u_n\}$ is *square-summable*. In fact, every such Fourier series is known to converge quasi-everywhere, and defines a real function of the required type. The norm of $f \in H^{1/2}(S^1)$ is the ℓ_2 norm $(\sum_{n=1}^{\infty} n|u_n|^2)^{1/2}$.

There are many connections between $H^{1/2}(S^1)$ and universal Teichmüller theory. We point out two salient ones.

Firstly, the Hilbert space comprising all square-integrable harmonic 1-forms on the unit disc Δ, is canonically and isometrically isomorphic to this Sobolev space $H^{1/2}(S^1)$. More precisely, integrate the 1-form to get a Dirichlet-finite harmonic function on Δ, and pass to the trace of this harmonic function on the boundary circle. The inverse map is given by the differential of the Poisson integral of any $f \in H^{1/2}(S^1)$. Now, $H^{1/2}(S^1)$ is invariant under the Hilbert transform, and it carries a natural symplectic structure arising from the cup product of harmonic forms:

$$\aleph(f,g) = \frac{1}{2\pi}\int_{S^1} f \cdot dg$$

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The Hilbert transform provides a canonical complex structure on $H^{1/2}(S^1)$ that is compatible with (i.e., positive with respect to) the symplectic form $\aleph$. Consequently, (see [7]), this space $H^{1/2}(S^1)$ can be identified with the "Hodge-theoretic Jacobi variety" of the unit disc.

Secondly, this Hilbert space $H^{1/2}(S^1)$ is *quasisymmetrically invariant* – indeed, q.s. homeomorphisms constitute precisely those homeomorphisms of S^1 that preserve the $H^{1/2}(S^1)$ space; (the action is, of course, "precomposition by ρ": $\rho^*(f) = f \circ \rho$, for $\rho \in Homeo(S^1)$). *Every quasisymmetric homeomorphism of S^1 acts as a bounded symplectic automorphism on $H^{1/2}(S^1)$.* Further, and that is crucial for us, among all q.s. homeomorphisms, precisely the *Möbius transformations* of S^1 act *unitarily* on $H^{1/2}(S^1)$.

That allows us the natural embedding, which had been called in [7] the universal period map:

$$\Pi : \mathcal{T}(\Delta) \longrightarrow \mathcal{S}(H^{1/2}) \tag{1.1}$$

mapping the universal Teichmüller space, $\mathcal{T}(\Delta)$, into the universal Siegel space $\mathcal{S}(H^{1/2}) = Sp(H^{1/2}(S^1))/U(H^{1/2}(S^1))$. It had been shown in [6] and [7] that Π is a *holomorphic embedding* of $\mathcal{T}(\Delta)$, and that Π is *equivariant* with respect to the Teichmüller modular group acting on $\mathcal{T}(\Delta)$, and the Siegel symplectic group $Sp(H^{1/2}(S^1))$ acting transitively on the homogeneous space $\mathcal{S}(H^{1/2})$. In fact, Π provided us with a faithful picture of every Teichmüller space (of arbitrary Fuchsian group) as embedded inside the universal Siegel space.

One can consider, for arbitrary (open or closed) Riemann surface X, the Hilbert space $\mathrm{Har}^1_{L^2}(X)$ of square-integrable $\mathbb{R}$-valued harmonic 1-forms on X. The Hodge star operator equips this with a complex structure. It is natural to think of $\mathrm{Har}^1_{L^2}(X)$ as the "universal covering of the Jacobian variety of X", in a Hodge theoretic sense. For the unit disc, this object is $H^{1/2}(S^1)$. Of course, this concept coincides with the universal covering of the usual Jacobian torus in the case of compact Riemann surfaces. (There is also a natural associated concept of a generalized Jacobi variety for any X, obtained by taking a natural quotient of $\mathrm{Har}^1_{L^2}(X)$ by $H_1(X, Z)$.)

