Introduction to Tauberian theory
A distributional approach

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Preface

These are lecture notes of a series of lectures I gave at the Institute of Mathematics of Aalto University and at the School of Mathematics of the University of Costa Rica, in the summer of 2011. The main aim of the lectures was to provide a modern introduction to Tauberian theory via distributional methods and some of its applications.

The central topic is Tauberian theorems for the Laplace transform of Schwartz distributions with applications to prime number theory and PDE with constant coefficients. It is also shown how to recover the classical Tauberian theorems from their distributional versions.

These notes consist of four chapters:

Chapter 1 is of introductory character. We state there a number of classical examples of Tauberian theorems. Their generalizations to the setting of generalization functions is the core of this text. We then explain a general scheme to attack problems in Tauberian theory from a distributional perspective. Basics from distribution theory are also recalled in this chapter.

In Chapter 2, we study one-dimensional Hardy-Littlewood type Tauberian theorems. We begin our incursion into Tauberian theorems for Laplace transforms by giving a simple proof of the celebrated Littlewood's Tauberian theorem for the converse of Abel's theorem on power series [29, 18, 27, 43]. We then proceed to show a general version of the Hardy-Littlewood Tauberian theorem for the distributional Laplace transform, such a version is due to Drozhzhinov and Zavialov [11]. We use the distributional version to easily recover several classical Tauberians of Hardy and Littlewood for power series and Stieltjes integrals.

Tauberian theorems in which complex-analytic or boundary properties of the transform play an important role are usually referred as *complex Tauberians*. Many of these complex Tauberians have been inspired by number theoretic questions; for example, the classical Ikehara [26] theorem was motivated by the search for a simple proof of the prime number theory. They have also important implications in PDE theory [2, 37]. We will study in Chapter 3 a generalization of the Wiener-Ikehara theorem, due to Korevaar [27], which relax the boundary requirements on the Laplace transform to a minimum [27, 36]. We shall also discuss applications to the theory of (Beurling) generalized primes [3, 36, 44].

Chapter 4 is devoted to multidimensional theory. We essential discuss generalizations of the Hardy-Littlewood theorem [48] for the multidimensional Laplace transform of distributions and measures supported by cones. As an illustration, we apply the results to obtain asymptotic properties of fundamental solutions to convolution equations, in particular, to hyperbolic operators with constant coefficients. This last chapter is based on results of Vladimirov, Drozhzhinov and Zavialov; we point out that they are explained in more detail in the beautiful monograph [48].

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Chapter 1

Introduction

Tauberian theory deals with the problem of obtaining asymptotic information about a function, classical or generalized one, from a priori knowledge of the asymptotic behavior of certain "averages" of the function. Such averages are usually given by an integral transform.

Tauberian theory provides striking methods to attack hard problems in analysis. The study of Tauberian type theorems has been historically stimulated by their potential applications in diverse fields of mathematics such as number theory, combinatorics, complex analysis, probability theory, and the analysis of differential operators [2, 5, 18, 26, 48, 51]. Even mathematical physics has pushed forward developments of the subject. Indeed, many of the theorems that we will discuss in future chapters had their origins in theoretical questions from quantum field theory [48]. Therefore, we may say that mathematical physics motivated the incorporation of *generalized functions* into the scope of Tauberian theory.

The nature of Tauberian theory is better self-explained through explicit examples. We state in this chapter a number of classical Tauberians. The core of the next chapters will be their generalizations to the context of Schwartz distributions. This chapter contains also some background material on distribution theory and Laplace transform (Section 1.5).

1.1 Littlewood's Theorem

Let us start by recalling Abel's theorem on power series (1826, [1]). It states that if the series $\sum_{n=0}^{\infty} c_n$ is convergent to the number β then

$$\lim_{r\to 1^-} \sum_{n=0}^{\infty} c_n r^n = \beta. \tag{1.1}$$

The proof of this well known theorem is very easy (cf. Subsection 1.2 below).

The theorem of Abel may be restated in terms of Abel summability, defined as follows. A (possible divergent) series $\sum_{n=0}^{\infty} c_n$ is said to be *Abel summable* to β if the power series $\sum_{n=0}^{\infty} c_n r^n$ has radius of convergence at least 1 and (1.1) is satisfied. In such a case one writes

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{A}),$$

the Abel sum of the series. Thus, Abel's theorem tells us that ordinary convergence implies Abel summability. Naturally, the converse is not true, in general, as shown by the divergent series $\sum_{n=0}^{\infty}(-1)^n$, which is Abel summable to $1/2$.

The genesis of Tauberian theory goes back to Tauber [39]. He showed in 1897 the following theorem that gives a sufficient condition for the converse of Abel's theorem.

Theorem 1.1 (Tauber's theorem, 1897)**.** *If* $\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{A})$ *and*

$$c_n = o\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.2}$$

then $\sum_{n=0}^{\infty} c_n$ *converges to* β*.*

Tauber's theorem is elementary and very simple to show. In 1910, Littlewood [29] gave his celebrated extension of Tauber's theorem, where he substituted the Tauberian condition (1.2) by the weaker one $c_n = O(1/n)$ and obtained the same conclusion of convergence as in Theorem 1.1.

Theorem 1.2 (Littlewood's theorem, 1910)**.** *If* $\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{A})$ *and*

$$c_n = O\left(\frac{1}{n}\right), \quad n \to \infty, \tag{1.3}$$

then $\sum_{n=0}^{\infty} c_n$ *converges to* β*.*

Despite the simplicity of its statement, it turns out that Littlewood's theorem is much deeper and difficult to prove than Theorem 1.1. Two years later [19], Hardy and Littlewood conjectured that the condition $nc_n > -K$ would be enough to ensure convergence; indeed, they provided a proof later in [20]. We will obtain proofs of these Hardy-Littlewood theorems in Section 2.2 as part of a more general Tauberian program for Laplace transforms of distributions.

Exercise 1.3. Show Tauber's theorem.

