

A NOTE ON MELLIN TRANSFORM AND DISTRIBUTIONS

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Abstract- Mellin Transform occurs in many branches of Applied Mathematics and Engineering. The Mellin transform is very much related to the Laplace and Fourier transforms and the theory for the ordinary functions is well established. In the distributional sense, first it was studied by Zemanian in [2]. In this work we try to extend to the wider class of distributions.

Key Words- Integral Transform, infinitely differentiable function, Distributions, Mellin Convolution.

1. INTRODUCTION

Integral Transforms are extensively used in solving several kinds of boundary problems and Integral Equations. Since Mellin Transform is special kind of Integral Transform first of all we have the following definition.

DEFINITION 1. The Transform

$$g(\alpha) = \int_a^b f(x) K(\alpha, x) dx$$

is called the Integral Transform and $K(\alpha, x)$ is called the Kernel of the transform. By changing the kernel we can have several different integral transforms.

Now we let $\mathbf{R}_+ = (0, \infty)$ be the set of all positive real the numbers and let $L^1(S)$ be the space of all (Lebesgue) measurable function with the norm

$$\|f\|_{L^1(S)} = \int_S |f(x)| dx$$

for $S \subset \mathbf{R}$. Then if $f: \mathbf{R}_+ \rightarrow \mathbf{C}$ is a function such that $f(x) x^{s-1} \in L^1(S)$ for some $s \in \mathbf{C}$ then the Mellin Transform is defined by

$$M[f : s] \int_0^\infty f(x) x^{s-1} dx = \int_0^\infty f(x) x^s \frac{dx}{x}, \quad s = a + it, \quad \text{see [4].}$$

We note that if the integral is bounded then the transform exists. But converse is not necessarily to be true. The Inverse Mellin transform is defined by the contour integral

$$M^{-1}[M(f:s)](x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} M(f:s) x^{-s} ds = f(x)$$

A simple example if $f(x) = e^{-x}$ the Mellin transform is the well know Gamma Function

$$\Gamma(s) = \int_0^{\infty} e^{-x} x^{s-1} dx$$

Then we can easily make a statement that the Mellin Transform of $f(x) = e^{-\lambda x}$ times any polynomials in the same variable x of the same degree in s , multiplied by the Gamma Function $\Gamma(s)$. in fact, if we let $P_n(x) = \sum_{k=0}^n a_{n,k} x^k$ be some polynomials then we obtain

$$\int_0^{\infty} P_n(x) e^{-\lambda x} x^{s-1} dx = \sum_{k=0}^n \int_0^{\infty} e^{-\lambda x} x^{k+s-1} dx = \frac{\Gamma(s)}{\lambda^{k+s-2}} \sum_{k=0}^n a_{n,k} (s)^k$$

The following theorem is easy and straightforward to prove:

THEOREM 1. If f is a Mellin transformable function then

$$M[x^\alpha f : s] = M[f : s + \alpha] \quad \text{and}$$

$$M[e^{-kx} x^\alpha : s] = \frac{\Gamma(s + \alpha)}{k^{s+\alpha}} \quad \text{for } s \neq -\alpha, -\alpha - 1, -\alpha - 2, \dots$$

The space A_s with its associated norm $\|f\|_{A_s}$ for some $s \in \mathbf{C}$ is defined by

$$A_s = \left\{ f : \mathbf{R}_+ \rightarrow \mathbf{C}; \|f(x)x^{s-1}\|_{L^1(\mathbf{R}_+)} \right\}$$

$$\|f\|_{A_s} = \|f(x)x^{s-1}\|_{L^1(\mathbf{R}_+)} = \int_0^{\infty} |f(x)| x^{s-1} dx < \infty$$

for some $s \in \mathbf{C}$. then the space $A_{(a,b)} = \bigcap_{s \in (a,b)} A_s$

for $a, b \in \mathbf{R}$, $a < b$. In the statistical application we may interpret the Mellin Transform $M[f : s]$ as the $(s - 1)$ the moment of $f(x)$, see [3]. In particular,

Area of $f(x)$	$= M[f : 1]$
First moment of $f(x)$	$= M[f : 2]$
Second moment of $f(x)$	$= M[f : 3]$

THEOREM 2. Mellin transform of $\sin(x^2)$ and $\cos(x^2)$ exist and

$$M[\sin x^2 : s] = \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}s\right) \sin\left(\frac{1}{4}\pi s\right) \quad \text{and}$$

$$M[\cos x^2 : s] = \left(\frac{1}{2}\right) \Gamma\left(\frac{1}{2}s\right) \cos\left(\frac{1}{4}\pi s\right)$$

Mellin transform have some important properties, which we summarize in the next theorem.