In the earlier work ([6], [7]), the following question had been left unsolved: to relate the the theory of the Jacobians of closed Riemann surfaces to the "universal Jacobian", $H^{1/2}(S^1)$, which is associated to the universal covering. In this paper we find and study a natural embedding of the Hilbert space $\mathrm{Har}^1_{L^2}(X)$, *for arbitrary Riemann surface X*, into the universal Jacobian $H^{1/2}(S^1)$. We obtain such an embedding,

$$\eta_X : \mathrm{Har}^1_{L^2}(X) \longrightarrow H^{1/2}(S^1) \tag{1.2}$$

the moment we fix a fundamental region representing X in the universal covering disc. Then we show that *the embedding can be deformed to a natural family* as the complex structure on X is deformed. *Theorem 4.5 below fits together all the embeddings into a holomorphic family* – giving a natural *holomorphic* vector bundle morphism of a holomorphic vector bundle over $\mathcal{T}(X)$ into the tautological vector bundle over the universal Siegel space. This bundle morphism on the total spaces covers the universal period mapping Π between the base spaces, and restricts to the desired embeddings on the fibers. Moreover, the vector bundle constructed over $\mathcal{T}(X)$ has a natural lifting on it, as holomorphic vector bundle automorphisms, of the biholomorphic action of the Teichmüller modular group.

Remark. The constructions developed here point out a rather remarkable result showing that the universal Teichmüller space, $\mathcal{T}(\Delta)$, has a natural holomorphic embedding into an infinite dimensional Grassmann manifold such that the non-trivial tautological bundle over that Grassmannian pulls back to a trivial, and actually canonically trivialized, holomorphic vector bundle over $\mathcal{T}(\Delta)$.

In Sections 6 and 7 we consider how the embeddings η_X and η_Y are related when $\pi : Y \to X$ is a finite covering space between Riemann surfaces. We find that the embeddings are related by commutative diagrams that allow us to proceed to *inductive limit constructions*. Indeed, as we go over a tower of finite coverings over an (arbitrary) base Riemann surface X, we obtain a direct system of universal covers of the corresponding Jacobians, as well as a concomitant direct system of the images of these $\widetilde{J(X)}$ via the η embeddings in $H^{1/2}(S^1)$. The *universal commensurability modular group,* $CM_\infty(X)$ (that acts as a modular group on the inductive limit of the Teichmüller spaces of the covering surfaces), operates on all these limit objects. The embeddings η fit together to give in the limit an equivariant map:

$$\eta_\infty : \overline{\mathrm{Hol}}_\infty(X) \longrightarrow H^{1/2}_\infty(X) \tag{1.3}$$

Finally these mappings further fit together as morphisms of appropriate vector bundles over $\mathcal{T}_\infty(X)$, – namely, they fit as the complex structure on X itself, or even on any of its finite coverings, is varied.

2. Closed embedding of $\mathbf{Har}^1_{L^2}(X)$ in $H^{1/2}(S^1)$

Let X be any Riemann surface, not necessarily compact, whose universal cover is the hyperbolic unit disc Δ, and let G denote an uniformizing Fuchsian group such that $X = \Delta/G$. We will let

$$p : \Delta \longrightarrow X \tag{2.1}$$

denote the holomorphic universal covering projection onto X.

In what follows, $\Omega^1_{L^2}(X)$ will stand for the *real* Hilbert space of measurable *square integrable* $\mathbb{R}$-valued 1-forms on X. Letting $\star$ denote the Hodge star operator on 1-forms, we can write the inner product on $\Omega^1_{L^2}(X)$ as

$$(\omega_1, \omega_2)_X = \int_X \omega_1 \wedge \star\omega_2. \tag{2.2}$$

Recall the fundamental theorem (see, for example, Chapter II.3 of [3]) that the Hilbert space $\Omega^1_{L^2}(X)$ has the following *orthogonal direct sum decomposition* into closed subspaces:

$$\Omega^1_{L^2}(X) = E_X \oplus \hat{E}_X \oplus \mathrm{Har}^1_{L^2}(X). \tag{2.3}$$

Here E_X ("exact") denotes the L^2 closure of the forms df, with $f \in C_0^\infty(X)$ (smooth functions with compact support); $\hat{E}_X$ ("co-exact") denotes the closure of the forms $\star df$, with f as before; and $\mathrm{Har}^1_{L^2}(X)$, which is the piece of principal interest to us, is the space of *harmonic* 1-forms on X that are square-integrable. The Hodge star maps E and $\hat{E}$ onto each other, while $\star$ acts as an automorphism, whose square is negative identity, on $\mathrm{Har}^1_{L^2}(X)$. *Thus, for arbitrary Riemann surface X, the space $Har^1_{L^2}(X)$ is a real Hilbert space equipped with a complex structure given by the Hodge star.*