1.2 Other Summability Methods

We can define other summability methods in the same spirit as that of Abel summability. Let φ be a function such that $\varphi(0) = 1$, $\lim_{x\to\infty} \varphi(x) = 0$, and its derivative φ' is integrable on the interval $[0, \infty)$. We say that the series $\sum_{n=0}^{\infty} c_n$ is (φ) summable to β if there is $\lambda_0 > 0$ such that

$$\sum_{n=0}^{\infty} c_n \varphi\left(\frac{n}{\lambda}\right) \quad \text{converges for all } \lambda > \lambda_0, \tag{1.4}$$

and

$$\lim_{\lambda\to\infty} \sum_{n=0}^{\infty} c_n \varphi\left(\frac{n}{\lambda}\right) = \beta. \tag{1.5}$$

Let us verify that this summability method is regular [18], in the sense that it sums convergent series to their actual values of convergence. We employ Stieltjes integrals for the proof of this fact. Write $s(x) = \sum_{n<x} c_n$ so that $\lim_{x\to\infty} s(x) = \beta$. Integration by parts then shows that

$$\lim_{\lambda\to\infty} \sum_{n=0}^{\infty} c_n \varphi\left(\frac{n}{\lambda}\right) = \lim_{\lambda\to\infty} \int_0^{\infty} \varphi\left(\frac{u}{\lambda}\right) \mathrm{d}s(u) = -\lim_{\lambda\to\infty} \int_0^{\infty} s(\lambda u)\, \varphi'(u)\, \mathrm{d}u = \beta\varphi(0) = \beta.$$

Different choices of the kernel φ lead to many familiar methods of summability. If $\varphi(u) = e^{-u}$ for $u > 0$, one then recovers the Abel method just discussed in Subsection 1.1. The choice $\varphi(u) =$

$(1-u)^{\kappa}$, where $\kappa > 0$, gives the kernel of Cesàro-Riesz summability of order κ; one writes in this case

$$\sum_{n=0}^{\infty} c_n = \beta \quad (\mathrm{C}, \kappa). \tag{1.6}$$

Cesàro summability is often introduced in a different but equivalent way. A non-trivial theorem of M. Riesz [34] (cf. [22]) shows that (1.6) holds if and only if

$$\lim_{n\to\infty} \frac{\Gamma(\kappa+1)}{n^{\kappa}} \sum_{j=0}^{n} \binom{j+\kappa}{j} c_{n-j} = \beta, \tag{1.7}$$

where Γ is the Euler gamma function. See [18] for the approach using the Cesàro means (1.7). However, the Riesz summability means given by $\varphi(u) = (1-u)^{\kappa}$ are more convenient from the distributional point of view [14]. An detailed study of Riesz means can be found in [8].

Another instance is provided by $\varphi(u) = u/(e^u - 1)$, $u > 0$, the kernel of Lambert summability which is so important in number theory [26, 51]. Remarkably, in 1928 Wiener used a Tauberian theorem for Lambert summability to deduce the prime number theorem [50].

It can be shown that Littlewood's Tauberian condition (1.3) is also a Tauberian condition for Cesàro and Lambert summability, namely, the conditions (1.4) and (1.5) along with (1.3) imply the convergence of the series. A deep analysis of Tauberian theorems for quite general kernels φ is given in the outstanding work of Wiener [26, 51] (see also [30, 32]). We will not pursue such a general study in these notes, we will rather focus in generalizations of Abel summability in the context of Schwartz distributions.

1.3 The Ikehara Theorem

In 1931 Ikehara showed the following Tauberian theorem for Dirichlet series [21]. His aim was to find a proof of the prime number theorem (PNT) in the form

$$\pi(x) = \sum_{\substack{p<x \\ p \text{ prime}}} 1 \sim \frac{x}{\log x} \quad \text{as } x \to \infty.$$

He succeeded in giving a proof of the PNT based solely on the non-vanishing of the Riemann zeta function on the line $\Re e\, s = 1$.

Theorem 1.4 (Ikehara, 1931)**.** *Let F be given by the Dirichlet series*

$$F(s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}, \quad \text{convergent for } \Re e\, s > 1,$$

where the coefficients satisfy the Tauberian condition $c_n \geq 0$. If there exists a constant β such that

$$F(z) - \frac{\beta}{z-1}$$

admits a continuous extension to the line $\Re e\, z = 1$, then

$$\sum_{n=1}^{N} c_n \sim \beta N \quad \text{as } N \to \infty.$$

Theorem 1.4 extends an early result of Landau [28]. Landau had also used his Tauberian theorem to deduce the prime number theorem, however, in his proof he needed additional information on the growth of the Riemann zeta function on $\Re e\, s = 1$. Such growth properties are avoided when one derives the prime number theorem from Ikehara's theorem. The derivation of the prime number theorem will be done in Section 3.4, where actually a more general result due to Beurling [3, 4] is shown.

Ikehara's theorem has also shown to be very valuable in the study of (pseudo)differential operators. See [2] for its use in semigroup theory and [37] for its applications to spectral theory of pseudodifferential operators.

We will consider a more general version of Theorem 1.4 in Chapter 3 and its applications to prime number theory in the context of Beurling's generalized numbers.

1.4 A Functional Analysis Scheme for Tauberian Problems

We describe in this subsection our general philosophy to attack Tauberian problems.

Let us start with some comments about the relationship between Littlewood's Tauberian condition (1.3) and convergence of series. We shall consider the Dirac delta measure δ_a, concentrated at $a \in \mathbb{R}$. It is the linear functional defined on continuous functions φ as

$$\langle \delta_a, \varphi \rangle = \varphi(a). \tag{1.8}$$

To a numerical series $\sum_{n=0}^{\infty} c_n$, we associate the formal delta series

$$f_\lambda = \sum_{n=0}^{\infty} c_n \delta_{\frac{n}{\lambda}}, \quad \lambda \in [1, \infty),$$

acting on a suitable vector space of continuous functions E as

$$\langle f_\lambda, \varphi \rangle = \sum_{n=0}^{\infty} c_n \varphi\left(\frac{n}{\lambda}\right). \tag{1.9}$$

If (1.9) converges for all $\varphi \in E$, then, clearly, f_λ automatically becomes a linear functional on E. The connection between (1.3) and the convergence of $\sum_{n=0}^{\infty} c_n$ is given by the following lemma, essentially contained in [42]. We use the notation $\mathcal{D}[0, \infty)$ for the space of C^∞-functions on $[0, \infty)$ which vanish off compact sets. Furthermore, $C[0, \infty)$ denotes the space of continuous functions on $[0, \infty)$.

Lemma 1.5. *Let E be a vector space such that $\mathcal{D}[0, \infty) \subseteq E \subset C[0, \infty)$ and (1.9) converges for each φ and $\lambda \in [1, \infty)$. Assume that $\lim_{\lambda\to\infty} f_\lambda = \beta\delta_0$ pointwise on E, i.e.,*

$$\lim_{\lambda\to\infty} \langle f_\lambda, \varphi \rangle = \beta\varphi(0), \quad \textit{for each } \varphi \in E, \tag{1.10}$$

Then, the Tauberian condition $c_n = O(1/n)$ implies $\sum_{n=0}^{\infty} c_n = \beta$.