THEOREM. 3 Let f be Mellin transformable function defined on \mathbf{R}_+ . Then if the differentiation under the integral sign is allowed,

$$\left(\frac{d}{ds}\right)^r M(f)(s) = \int_0^\infty f(x) \log^r x x^{s-1} dx = M[\log^r f : s-1]$$

$$M(f')(s) = \int_0^\infty f'(x) x^{s-1} dx = -(s-1)M[f : s-1]$$

$$M(f'')(s) = \int_0^\infty f''(x) x^{s-1} dx = (s-1)(s-2)M[f : s-2]$$

$$M(xf')(s) = \int_0^\infty xf'(x) x^{s-1} dx = -sM[f : s]$$

$$M(x^2 f'')(s) = \int_0^\infty x^2 f''(x) x^{s-1} dx = s(s+1)M[f : s]$$

On using the above theorem we can easily obtain the following results:

THEOREM 4. The Mellin transforms of $\log(x) \sin(x^2)$ and $\log(x) \cos(x^2)$ exist and

$$M[\log(x) \sin(x^2) : s] = \frac{1}{4} \varphi\left(\frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s\right) \sin\left(\frac{1}{4}\pi s\right) + \frac{\pi}{8} \Gamma\left(\frac{1}{2}s\right) \cos\left(\frac{1}{4}\pi s\right)$$

$$M[\log(x) \cos(x^2) : s] = \frac{1}{4} \varphi\left(\frac{1}{2}s\right) \Gamma\left(\frac{1}{2}s\right) \cos\left(\frac{1}{4}\pi s\right) - \frac{\pi}{8} \Gamma\left(\frac{1}{2}s\right) \sin\left(\frac{1}{4}\pi s\right)$$

In fact the results in theorem 3 are easily extended to the further derivatives in the following theorem.

THEOREM. 5 The f be n -times continuously differentiable, further f all its derivatives Mellin transformable function defined on \mathbf{R}_+ for all $n \in \mathbf{N}$. Then if differentiable under the integral sign allowed,

$$M(f^{(n)})(s) = \int_0^\infty f^{(n)}(x) x^{s-1} dx = \frac{(-1)^n (s-1)!}{(s-n-1)!} M[f : s-n]$$

$$M(x^n f^{(n)})(s) = \int_0^\infty x^n f^{(n)}(x) x^{s-1} dx = (-1)^n \frac{(s+n-1)!}{(s-1)!} M[f : s]$$

In the next definition we give the Mellin convolution structure.

DEFINITION 2 The Mellin convolution product $f * g$ of two given functions $f, g; \mathbf{R}_+ \rightarrow \mathbf{C}$ is defined by

$$(f * M g)(s) = \int_0^\infty f\left(\frac{x}{u}\right)g(u) \frac{du}{u} = \int_0^\infty \left(\frac{1}{u}\right)^s f\left(\frac{x}{u}\right)g(u)u^s \frac{du}{u}$$

provided that the integral exist. As we can see easily the convolution product $f * g$ in the Mellin sense can be expressed as the Mellin Transform of $\left(\frac{1}{u}\right)^s f\left(\frac{x}{u}\right)g(u)$.

We note that

1. If $f, g \in A_s$ then the convolution $f * g$ exist almost everywhere on \mathbf{R}_+ and belongs to A_s and further

$$\|f * g\|_{A_s} \leq \|f\|_{A_s} \cdot \|g\|_{A_s}$$

2. If $x^c f(x)$ is uniformly continuous and bounded function on \mathbf{R}_+ then $f * g$ is also bounded on \mathbf{R}_+ .
3. If $f, g \in A_s$ and $s = c + it, t \in \mathbf{R}$, then

$$M[f * g : s] = M[f : s]M[g : s]$$

4. The convolution product is commutative and associative, that is, for $f_1, f_2, f_3 \in A_s$ then

$$f_1 * f_2 = f_2 * f_1, (f_1 * f_2) * f_3 = f_1 * (f_2 * f_3), \text{ see [4].}$$

In the next section we will generalize this Mellin Transform to the space of linear functionals E' which are the distributions with bounded support in $(0, \infty)$.