Let

$$P^X_{har} = P_{har} : \Omega^1_{L^2}(X) \longrightarrow \mathrm{Har}^1_{L^2}(X). \tag{2.4}$$

be the orthogonal projection onto the harmonic subspace. Our first and foremost consideration is to provide a natural mapping that associates to any square-integrable harmonic form, ω, on X, a harmonic form on the universal covering disc which is also *square-integrable* on Δ. [The pullback by an infinite degree covering of a L^2 form is, of course, never L^2, (except for the trivial form).]

We shall do this utilizing the projection P_{har} for the disc.

Remark. Since every harmonic form is canonically the sum of a holomorphic and an anti-holomorphic form, we can canonically identify $\mathrm{Har}^1_{L^2}(X)$ with either the square integrable holomorphic 1-forms, or its conjugate space, $\overline{\mathrm{Hol}}^1_{L^2}(X)$. The projection to the conjugate holomorphic part is actually $\mathbb{C}$-linear. (The other projection is conjugate linear.) We shall, without further comment, identify $\mathrm{Har}^1_{L^2}(X)$ with $\overline{\mathrm{Hol}}^1_{L^2}(X)$.

The embedding η_F: Let F be any region in Δ which is covered by a finite number (say N) of fundamental domains for the group G. We assume that the relative boundary of F in the disc, ∂F, is a null-set for two-dimensional planar Lebesgue measure. In the cases of chief interest to us, F will be a fundamental polygon, or a finite union of fundamental polygons, for the Fuchsian group G. Such regions have piecewise analytic arcs as the relative boundary.

Consider the pullback $p^*\omega$, restrict it to F, and extend that by setting it equal to zero outside F. Clearly, this "cutoff of the pullback" $(p^*\omega)\chi_F$, is a measurable and square-integrable form on the disc Δ. We define:

$$\eta_F : \mathrm{Har}^1_{L^2}(X) \longrightarrow \mathrm{Har}^1_{L^2}(\Delta)$$

$$\eta_F(\omega) = P^\Delta_{har}(p^*\omega.\chi_F) \tag{2.5}$$

As explained in the Introduction, there are canonical isometric isomorphisms of the following Hilbert spaces:

$$\text{Har}^1_{L^2}(\Delta) \cong Dir(\Delta) \cong H^{1/2}(S^1) \tag{2.6}$$

Here $Dir(\Delta)$ denotes the space of Dirichlet-finite harmonic functions on the disc having mean value zero. The Hodge star operator for forms on the disc equips $\text{Har}^1_{L^2}(\Delta)$ with a complex structure which, under the isomorphism with $H^{1/2}(S^1)$, gets translated into the *Hilbert transform* on these circle functions.

Proposition 2.7. *Let the cutoff region $F \subset \Delta$ be simply connected, and the interior of its closure. Then η_F is a closed embedding of $Har^1_{L^2}(X)$ into $H^{1/2}(S^1)$, which is complex linear with respect to the Hodge star and the Hilbert transform complex structures, (on domain and target, respectively). The operator norm of η_F is bounded by $\sqrt{N}$.*

Idea of Proof. Express the projection operator from L^2 forms to harmonic/holomorphic L^2 forms by using the Bergman kernel projector (as an integral operator). Fundamental properties of the Bergman kernel for the disc then imply the statements. □

3. Deformation of the embedding for deformation of complex structure

The universal Siegel space, $\mathcal{S}(H^{1/2})$, is a complex Banach manifold comprising all the complex structures on the real Hilbert space $H^{1/2}(S^1)$ whose eigenspace for the eigenvalue $\sqrt{-1}$ (in the complexified Hilbert space) are positive polarizing isotropic subspaces with respect to the canonical symplectic form $\aleph$. Another description of $\mathcal{S}(H^{1/2})$, (amongst several useful ones) is the homogeneous space $Sp(H^{1/2}(S^1))/U$. The relationships among the various descriptions are spelled out in [7].