Proof. Let $M > 0$ be such that $n\,|c_n| \le M$. Given $\sigma > 1$ arbitrary, find $\varphi_\sigma \in \mathcal{D}[0, \infty)$ such that $0 \le \varphi_\sigma \le 1$, $\varphi_\sigma(x) = 1$ for $x \in [0, 1]$, and $\operatorname{supp} \varphi_\sigma \subset [0, \sigma)$, then, (1.10) with $\varphi = \varphi_\sigma$ gives

$$\limsup_{\lambda\to\infty} \left| \sum_{0\le n\le\lambda} c_n - \beta \right| \le \limsup_{\lambda\to\infty} \left| \sum_{1<\frac{n}{\lambda}<\sigma} \frac{M}{n} \varphi_\sigma\left(\frac{n}{\lambda}\right) \right| = M \int_1^\sigma \frac{\varphi_\sigma(u)}{u}\, \mathrm{d}u < M(\sigma - 1),$$

and so, taking $\sigma \to 1^+$, we conclude $\sum_{n=0}^{\infty} c_n = \beta$. ☐

Therefore, Lemma 1.5 tells that if $\sum_{n=0}^{\infty} c_n$ is (φ) summable to β for every kernel $\varphi \in E$, then the Tauberian condition (1.3) is sufficient to obtain convergence. Now, the typical Tauberian problem may be rephrased as follows: to pass from (φ) summability for one specific kernel φ to (φ) summability for all kernels in a function space E satisfying $\mathcal{D}[0,\infty) \subseteq E \subset C[0,\infty)$.

So far, we have not spoken about any topological structure on E. However, as any student in a very first course of functional analysis learns, the main tool to deduce (1.10) is the Banach-Steinhaus theorem together with weak convergence over a dense subset. The following theorem will be therefore very important in the sequel, it is nothing but a consequence of the Banach-Steinhaus theorem.

Given a topological vector space E, the set E' is as usual the space of continuous linear functional over E. We assume that the reader is familiar with the notion of Fréchet spaces [35, 40]. In our context E will be a Schwartz space of test functions and E' a distribution space (cf. Sections 1.5 and 4.1).

Theorem 1.6. *Let E be a Fréchet space (or more generally a barreled space [40]). Denote by E' its dual space. Let $\{f_\lambda\}_{\lambda\in[1,\infty)}$ be a net in E'. Necessary and sufficient conditions for the existence of the limits*

$$\lim_{\lambda\to\infty} \langle f_\lambda, \varphi \rangle \,, \tag{1.11}$$

for each $\varphi \in E$, are:

(i) The existence of a dense subset $\mathfrak{B} \subset E$ such that the limit (1.11) exists for all $\varphi \in \mathfrak{B}$.

(ii) $\{f_\lambda\}_{\lambda\in[1,\infty)}$ is weakly bounded in E': For each $\varphi \in E$, $\langle f_\lambda, \varphi \rangle = O(1)$, namely, there is M_φ such that $\langle f_\lambda, \varphi \rangle \leq M_\varphi$ for $\lambda \in [1,\infty)$.

In such a case, there is a $g \in E'$ such that $\lim_{\lambda\to\infty} f_\lambda = g$ weakly, i.e.,

$$\lim_{\lambda\to\infty} \langle f_\lambda, \varphi \rangle = \langle g, \varphi \rangle \quad \text{for each } \varphi \in E.$$

Proof. That the conditions are necessary is obvious. Let us show the sufficiency. Condition (ii) is equivalent, by the Banach-Steinhaus theorem, to the equicontinuity of $\{f_\lambda\}_{\lambda\in[1,\infty)}$. Thus, there exists a open neighborhood of the origin $V \subset E$ and a constant M (independent on ψ below) such that

$$|\langle f_\lambda, \psi \rangle| \leq M \quad \text{for all } \psi \in V. \tag{1.12}$$

Fix an arbitrary $\varphi_0 \in E$. We show that $\langle f_\lambda, \varphi_0 \rangle$ is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Then, by the density of $\mathfrak{B}$, we can find $\varphi \in \mathfrak{B}$ such that $\varphi_0 - \varphi \in \varepsilon V$. Moreover, (i) ensures the existence of $\lambda_0 > 0$ such that $|\langle f_{\lambda_1} - f_{\lambda_2}, \varphi \rangle| < \varepsilon$ for all $\lambda_1, \lambda_2 \in (\lambda_0, \infty)$. Hence, since $\psi = \varepsilon^{-1}(\varphi_0 - \varphi) \in V$, it follows from (1.12) that

$$|\langle f_{\lambda_1}, \varphi_0 \rangle - \langle f_{\lambda_2}, \varphi_0 \rangle| \leq |\langle f_{\lambda_1}, \varphi_0 - \varphi \rangle| + |\langle f_{\lambda_2}, \varphi_0 - \varphi \rangle| + |\langle f_{\lambda_1} - f_{\lambda_2}, \varphi \rangle| \leq 2\varepsilon M + \varepsilon,$$

this shows the existence of $\langle g, \varphi_0 \rangle := \lim_{\lambda\to\infty} \langle f_\lambda, \varphi_0 \rangle$ for each $\varphi_0 \in E$. It is clear that g is a linear functional; however, we still need to show that it is continuous. For it, it is enough to take limit in (1.12) for fixed $\psi \in V$, so that we conclude

$$|\langle g, \psi \rangle| \leq M \quad \text{for all } \psi \in V.$$

This yields $g \in E'$. □

Let us say some further words about our strategy to show Tauberian theorems. We exemplify it with the case of Littlewood's theorem. Let $\mathfrak{D}$ be the linear span of functions of the form $e^{-\sigma u}$, $\sigma > 0$. Consider the linear functionals $\{f_\lambda\}_{\lambda\in[1,\infty)}$ given by (1.9). What Abel summability of $\sum_{n=0}^{\infty} c_n$ tells is that (1.10) holds precisely for all $\varphi \in \mathfrak{D}$. Thus, by Lemma 1.5 and Theorem 1.6, Littlewood's theorem would follow at once if we are able to find a suitable Fréchet space E such that $\{f_\lambda\}_{\lambda\in[1,\infty)}$ is weakly bounded in E' and $\mathfrak{D}$ is dense in E. We could go on and prove Littlewood's theorem right now; however, we choose to postpone the details until Section 2.2 after recalling some basics from distribution theory.