2. THE SPACE OF INFINITELY DIFFERENTIABLE FUNCTIONS

By the $E_{p,q}$ we define the linear space of infinitely differentiable function ϕ defined on $\mathbf{R}_+ = (0, \infty)$ such that there exist two positive numbers r_1, r_2 for

$$\lim_{x \rightarrow 0^+} x^{k+1-p} \phi^{(k)}(x) = 0$$

$$\lim_{x \rightarrow \infty} x^{k+1-r_2-q} \phi^{(k)}(x) = 0$$

and we consider $\phi(x) = 0$ for all $x < 0$. Of course $E_{p,q}$ is not empty and it is a linear space.

EXAMPLE 1 If we define

$$\phi(x) = \cos x = \begin{cases} \cos x & x \geq 0 \\ 0 & x < 0 \end{cases}$$

then it is obvious that $\phi(x) \in E_{p,q}$.

EXAMPLE 2 If is defined as follows

$$\phi(x) = x_+^{s-1} = \begin{cases} x^{s-1} & x > 0 \\ 0 & x < 0 \end{cases}$$

then $\phi(x) \in E_{p,q}$ if $p < s < q$. Then we define

$$h_{p,q}(x) = \begin{cases} x^{-p} & 0 < x < 1 \\ x^q & x \geq 1 \end{cases}$$

and

$$\gamma_{k,p,q}(\phi) = \sup_{x>0} h_{p,q}(x) x^{k+1} |\phi^{(k)}(x)|$$

the are all bounded and positively defined. In particular if we set $k = 0$ then satisfy the properties of norm. Then we define the convergence in $E_{p,q}$ that infinite sequence $\phi_n(x)$ convergence to 0 in sense of $E_{p,q}$ if and only if $\gamma_{h,p,q}(\phi_n) \rightarrow 0$ for each $k \in \mathbb{N}$. Then if we let $D(S)$ be the space of infinitely differentiable functions $\phi(x)$ having compact support then one easily prove that $D(S) \subset E_{p,q}$ and it is dense in $E_{p,q}$. By the $E'_{p,q}$ we define the linear space of continuous linear functionals on $E_{p,q}$ which is zero on the interval $(-\infty, 0)$. In fact the space $E'_{p,q}$ is the dual of the $E_{p,q}$. That is, for every $f \in E'_{p,q}$ if and only if the following conditions are satisfied.

1. $\langle f(x), \phi(x) \rangle$ is defined for each $\phi \in E_{p,q}$.
2. $\langle f(x), \phi(x) \rangle = 0$ if $\phi(x) = 0$ for $x > 0$.
3. $\langle f(x), a_1\phi_1 + a_2\phi_2 \rangle = \langle f(x), a_1\phi_1 \rangle + \langle f(x), a_2\phi_2 \rangle$ for $a_1, a_2 \in \mathbb{R}$.
4. $\langle f(x), \phi_n(x) \rangle \rightarrow 0$ if $\phi_n(x) \rightarrow 0$ in $E_{p,q}$ as $n \rightarrow \infty$.

If $\phi \in E_{p,q}$ and $f \in E'_{p,q}$ then we define by an integral,

$$\langle f, \phi \rangle = \int_0^\infty f(x) \phi(x) dx$$

Since theory of distributions is a linear theory. We can extend some operations which are valid for ordinary functions to $E'_{p,q}$ such operations are called regular operations such as addition, multiplication by scalars. For example if f_1, f_2 in $E'_{p,q}$ and a is a scalar then

1. $\langle f_1(x) + f_2(x), \phi(x) \rangle = \langle f_1, \phi(x) \rangle + \langle f_2, \phi(x) \rangle$ for each every $\phi \in E_{p,q}$
2. $\langle af(x), \phi(x) \rangle = a \langle f(x), \phi(x) \rangle$ for every $\phi \in E_{p,q}$ for $x > 0$.

Other operations can be defined only for particular distributions for certain restricted subclasses of distributions; these are called irregular operations such as convolution, multiplication and change of variables, see [1].