The universal Teichmüller space $\mathcal{T}(\Delta)$ is the space of quasisymmetric homeomorphisms of S^1 modulo the Möbius transformations. $\mathcal{T}(\Delta)$ is naturally a complex Banach manifold (in fact, a contractible domain of holomorphy), where the complex structure comes from the space of Beltrami differentials on the disc. (See, for instance, [4] or [5] for this material.)

The "universal period" (or "universal polarization") mapping gives a faithful holomorphic embedding of $\mathcal{T}(\Delta)$ into $\mathcal{S}(H^{1/2})$. Recall that this goes as follows. Given any point $[\mu]$ of $\mathcal{T}(\Delta)$, represented by a Beltrami coefficient μ on Δ, one associates the bounded linear symplectic automorphism of $H^{1/2}(S^1)$:

$$T_{[\mu]} : H^{1/2}(S^1) \to H^{1/2}(S^1) \tag{3.1}$$

given by pre-composition ("pullback") of $H^{1/2}(S^1)$ functions by the quasisymmetric homeomorphism of the circle which arises as the boundary values of the μ-conformal automorphism w_μ of the disc. (Here w_μ is the quasiconformal self-map of Δ solving the Beltrami equation $w_{\bar{z}} = \mu w_z$, and fixing $\{1, -1, i\}$.)

The U-coset of $T_{[\mu]}$ determines the point $\Pi([\mu])$ of the Siegel space. Alternatively, if we *conjugate* the reference complex structure (i.e., the Hilbert transform), $J_0 : H^{1/2}(S^1) \to H^{1/2}(S^1)$ by this symplectomorphism, we get the *μ-deformed complex structure operator* on $H^{1/2}(S^1)$:

$$J_{[\mu]} = T_{[\mu]} \circ J_0 \circ {T_{[\mu]}}^{-1} \tag{3.2}$$

This is again compatible (i.e., positive) with respect to the symplectic form, and describes the desired point $\Pi([\mu])$.

Now we bring back into play the Fuchsian group G and its Teichmüller space $\mathcal{T}(G) = \mathcal{T}(X)$. Let us fix a fundamental polygon F for G. Notice that fixing such an F requires simply the choice of one base point in the universal cover of X. Indeed, we may take F to be the Dirichlet polygon in Δ centered at the chosen point.

Let μ be a G-invariant Beltrami coefficient on the disc, namely $[\mu]$ represents a point of $\mathcal{T}(G)(\subset \mathcal{T}(\Delta))$. We let X_μ denote the Riemann surface with the μ complex structure.

Definition/Lemma 3.3. *There is a closed embedding*

$$\eta_{[\mu]} : Har^1_{L^2}(X) \longrightarrow H^{1/2}(S^1),\ \eta_{[\mu]} = T_{[\mu]} \circ \eta_F. \tag{3.3}$$

The map $\eta_{[\mu]}$ depends only on the Teichmüller class of μ: since the symplectomorphism $T_{[\mu]}$ depends only on the boundary homeomorphism of the quasiconformal automorphism w_μ of Δ. The image of $\eta_{[\mu]}$ is a closed complex subspace of the Hilbert space $H^{1/2}(S^1)$ equipped with the $J_{[\mu]}$ complex structure.

4. Vector bundles over Siegel and Teichmüller spaces

The discussion regarding the period mapping in references [6], [7], tells us that the image of $\eta_{[\mu]}$ will represent the square-integrable harmonic forms for the deformed Riemann surface X_μ. Our object is to show that the *embeddings $\eta_{[\mu]}$ vary holomorphically* with deformation of complex structure on X. That result is best formulated in terms of certain natural holomorphic vector bundles that live over the Teichmüller spaces, and over the universal Siegel space, respectively.

Bundle of eigenspaces over Siegel space: We construct the tautological vector bundle, $\mathcal{V}$, over $\mathcal{S}(H^{1/2})$. Let J be a positive polarizing complex structure on $H^{1/2}(S^1)$; by definition J determines a point in $\mathcal{S}(H^{1/2})$. The fiber of $\mathcal{V}$ over the point $J \in \mathcal{S}(H^{1/2})$ will be canonically identified with the *complex* Hilbert space $(H^{1/2}(S^1), J)$.