Finally, it should be mentioned that the functional analysis scheme just described above is also present in the methods developed by Karamata [24, 25] and Wiener [51] (cf. [26]).

1.5 Preliminaries on Distribution Theory

We briefly introduce in this section some concepts from distribution theory. We refer to [38, 47] for the theory of distributions.

The test function space $\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing smooth test functions, that is, those C^∞-functions over the real such that for each $m, k \in \mathbb{N}$

$$\sup_{x\in\mathbb{R}} \left| x^m \varphi^{(k)}(x) \right| < \infty.$$

It is a Fréchet space, topologized by means of the countable family of norms

$$\|\varphi\|_m = \sup_{0\le j\le m,\, x\in\mathbb{R}} (1+|x|)^m \left|\varphi^{(j)}(x)\right|, \quad m \in \mathbb{N}. \tag{1.13}$$

We shall use the following Fourier transform

$$\hat{\varphi}(t) = \mathcal{F}\{\varphi(u); t\} = \int_{-\infty}^{\infty} \varphi(u) e^{-itu} \mathrm{d}u, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

The Fourier inverse transform is given by

$$\mathcal{F}^{-1}\{\varphi(t); u\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(t) e^{iut} \mathrm{d}t, \quad \varphi \in \mathcal{S}(\mathbb{R}).$$

The Fourier transform is one to one and onto. It is also continuous in the topology of $\mathcal{S}(\mathbb{R})$, as may be easily deduced from the formulas

$$\widehat{(\varphi^{(k)})}(t) = (it)^k \hat{\varphi}(t) \quad \text{and} \quad \widehat{(u^k\varphi)}(t) = i^k \hat{\varphi}^{(k)}(t).$$

Its dual, the space of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R})$. The evaluation of $f \in \mathcal{S}'(\mathbb{R})$ at $\varphi \in \mathcal{S}(\mathbb{R})$ is denoted by $\langle f, \varphi\rangle$. Thus, $f \in \mathcal{S}'(\mathbb{R})$ if:

- $\langle f, a\varphi + \psi\rangle = a\langle f, \varphi\rangle + \langle f, \psi\rangle$, for all $a \in \mathbb{C}$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$.
- $\lim_{n\to\infty} \langle f, \varphi_n\rangle = \langle f, \lim_{n\to\infty} \varphi_n\rangle$ whenever $\{\varphi\}_{n=0}^{\infty}$ is convergent in $\mathcal{S}(\mathbb{R})$.

It is sometimes convenient to write a "dummy variable" in the evaluation and think of $\langle f(u), \varphi(u)\rangle$ as some kind of integral so that we view f as a generalized function. Indeed, any locally integrable function f of at most polynomial growth can be regarded as tempered distribution $f \in \mathcal{S}'(\mathbb{R})$ via the identification

$$\langle f(u), \varphi(u)\rangle = \int_{-\infty}^{\infty} f(u)\varphi(u) \mathrm{d}u.$$

Distributions determined in these manner are called regular distributions.

We can define many useful operations on $\mathcal{S}'(\mathbb{R})$ by duality, as long as they are continuous operations on the test function space $\mathcal{S}(\mathbb{R})$. Given a generic $f \in \mathcal{S}'(\mathbb{R})$, we define

- Its derivative $f^{(k)} \in \mathcal{S}'(\mathbb{R})$ as

$$\left\langle f^{(k)}(u), \varphi(u) \right\rangle = (-1)^k \left\langle f(u), \varphi^{(k)}(u) \right\rangle .$$

- Its Fourier transform $\hat{f} \in \mathcal{S}'(\mathbb{R})$ as

$$\left\langle \hat{f}(u), \varphi(u) \right\rangle = \langle f(t), \hat{\varphi}(t) \rangle .$$

- For numbers $a, b \in \mathbb{R}$, the linear change of variables $f(au+b) \in \mathcal{S}'(\mathbb{R})$ as

$$\langle f(au+b), \varphi(u) \rangle = \frac{1}{|a|} \left\langle f(t), \varphi\left(\frac{t-b}{a}\right) \right\rangle .$$

- If $\psi \in \mathcal{S}(\mathbb{R})$, the distribution $\psi f \in \mathcal{S}'(\mathbb{R})$ is defined as

$$\langle \psi(u) f(u), \varphi(u) \rangle = \langle f(u), \psi(u)\varphi(u) \rangle$$

These four operations on distributions are in accordance with those for ordinary functions. For instance, the definition of the distributional derivative is nothing but integration by parts, while that of Fourier transform coincides with Parseval's identity when $f \in L^2(\mathbb{R})$.

Convergence of nets $f_\lambda \to f$ in $\mathcal{S}'(\mathbb{R})$ is interpreted in the weak sense, namely,

$$f_\lambda \to f \text{ in } \mathcal{S}'(\mathbb{R}) \quad \text{if and only if} \quad \langle f_\lambda, \varphi \rangle \to \langle f, \varphi \rangle \quad \text{for each } \varphi \in \mathcal{S}(\mathbb{R}).$$

The operations defined above become automatically continuous on $\mathcal{S}'(\mathbb{R})$, thus if $f_\lambda \to f$ in $\mathcal{S}'(\mathbb{R})$ then $\hat{f}_\lambda \to \hat{f}$ and $f_\lambda^{(k)} \to f^{(k)}$ in $\mathcal{S}'(\mathbb{R})$.

We denote by $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ the subspace of compactly supported test functions.

A distribution $f \in \mathcal{S}'(\mathbb{R})$ is said to vanish on the interval (a, b) if $\langle f, \varphi \rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ supported in (a, b). One can then defined $\operatorname{supp} f$ as the complement of the biggest open set where f vanishes.

Let us discuss some examples of particular distributions. These examples will be used in the future.

Example 1.7. *Dirac delta.* We defined the Dirac deltas in Section 1.4 through (1.8). From now on, we shall use the notation $\delta = \delta_0$, so that δ_a is simply given by translation of δ, that is, $\delta_a(u) = \delta(u-a)$. Observe also that δ transforms under linear change of variables as $\delta(au-b) = |a|^{-1}\,\delta(u - a^{-1}b)$.