DEFINITION 3 If $f, g \in E'_{p,q}$ then we define the Mellin Convolution

$$M[f * g : s] = \langle (f * g)(x), \phi(x) \rangle = \langle f(u), \langle g(t), \phi(ut) \rangle \rangle$$

for all $\phi \in E_{p,q}$.

EXAMPLE 3 Let $f, g \in E'_{p,q}$ and $\delta(x-n)$ then

$$\begin{aligned} \langle f * \delta(x-n), \phi(x) \rangle &= \langle f(u), \langle \delta(x-n), \phi(ux) \rangle \rangle \\ &= \langle f(u), \phi(nx) \rangle \end{aligned}$$

Thus we can see that in particular case when $n = 1$, the $\delta(x-1)$ acts as identity and always exists. In the example if we make substitution $nx = t$ then we have

$$\begin{aligned} \langle f * \delta(x-n), \phi(x) \rangle &= \langle f(u), \langle \delta(x-n), \phi(ux) \rangle \rangle \\ &= \langle f(u), \phi(nx) \rangle \\ &= \frac{1}{n} \left\langle f\left(\frac{t}{n}\right), \phi(t) \right\rangle \end{aligned}$$

then we obtain

$$f * \delta(x-a) = \frac{1}{a} f_{\frac{t}{a}} \text{ for } a > 0$$

The following theorem holds in [1].

THEOREM 7 The Mellin convolution is commutative and associative that is, $f, g, h \in E'_{p,q}$ then

$$f * g = g * f, (f * g) * h = f * (g * h)$$

Let $M(f)$ and $M(g)$ be exists for $f, g \in E'_{p,q}$ then the following equation also hold in distributional sense.

$$M(f * g) = M(f) M(g)$$

3. SOME APPLICATIONS OF MELLIN TRANSFORM

The Mellin Transform has applications in various areas, including digital data structures, probabilistic algorithms, asymptotics of Gamma-related functions, coefficients of Dirichlet series, asymptotic estimation of integral forms, asymptotic of algorithms and communication theory.

The Mellin transform might be used in solving the ordinary differential equations with polynomial coefficients which are usually difficult to be solved by using the Laplace transform.

Let us consider the general form of the linear ordinary differential equation with polynomial-coefficient which is given by the equation

$$P(D)y = f(x)$$

where $P(D)$ is the differential operator with polynomial coefficients. There is no general method that can solve all kind of differential equations. Each might require different methods. By application of Mellin transform we reduce ordinary differential equations to the difference equation. In fact when we try to solve this differential equation, we might have either of the following cases, see [3].

1. The solution y is a smooth function such that the operation can be performed in the classical sense and the resulting equation is an identity. Then y is a classical solution.
2. The solution y is not smooth enough, so that the operation can not be performed but satisfies as a distributions.
3. The solution y is a singular distribution then the solution is a distributional solution.

In particular if we choose here to discuss the following differential equation

$$f'' - x^2 f' + xf = \delta(x-a)$$

Then by applying the Mellin transform to the both side and on using the theorem 6 we have

$$(s-10)(s-2)M(f:s-2) + \frac{1}{s+2}M(f:s+3) + M(f:s+1) = \frac{1}{a}$$

By solving the difference equation we obtain

$$M(f:s) = \frac{1}{a} \frac{1}{\left(\frac{(s-1)(s-2)}{x^2} + \frac{x^4}{s+2} + x^2 \right)}$$

Then one can apply the Inverse Mellin transform by using the complex inversion integral in order to cover the $f(x)$ explicitly as the solution.

The Mellin transform is also applicable to series in order to get summation for infinite series. For example if we want to get the sum of

$$h(x) = \sum_{i=1}^{\infty} f(kx)$$

where $x \in (0, \infty)$. By taking the Mellin Transform on both sides one can obtain

$$M(h:s) = \sum_{i=1}^{\infty} \frac{1}{k^s} M(f:s) = \zeta(s)M(f:s)$$

where $\zeta(s)$ is the Riemann Zeta function. Then by applying the Inverse Mellin transform it gives

$$h(x) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} \zeta(s) M(f : s) x^{-s} ds, \text{ see [6].}$$

4. REFERENCES

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