Note that the inner product on $(H^{1/2}(S^1), J)$ is given by $\langle u, u\rangle = \aleph(Ju, u)$. This will provide natural hermitian structure on the fibers of $\mathcal{V}$.

Recall the decomposition of the complexification of $H^{1/2}(S^1)$:

$$H^{1/2}(S^1) \otimes \mathbb{C} = E_+ \oplus E_-,\ E_- = \bar{E_+}, \tag{4.1}$$

where E_+ consists of complex valued $H^{1/2}$ functions which are boundary values of Dirichlet-finite conjugate holomorphic functions; E_+ is precisely the $+i$

eigenspace of the Hilbert transform operator J_0, (extended to $H^{1/2}(S^1) \otimes \mathbb{C}$ by $\mathbb{C}$-linearity). The boundary values of holomorphic functions on the disc whose derivatives are square integrable, comprise the $-i$ eigenspace, E_-.

We utilize the holomorphic embedding of $\mathcal{S}(H^{1/2})$ in a complex analytic Grassmannian which is discussed in [7]. Introduce

$$\mathcal{P} : \mathcal{S}(H^{1/2}) \longrightarrow Gr = Gr(E_+, H^{1/2}(S^1) \otimes \mathbb{C}) \tag{4.2}$$

parametrizing subspaces that are "comparable to" E_+ in the complexified Hilbert space $H^{1/2}(S^1) \otimes \mathbb{C}$. This Grassmannian is a complex Banach manifold modeled on the Banach space of all bounded complex linear operators from E_+ to E_-. The subspaces we are parametrizing are the *graphs* of such operators. The tautological vector bundle over Gr, a holomorphic bundle, is defined standardly by

$$\text{Taut} = \{(v, \Gamma) \in (H^{1/2}(S^1) \otimes \mathbb{C}) \times Gr : \ v \in \Gamma\}. \tag{4.3}$$

One now obtains $\mathcal{V}$ as the restriction of the tautological bundle:

Proposition 4.4. *The map $\mathcal{P} : \mathcal{S}(H^{1/2}) \longrightarrow Gr$, which associates to the point $J \in \mathcal{S}(H^{1/2})$ the $+i$ eigenspace for J is a holomorphic embedding. The image comprises the positive polarizing isotropic subspaces in $H^{1/2}(S^1) \otimes \mathbb{C}$.*

The pullback bundle, $\mathcal{P}^*(\text{Taut})$ over $\mathcal{S}(H^{1/2})$, is a holomorphic and hermitian vector bundle $\mathcal{V} \to \mathcal{S}(H^{1/2})$, whose fibers are complex Hilbert spaces. The symplectic automorphism group on $\mathcal{S}(H^{1/2})$ has a natural lifted action, by holomorphic vector bundle automorphisms, acting as isometries, on $\mathcal{V}$.

Utilizing the fact that the equivariant map Π induces a monomorphism of the group of right translations on $\mathcal{T}(\Delta)$ into $Sp(H^{1/2}(S^1))$, we derive:

Theorem 4.5. *The universal Teichmüller space $\mathcal{T}(\Delta)$ carries a holomorphic vector bundle $\mathcal{W} = \Pi^*\mathcal{V}$, arising as the pullback by the period mapping Π of the bundle $\mathcal{V} \to \mathcal{S}(H^{1/2})$. But $\mathcal{T}(\Delta)$ is a group, in which the right translations act as biholomorphic automorphisms; these automorphisms lift as holomorphic bundle automorphisms of $\mathcal{W}$ acting as hermitian isometries on the fibers. This structure provides a canonical trivialization of the bundle $\mathcal{W}$.*

5. Morphism of vector bundle over $\mathcal{T}(X)$ into $\mathcal{V}$

Let X be Δ/G, for arbitrary Fuchsian G, and let F be a fundamental domain for G as in sections 2 and 3. Let N denote the image of η_F (as a closed complex subspace in $(H^{1/2}(S^1), J_0)$). By transporting N via the liftings of the right-translations to all fibers of $\mathcal{W}$, we obtain a certain holomorphic subbundle, say $N^\sharp$, of $\mathcal{W}$. Since the Teichmüller space $\mathcal{T}(X) = \mathcal{T}(G)$ sits holomorphically embedded in $\mathcal{T}(\Delta)$, we obtain a holomorphic vector bundle $N_X^\sharp$ over $\mathcal{T}(X)$ by restricting $N^\sharp$.