Example 1.8. *Heaviside function.* The Heaviside function H is the characteristic function of the interval $[0, \infty)$. Its evaluation at test functions is then given by $\langle H(u), \varphi(u) \rangle = \int_0^\infty \varphi(u)\mathrm{d}u$, and since $-\int_0^\infty \varphi'(u)\mathrm{d}u = \varphi(0)$, we obtain $H' = \delta$.

Example 1.9. *The distributions u_+^α.* If $\Re e\, \alpha > -1$, the distribution u_+^α is a regular distribution given by $u_+^\alpha = u^\alpha H(u)$. Its action on test functions is simply given by the integral

$$\left\langle u_+^\alpha, \varphi(u) \right\rangle = \int_0^\infty u^\alpha \varphi(u) \mathrm{d}u\,; \tag{1.14}$$

when $\Re e\, \alpha < -1$, $\alpha \notin \mathbb{Z}_-$, then u_+^α is defined as

$$\frac{u_+^\alpha}{\Gamma(\alpha+1)} = \frac{\left(u_+^{\alpha+n}\right)^{(n)}}{\Gamma(\alpha+n+1)}\,, \tag{1.15}$$

where $n = [-\alpha]$. Therefore, u_+^α is well defined for $\alpha \in \mathbb{C} \setminus \mathbb{Z}_-$. The expression (1.15) is meaningful for $\alpha = -k \in \mathbb{Z}_-$; indeed, since $u_+^0 = H$,

$$\left.\frac{u_+^\alpha}{\Gamma(\alpha+1)}\right|_{\alpha=-k} = \delta^{(k-1)}(u). \tag{1.16}$$

Example 1.10. *Functions of local bounded variation.* Assume that S is a function of local bounded variation (i.e., of bounded variation on each compact interval) which has at most polynomial growth. Then integration by parts yields $S' = \mathrm{d}S$, in the sense

$$\left\langle S'(u), \varphi(u) \right\rangle = \int_{-\infty}^{\infty} \varphi(u) \mathrm{d}S(u). \tag{1.17}$$

The Stieltjes integral (1.17) may not be absolutely convergent, it is rather an improper integral (e.g., consider $S(x) = e^{ie^x}$). The case $S(x) := \sum_{0 \le \lambda_n < x} c_n$ is of particular interest for us; in this case we obtain

$$S'(u) = \sum_{n=0}^{\infty} c_n \delta(u - \lambda_n).$$

Exercise 1.11. A distribution $f \in \mathcal{S}'(\mathbb{R})$ is said to be homogeneous of degree $\alpha \in \mathbb{R}$ if $f(au) = a^\alpha f(u)$ for all $a > 0$, namely, for each test function

$$\langle f(au), \varphi(u) \rangle = \frac{1}{a} \left\langle f(u), \varphi\left(\frac{u}{a}\right) \right\rangle = a^\alpha \langle f(u), \varphi(u) \rangle\,, \quad \text{for all } a > 0. \tag{1.18}$$

Show that if $\alpha \notin \mathbb{Z}_-$, then u_+^α homogeneous of degree α, while $\delta^{(k-1)}(u)$ is homogeneous of degree $-k \in \mathbb{Z}_-$, that is,

$$(au)_+^\alpha = a^\alpha u_+^\alpha \quad \text{and} \quad \delta^{(k-1)}(au) = \frac{\delta^{(k-1)}(u)}{a^k}, \quad a > 0. \tag{1.19}$$

Exercise 1.12. Show that if $g \in \mathcal{S}(\mathbb{R})$ is homogeneous of degree α, then it satisfies the differential equation

$$ug'(u) = \alpha g(u) \tag{1.20}$$

(**Hint**: Differentiate (1.18) with respect to a and then set $a = 1$).

Exercise 1.13. Show that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$.

1.5.1 Laplace Transforms

We follow [47] and define the Laplace transform for tempered distributions with supports in $[0,\infty)$.

The spaces $\mathcal{D}[0,\infty)$ and $\mathcal{S}[0,\infty)$ consist of restrictions of elements of $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ to $[0,\infty)$, respectively. The space $\mathcal{S}[0,\infty)$ has a canonical topology, given by the norms (1.13) where only $x \geq 0$ is taken into account.

Analogously, $\mathcal{S}'[0,\infty) \subset \mathcal{S}'(\mathbb{R})$ denotes the subspace of distributions supported in $[0,\infty)$. It is canonically isomorphic to the dual of $\mathcal{S}[0,\infty)$. The identification is as follows. Let $f \in \mathcal{S}'(\mathbb{R})$ be such that $\operatorname{supp} f \subseteq [0,\infty)$ and $\varphi \in \mathcal{S}[0,\infty)$, find any $\psi \in \mathcal{S}(\mathbb{R})$ so that $\psi(x) = \varphi(x)$ for all $x \in [0,\infty)$; one then defines $\langle f, \varphi \rangle := \langle f, \psi \rangle$, such a definition does not depend on the extension ψ.

The definition of the Laplace transform on $\mathcal{S}'[0,\infty)$ is now easy. Observe that if $\Re e\, s > 0$, then the function $e^{-su} \in \mathcal{S}[0,\infty)$. Given $f \in \mathcal{S}'[0,\infty)$, its Laplace transform is defined as

$$\mathcal{L}\{f; s\} = \left\langle f(u), e^{-su} \right\rangle, \quad \Re e\, s > 0. \tag{1.21}$$

It is analytic in s on the half-plane $\Re e\, s > 0$. We write $s = \sigma + it$ for complex variables.

The relation between the Laplace and Fourier transforms [47, 7] is given by the following two propositions.

Proposition 1.14. *Let $f \in \mathcal{S}'[0,\infty)$. Then,*

$$\mathcal{L}\{f; \sigma + it\} = \mathcal{F}\left\{f(u)e^{-\sigma u}; t\right\}, \quad \text{for } \sigma > 0. \tag{1.22}$$

Moreover,

$$\hat{f}(t) = \lim_{\sigma \to 0^+} \mathcal{L}\{f; \sigma + it\} \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{1.23}$$

In particular, the Laplace transform is injective.

Proof. The equality (1.22) means that if $\varphi \in \mathcal{S}(\mathbb{R})$

$$\int_{-\infty}^{\infty} \left\langle f(u), \varphi(t) e^{-\sigma u} e^{-itu} \right\rangle \mathrm{d}t = \left\langle f(u), e^{-\sigma u} \int_{-\infty}^{\infty} \varphi(t) e^{-itu} \mathrm{d}t \right\rangle. \tag{1.24}$$

The fact that one can interchange the integral and the dual paring is left as an exercise for the reader (cf. Exercise 1.21).

Next, we clearly have $\lim_{\sigma \to 0^+} e^{-\sigma u} f(u) = f(u)$ in $\mathcal{S}'[0,\infty)$; therefore, 1.23 follows by taking limit in (1.22) and using that the Fourier transform is continuous on $\mathcal{S}'(\mathbb{R})$.

That the Laplace transform is injective follows now from (1.23) and the fact that the Fourier transform is one to one. □

Thus, $\hat{f}$ is the distributional boundary value, in the sense of (1.23), of $\mathcal{L}\{f; s\}$ on the line $\Re e\ s = 0$. This fact yields the following corollary (a consequence of the Hanh-Banach theorem [35, 40]).

Corollary 1.15. *Let $\mathfrak{D}$ be the linear span of the set $\{e^{-su} :\ s > 0\}$. Then, $\mathfrak{D}$ is dense in $\mathcal{S}[0,\infty)$.*

Proof. By the Hahn-Banach theorem, the assertion is equivalent to show that if $f \in \mathcal{S}'[0,\infty)$ and $\langle f, \varphi \rangle = 0$ for all $\varphi \in \mathfrak{D}$, then f must necessarily be the zero functional. Thus assume that the functional f vanishes identically on $\mathfrak{D}$. This gives in particular that $\mathcal{L}\{f; s\} = 0$ for all $(0,\infty)$, but since it is analytic, it must identically vanish on $\Re e\ s > 0$. Proposition 1.14 now implies that $f = 0$. □

Observe that (1.22) yields an inversion formula for the Laplace transform.

Corollary 1.16. *Let $f \in \mathcal{S}'[0,\infty)$. Then, for any fix $\sigma > 0$,*

$$f(u) = e^{\sigma u}\mathcal{F}_t^{-1}\left\{\mathcal{L}\left\{f;\sigma + it\right\};u\right\}, \tag{1.25}$$

where the inverse Fourier transform is taken with respect to the variable t.

We end this subsection by estimating the growth properties of the Laplace transform.

Proposition 1.17. *Given $f \in \mathcal{S}'[0,\infty)$, there exist $m, l \in \mathbb{N}$ and $C > 0$ such that*

$$|\mathcal{L}\left\{f;s\right\}| \leq C\frac{(1+|s|)^l}{\sigma^m}, \quad \textit{for } \sigma > 0 \ (\Re e\ s = \sigma). \tag{1.26}$$

Proof. Since f is a continuous linear functional on $\mathcal{S}[0,\infty)$, there exists $m \in \mathbb{N}$ and $M > 0$ such that

$$|\langle f(u), \varphi(u)\rangle| \leq M \sup_{0\leq j\leq m,\, u\in[0,\infty)} (1+u)^m \left|\varphi^{(j)}(u)\right|, \quad \text{for all } \varphi \in \mathcal{S}[0,\infty).$$

Setting $\varphi(u) = e^{-su}$ in the above inequality, we obtain

$$\begin{aligned}|\mathcal{L}\left\{f;s\right\}| &\leq M \sup_{0\leq j\leq m,\, u\in[0,\infty)} (1+u)^m \left|s^j e^{-su}\right| \leq M(1+|s|)^m \sup_{u\in[0,\infty)} (1+u)^m e^{-\sigma u} \\ &= M\frac{(1+|s|)^m}{\sigma^m} \sup_{u\in[0,\infty)} (\sigma+u)^m e^{-u} \leq C\frac{(1+|s|)^k}{\sigma^m},\end{aligned}$$

where $k = 2m$ and $C = M\sup_{u\in[0,\infty)}(1+u)^m e^{-u}$. □

Remark 1.18. The converse of Proposition 1.17 also holds [47]: If a function $F(s)$, analytic on $\Re e\ s > 0$, satisfies a estimate (1.26), then there corresponds a unique $f \in \mathcal{S}'[0,\infty)$ such that $F(s) = \mathcal{L}\left\{f;s\right\}$.

Exercise 1.19. Show the formulas

$$\mathcal{L}\left\{u_+^\alpha;s\right\} = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \text{and} \quad \mathcal{L}\left\{\delta^{(k)}(u);s\right\} = s^k.$$

Exercise 1.20. (*Homogeneous distributions*) Let $g \in \mathcal{S}'[0,\infty)$ be homogeneous of degree α. Show that g must be of the form

$$g(u) = \begin{cases} Cu_+^\alpha, & \text{if } \alpha \notin\in \mathbb{Z}_-, \\ C\delta^{(k-1)}(u), & \text{if } \alpha = -k \in \mathbb{Z}_-, \end{cases} \tag{1.27}$$

for some constant C (**Hint**: Apply Laplace transform to (1.20) in Exercise 1.12, solve the differential equation, and then use Exercise 1.19 and the injectivity of the Laplace transform).

Exercise 1.21. Show (1.24) (**Hint**: Define the sequence $\psi_N \in \mathcal{S}[0,\infty)$ by

$$\psi_N(u) = e^{-\sigma u}\sum_{n=-N}^{N}\frac{1}{N}\varphi\left(\frac{k}{N}\right)e^{-iu\frac{k}{N}}$$

and show that $\lim_{N\to\infty}\psi_N(u) = e^{-\sigma u}\int_{-\infty}^{\infty}\varphi(t)e^{-itu}\mathrm{d}t$ in the topology of $\mathcal{S}[0,\infty)$).

Exercise 1.22. Show that if $\varphi \in \mathcal{D}(\mathbb{R})$, then

$$F(s) = \mathcal{L}\{\varphi; s\} = \int_{-\infty}^{\infty} \varphi(u) e^{-su} \mathrm{d}u$$

is an entire function in $s \in \mathbb{C}$. In addition, show that if $\operatorname{supp} \varphi \subseteq [-a, a]$, then for all $m, k \in \mathbb{N}$, there are constants $M_{m,k} > 0$ such that

$$\left| F^{(m)}(s) \right| < M_{m,k} \frac{e^{a|\Re e\, s|}}{(1 + |s|)^k} \ .$$

Chapter 2

Hardy-Littlewood Type Theorems

Hardy and Littlewood provided in [20] an extension of Theorem 1.2 to asymptotic behavior. We see that the one-sided version of Theorem 1.2 is the case $\alpha = 0$ of the following theorem. A more general version of Theorem 2.1 for Stieltjes integrals holds (cf. Subsections 2.3.2).