If $[\mu]$ is a point of $\mathcal{T}(X)$, then the "deformed embedding" $\eta_{[\mu]} = T_{[\mu]} \circ \eta_F$ (see (3.3)), is seen to have image equal to the fiber of $N_X^\sharp$ over the point $[\mu]$. Hence, the entire family of embeddings $\eta_{[\mu]}$, as $[\mu]$ varies over all of $\mathcal{T}(X)$, fit together to give a holomorphic isomorphism $\eta^\natural : \mathrm{Har}^1_{L^2}(X) \times \mathcal{T}(X) \to N^\sharp_X$. One proves:

Theorem 5.1. *The holomorphic vector bundle $N^\sharp_X \to \mathcal{T}(X)$ carries a lifting of the action of the modular group $Mod(\mathcal{T}(X))$; the lifts of modular transformations act as holomorphic and isometric bundle automorphisms.*

One has a commuting square, with the holomorphic morphism $\eta^\sharp$ covering the universal period mapping Π (restricted to $\mathcal{T}(X)$):

$$\begin{array}{ccc} \overline{Hol}^1_{L^2}(X) \times \mathcal{T}(X) & \xrightarrow{\eta^\sharp} & \mathcal{V} \\ \downarrow & & \downarrow \\ \mathcal{T}(X) & \xrightarrow{\Pi} & \mathcal{S}(H^{1/2}) \end{array}$$

The morphism $\eta^\sharp$ is a closed injection on each fiber, and is equivariant with respect to the actions of the Teichmüller modular group (on the domain) and the Siegel symplectic group (on the range).

Remark. In case $X = \Delta$, the bundle $N^\sharp_X$ is exactly the $\mathcal{W}$ of Theorem 4.5.

6. Coverings of surfaces and the η embeddings

The aim now is to show the naturality with which the various η embeddings fit together as we go through finite coverings of Riemann surfaces. Let $q : Y \to X$ be any finite unbranched covering map, determined by a subgroup H, (of some finite index n), in the Fuchsian group G (recall, G uniformized X). We will call the covering "enhanced" if a set of n coset representatives for G/H, say $g_1, \cdots, g_n$, (we assume $g_1 = 1$) are given. We emphasize that X need *not* be compact.

Having fixed the fundamental polygon F for X, we obtain a corresponding *union of translates of* F: namely $F_Y = g_1 F \cup \cdots \cup g_n F$, so that F_Y is a fundamental region in the disc representing Y. We have the η embeddings of the Hilbert spaces of (conjugate) holomorphic (or harmonic) square-integrable forms for each surface into $H^{1/2}$:

$$(6.1)\quad \eta_X := \eta_F : \overline{\mathrm{Hol}}^1_{L^2}(X) \longrightarrow H^{1/2}(S^1),\ \eta_Y := \eta_{F_Y} : \overline{\mathrm{Hol}}^1_{L^2}(Y) \longrightarrow H^{1/2}(S^1)$$

Remark. The domain F_Y may not be connected, but that will not affect our inductive limit constructions. It is possible to choose the representatives so that the translates of F are contiguous.

The Lemma is the following relationship between the two η mappings:

Lemma 6.2. *Let* $q^* : \overline{Hol}^1_{L^2}(X) \to \overline{Hol}^1_{L^2}(Y)$ *denote the pullback of* 1*-forms induced by the finite cover. The following diagram commutes:*

$$\begin{array}{ccc} \overline{Hol}^1_{L^2}(X) & \xrightarrow{\eta_X} & H^{1/2}(S^1) \\ \big\downarrow q^* & & \big\downarrow \mathcal{A}_q \\ \overline{Hol}^1_{L^2}(Y) & \xrightarrow{\eta_Y} & H^{1/2}(S^1) \end{array}$$

where $\mathcal{A}_q(f) = \sum_{i=1}^n U^*_{g_i}(f)$*, is the sum of the adjoints of* n *unitary operators on* $H^{1/2}(S^1)$*. Here,* $U_g(f) = f \circ g$*, is the unitary operator corresponding to the Möbius transformation* g *of* S^1*. Unitarity implies,* $U^*_g = U_{g^{-1}}$*.*