Theorem 2.1 (Hardy and Littlewood)**.** *Let $\sum_{n=0}^{\infty} c_n r^n$ be convergent for $|r| < 1$. Suppose that for some number $\alpha \geq 0$*

$$F(r) = \sum_{n=0}^{\infty} c_n r^n \sim \frac{C}{(1-r)^{\alpha}} \quad \text{as } r \to 1^- \tag{2.1}$$

(i.e., $\lim_{r \to 1^-} (1-r)^{\alpha} F(r) = C$). If

$$n c_n \geq -M n^{\alpha}, \quad \text{for all } n \geq 1, \tag{2.2}$$

then,

$$S_n = \sum_{k=0}^{n} c_k \sim C' n^{\alpha} \quad \text{as } n \to \infty, \quad \text{where } C' = \frac{C}{\Gamma(\alpha+1)}\,. \tag{2.3}$$

The corresponding Abelian counterpart to Theorem 2.1 is not difficult to show:

Proposition 2.2. *Suppose $S_n = \sum_{k=0}^{n} c_k \sim C' n^{\alpha}$. Then (2.1) holds with $C = C'\Gamma(\alpha+1)$.*

Proof. Write $S(x) = \sum_{n<x} c_n$ so that $S(x) \sim C'$ as $x \to \infty$. Set $r = e^{-y}$, then $y \to 0^+$ as $r \to 1^-$. By applying first integration by parts and then the dominated convergence theorem, we have

$$\sum_{n=0}^{\infty} c_n e^{-yn} = \int_0^{\infty} e^{-yu} \mathrm{d}S(u) = \int_0^{\infty} S\left(\frac{u}{y}\right) e^{-u} \mathrm{d}u$$
$$\sim \frac{1}{y^{\alpha}} \int_0^{\infty} u^{\alpha} e^{-u} \mathrm{d}u = \frac{C'\Gamma(\alpha+1)}{(-\log r)^{\alpha}} \sim \frac{C'\Gamma(\alpha+1)}{(1-r)^{\alpha}} \quad \text{as } r \to 1^-.$$

□

The aim of this chapter is to study analogs to Theorem 2.1 that are applicable to tempered distributions. Such distributional versions are originally due to Drozhzhinov and Zavialov [11]. We shall show that they include Theorem 2.1 as a particular instance. Multidimensional version will be considered in Chapter 4. Let us mention that all the results of this chapter can be extended to include comparison with Karamata regularly varying functions (cf. see [5, 26] for Stieltjes integrals and [48] for generalized functions). We also refer to [15, 16, 43] for distributional methods in Tauberian theorems for Abel summability.

2.1 Distributional Asymptotics

Let $f \in L^1_{\text{loc}}[0,\infty)$ (a locally integrable function). Assume that $f(x) \sim Cx^\alpha$ as $x \to \infty$, where $\alpha > -1$. Then, for each $\varphi \in \mathcal{S}(\mathbb{R})$

$$\int_0^\infty f(\lambda u)\varphi(u)\mathrm{d}u \sim C\lambda^\alpha \int_0^\infty u^\alpha \varphi(u)\mathrm{d}u = C\lambda^\alpha \left\langle u_+^\alpha, \varphi(u)\right\rangle \quad \text{as } \lambda \to \infty. \tag{2.4}$$

The idea of the distributional asymptotics is simple and natural. We use an analog to (2.4) as the definition for the asymptotic behavior of a distribution.

Definition 2.3. Let $f \in \mathcal{S}'[0,\infty)$. We say that f has (distributional) asymptotic behavior with respect to λ^α if there exists $g \in \mathcal{S}'[0,\infty)$ such that

$$\langle f(\lambda u), \varphi(u)\rangle \sim \lambda^\alpha \langle g(u), \varphi(u)\rangle \quad \text{as } \lambda \to \infty, \quad \text{for each } \varphi \in \mathcal{S}(\mathbb{R}). \tag{2.5}$$

In such a case, we write

$$f(\lambda u) \sim \lambda^\alpha g(u) \quad \text{as } \lambda \to \infty. \tag{2.6}$$

The distribution g in (2.6) is forced to be homogeneous, as stated in the following proposition.

Proposition 2.4. *If (2.6) is satisfied and $g \neq 0$, then g has the form (1.27).*

Proof. Take $\varphi \in \mathcal{S}(\mathbb{R})$, then, by (2.5), for each $a > 0$,

$$\begin{aligned}\langle g(au), \varphi(u)\rangle &= \left\langle g(u), \frac{1}{a}\varphi\left(\frac{u}{a}\right)\right\rangle = \lim_{\lambda\to\infty} \frac{\left\langle f(\lambda u), \frac{1}{a}\varphi(\frac{u}{a})\right\rangle}{\lambda^\alpha} = a^\alpha \lim_{\lambda\to\infty}\frac{\langle f(a\lambda u), \varphi(u)\rangle}{(a\lambda)^\alpha} \\ &= a^\alpha \langle g(u), \varphi(u)\rangle .\end{aligned}$$

□

We have seen that the ordinary asymptotic behavior of functions implies the distributional one. The converse is not true in general.

Example 2.5. The function $f(u) = (1+\sin u)H(u)$ has the following distributional limit

$$\lim_{\lambda\to\infty} f(\lambda u) = H(u).$$

However, it is clear that $\lim_{x\to\infty}(1+\sin x)$ does not exist. Let us verify that f has the distributional limit; indeed, if $\varphi \in \mathcal{S}(\mathbb{R})$,

$$\int_0^\infty (1+\sin\lambda u)\varphi(u)\mathrm{d}u = \int_0^\infty \varphi(u)\mathrm{d}u + \frac{\varphi(0)}{\lambda} + \frac{1}{\lambda}\int_0^\infty \cos(\lambda u)\varphi'(u)\mathrm{d}u = \int_0^\infty \varphi(u)\mathrm{d}u + O\left(\frac{1}{\lambda}\right).$$

It is therefore of vital importance to find conditions that allow us to come back from distributional asymptotics to classical ones. The following proposition goes in that direction. It can be interpreted as a Tauberian theorem, where the Tauberian hypothesis is given by monotonicity.