Idea of proof. The lemma rests on basic invariance properties enjoyed by the Bergman kernel for the disc; one utilizes the fact that both η_X and η_Y may be expressed as integral operators using that kernel. □

Remark. It is also possible to form a direct system of η embeddings for the tower of finite covers by *fixing* a region F in the universal covering disc for all the surfaces. Then one has $\eta_X = \eta_Y \circ q^*$; consequently the inductive limit of the images of the η embeddings (in $H^{1/2}(S^1)$) is constructible. A parallel theory to the one being described emerges.

Towers of enhanced coverings: It is easy to see that the composition of two finite enhanced coverings, $r : Z \to Y$ (say degree m), $q : Y \to X$ (say degree n), is itself naturally an enhanced cover. Let $\{g_1, \cdots, g_n\}$ be the coset representatives for q, and $\{h_1, \cdots, h_m\}$ be those chosen for r. Then, in fact, the mn elements of G obtained as the pairwise products $\{h_j g_i\}$ constitute coset representatives for the composite covering $q \circ r$.

Therefore enhanced finite coverings over any Riemann surface X constitute a *directed set*, where one cover dominates another if there is a factoring in the category of enhanced covers.

It is important now to observe that the linear operators $\mathcal{A}_q$ satisfy the compatibility condition:

$$\mathcal{A}_r \circ \mathcal{A}_q = \mathcal{A}_{q \circ r}. \tag{6.3}$$

Consequently, we can construct, over the directed set of enhanced finite covers over X*, the direct systems of Hilbert spaces of square integrable forms, as well as the direct system of the images (by* η*) of these spaces within* $H^{1/2}(S^1)$*.*

One obtains several inter-related limit objects. Define:

$$\overline{\mathrm{Hol}}_\infty(X) = \varinjlim_{\{Y \to X\}} \overline{\mathrm{Hol}}^1_{L^2}(Y) \tag{6.4}$$

Since the inductive limit of Banach spaces is also a Banach space, we note that

$\overline{\mathrm{Hol}}_\infty(X)$ is a complex Banach space. Now define:

$$H^{1/2}_\infty(X) = \lim_{\substack{\longrightarrow \\ \{Y \to X\}}} \eta_Y(\overline{\mathrm{Hol}}^1_{L^2}(Y)) \tag{6.5}$$

The connecting maps for the inductive system in (6.5) are, of course, the $\mathcal{A}_q$ (which are injections when restricted to the η images).

Clearly, the maps η_Y give a map between the inductive systems themselves, and hence induces a complex linear map on the inductive limits:

$$\eta_\infty : \overline{\mathrm{Hol}}_\infty(X) \longrightarrow H^{1/2}_\infty(X) \tag{6.6}$$

Let us also introduce the direct limit of the Teichmüller spaces of the covering surfaces Y over X:

$$\mathcal{T}_\infty(X) = \varinjlim \mathcal{T}(Y) \tag{6.7}$$

This complex analytic "ind-space", (see [8]), is a stratified object that embeds in the universal Teichmüller space $\mathcal{T}(\Delta)$ as the directed union of the Teichmüller spaces of the finite index Fuchsian subgroups (corresponding to the covers Y) of the original Fuchsian group G. Its closure in $\mathcal{T}(\Delta)$ is a complex Banach manifold. In the case of X *compact*, a study was made of this ind-space, and of bundles over it, in the works [2], [1]. $\mathcal{T}_\infty(X)$ was called the *commensurability Teichmüller space.*

The action of the commensurability groups: Any element of the universal commensurability modular group, $CM_\infty(X)$, (see [2], [1]), arises from an undirected polygon whose edges represent pointed unramified coverings, *in the topological category*, starting at and returning to X. For X connected and compact, this group was shown to operate faithfully, as a group of biholomorphic automorphisms, on the commensurability Teichmüller space, $\mathcal{T}_\infty(X)$.

Now, for essentially set-theoretic reasons, such a phenomenon continues to hold true for the limit objects we introduced in (6.4), (6.5), (and actually, very generally, in other similar situations). Namely, a group of automorphisms of the limit object is created by considering undirected polygons *in the holomorphic category.*

Let F be a contravariant functor that associates to the objects and morphisms of the inverse system constituted by some finite holomorphic coverings over the Riemann surface X, objects and morphisms from any category, (for instance the categories in (6.4) (6.5)). Denote by $F_\infty(X) = \varinjlim F(Y)$.

Lemma 6.8. *Given any morphism in the inverse system $q : Y \to X$, we get a natural induced map of the direct system of F-objects over Y to the direct system of F-objects over X. (Since the system above Y is cofinal with that above X.) The corresponding map of direct limits:*

$$F_\infty(q) : F_\infty(Y) \longrightarrow F_\infty(X)$$

is a bijection.

Since each $F_\infty(q)$ is invertible, we obtain the desired group of automorphisms on $F_\infty(X)$, defined by following around undirected cycles of covering morphisms in the *holomorphic* category, starting and ending at X. For a given Riemann surface X, this turns out to be the naturally associated "commensurability automorphism group", $\mathrm{ComAut}(X)$, of that Riemann surface; it is the group of finite holomorphic self-correspondences of X.

Remark. There is an intimate relationship between the commensurability groups in the two categories, which will be taken up in future publications. The group $\mathrm{ComAut}(Y)$, for any Riemann surface Y covering X, embeds in $CM_\infty(X)$ as the isotropy subgroup of the action of $CM_\infty(X)$ at the point of $\mathcal{T}_\infty(X)$ represented by Y.

It can be proved in this generality, that each commensurability group we are working with may be described as the fundamental group of an appropriate 2-complex obtained from the graph of the inverse system by filling in a 2-cell wherever there is a commuting triangle of covering morphisms. This connects up $CM_\infty(X)$, and $\mathrm{ComAut}(X)$, with the virtual automorphism group of the fundamental group of X, as in previous work ([2], [1]).

Remark. The solenoidal surface obtained as the inverse limit of the coverings: $H_\infty(X) = \varprojlim Y$, has a mapping class group related intimately to $CM_\infty(X)$, – even in this more general situation where the surface X is not necessarily compact. The Teichmüller space of the solenoid can be defined to be the closure in $\mathcal{T}(\Delta)$ of $\mathcal{T}_\infty(X)$.

7. Vector bundles on $\mathcal{T}_\infty(X)$ and the limit map η_∞

As in sections 4 and 5, the idea even with these direct limit Banach spaces, is to transport the subspace $H^{1/2}_\infty(X)$ into each fiber of the vector bundle $\mathcal{W}$ that we had obtained over the universal Teichmüller space, by utilizing the liftings of the right translation biholomorphisms. Here we utilize the natural embedding of $\mathcal{T}_\infty(X)$ into $\mathcal{T}(\Delta)$ (described above). If $[\mu]$ represents a point of $\mathcal{T}_\infty(X)$, one transports the basic fiber $H^{1/2}_\infty(X)$ (over $[0]$) to the space $T_{[\mu]}(H^{1/2}_\infty(X))$ as the fiber above $[\mu]$.

Theorem 7.1. *There is a holomorphic subbundle $\mathcal{W}_\infty(X)$ of $\mathcal{W} \to \mathcal{T}(\Delta)$ whose fiber above the base point is $H^{1/2}_\infty(X)$. The direct limit form of the embeddings η, composed with the symplectomorphisms $T_{[\mu]}$, namely, $T_{[\mu]} \circ \eta_\infty$, provides us a holomorphic bundle map of $\overline{Hol}_\infty(X) \times \mathcal{T}_\infty(X)$ to the restriction of $\mathcal{W}_\infty(X)$ over the commensurability Teichmüller space $\mathcal{T}_\infty(X)$.*

The entire construction is equivariant with respect to the actions of the commensurability groups explained above. The action of $CM_\infty(X)$, by biholomorphic automorphisms on $\mathcal{T}_\infty(X)$, lifts naturally to these bundles, and the product group $CM_\infty(X) \times ComAut(X)$, acts on the bundle $\mathcal{W}_\infty(X)$ restricted over $\mathcal{T}_\infty(X)$.

Details and proofs will appear elsewhere.

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