Proposition 2.6. *Let S be a non-decreasing function that vanishes for $x \leq 0$ and has distributional asymptotic behavior*

$$S(\lambda u) \sim C\lambda^\alpha u_+^\alpha \quad \text{as } \lambda \to \infty, \tag{2.7}$$

for some $\alpha \geq 0$. Then, it has the ordinary asymptotic behavior

$$S(x) \sim Cx^\alpha \quad \text{as } x \to \infty. \tag{2.8}$$

Proof. We only give the proof when $\alpha > 0$, the case $\alpha = 0$ is easy and it is left to the reader (cf. Exercise 2.9). Differentiating (2.7), cf. Exercise 2.8, we have

$$S'(\lambda u) \sim C\alpha\lambda^{\alpha-1}x_{+}^{\alpha-1} \quad \text{as } \lambda \to \infty;$$

the above asymptotic relation means that

$$\left\langle S'(\lambda u), \varphi(u)\right\rangle = \frac{1}{\lambda}\int_0^\infty \varphi\left(\frac{u}{\lambda}\right) \mathrm{d}S(u) \sim C\alpha\lambda^{\alpha-1}\int_0^\infty \varphi(u)u^{\alpha-1}\mathrm{d}u \quad \text{as } \lambda \to \infty,$$

for each test function. We now select suitable test functions. Let $0 < \sigma < 1$ be arbitrary. Pick $\varphi = \varphi_1 \in \mathcal{D}[0,\infty)$ with the properties $0 \leq \varphi_1 \leq 1$, $\operatorname{supp}\varphi_1 \subseteq [0, 1+\sigma]$ and $\varphi_1(x) = 1$ on $[0,1]$. Then,

$$\begin{aligned}
\limsup_{\lambda\to\infty} \frac{S(\lambda)}{\lambda^\alpha} - C &= \limsup_{\lambda\to\infty} \frac{1}{\lambda^\alpha}\int_0^\lambda \mathrm{d}S(u) - C \\
&\leq \lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha}\int_0^\infty \varphi_1\left(\frac{u}{\lambda}\right)\mathrm{d}S(u) - C \\
&= \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha-1}}\left\langle S'(\lambda u), \varphi_1(u)\right\rangle - C \\
&= C\alpha\int_0^{1+\sigma} u^{\alpha-1}\varphi_1(u)\mathrm{d}u - C \leq C\alpha\int_1^{1+\sigma} u^{\alpha-1}\mathrm{d}u.
\end{aligned}$$

Similarly, choosing the test function $\varphi = \varphi_2$ with the properties $0 \leq \varphi_2 \leq 1$, $\operatorname{supp}\varphi_2 \subseteq [0,1]$ and $\varphi_2(x) = 1$ on $[0, 1-\sigma]$, we come to the conclusion

$$\liminf_{\lambda\to\infty} \frac{S(\lambda)}{\lambda^\alpha} - C \geq -C\int_{1-\sigma}^1 u^{\alpha-1}\mathrm{d}u.$$

Since σ is arbitrary, we obtain (2.8). □

The asymptotic behavior $f(\lambda u) \sim \beta\lambda^{-1}\delta(u)$ is related to problems involving numerical series. Actually, Lemma 1.5 can be extended to the following one-sided proposition.

Proposition 2.7. *Let $f(u) = \sum_{n=0}^\infty c_n\delta(u-n) \in \mathcal{S}'[0,\infty)$ have the asymptotic behavior*

$$f(\lambda u) = \sum_{n=0}^\infty c_n\delta(\lambda u - n) \sim \beta\frac{\delta(u)}{\lambda} \quad \textit{as } \lambda \to \infty. \tag{2.9}$$

If $nc_n \geq -M$ for some M, then $\sum_{n=0}^\infty c_n = \beta$.

Proof. We can assume that $c_0 = 0$. Set $S(x) = \sum_{n<x} c_n$. Choose as before $\varphi = \varphi_1 \in \mathcal{D}[0,\infty)$ with the properties $0 \leq \varphi_1 \leq 1$, $\operatorname{supp}\varphi_1 \subseteq [0, 1+\sigma]$ and $\varphi_1(x) = 1$ on $[0,1]$. Then,

$$\begin{aligned}
\limsup_{\lambda\to\infty} S(\lambda) - \beta &= \limsup_{\lambda\to\infty} \sum_{0<n<\lambda}\left(c_n + \frac{M}{n}\right) - \beta - M\sum_{n<\lambda}\frac{1}{n} \\
&\leq \lim_{\lambda\to\infty}\sum_{n=1}^\infty c_n\varphi_1\left(\frac{n}{\lambda}\right) - \beta + M\sum_{n=1}^\infty \frac{1}{n}\varphi_1\left(\frac{n}{\lambda}\right) - M\sum_{n<\lambda}\frac{1}{n} \\
&= M\lim_{\lambda\to\infty}\sum_{1\leq\frac{n}{\lambda}\leq\sigma}\frac{1}{n}\varphi_1\left(\frac{n}{\lambda}\right) = M\int_1^{\sigma+1}\frac{\varphi_1(u)}{u}\mathrm{d}u \\
&\leq M\sigma.
\end{aligned}$$

Similarly, choosing the test function $\varphi = \varphi_2$ with properties $0 \le \varphi_2 \le 1$, $\operatorname{supp}\varphi_2 \subseteq [0,1]$ and $\varphi_2(x) = 1$ on $[0, 1-\sigma]$, we obtain

$$\liminf_{\lambda\to\infty} S(\lambda) - \beta \ge -M \int_{1-\sigma}^{1} \frac{\mathrm{d}u}{u}.$$

Since σ is arbitrary, we obtain $\sum_{n=1}^{\infty} c_n = \beta$. □

In the future, we shall also make use of the big O landau symbol in the distributional sense, so we write

$$f(\lambda u) = O(\lambda^{\alpha}) \quad \text{as } \lambda \to \infty,$$

if it holds after evaluation at each test function, i.e., for each test function $\varphi \in \mathcal{S}(\mathbb{R})$

$$\langle f(\lambda u), \varphi(u)\rangle = O(\lambda^{\alpha}) \quad \text{as } \lambda \to \infty.$$

We refer to [14, 32, 41, 48] for a more complete account about properties of the distributional asymptotics.

Exercise 2.8. Use the definition of distributional derivative to show that (2.6) can be differentiated with respect to u, namely, if (2.6) holds and $k \in \mathbb{N}$, then,

$$f^{(k)}(\lambda u) \sim \lambda^{\alpha-k} g^{(k)}(u) \quad \text{as } \lambda \to \infty. \tag{2.10}$$

Observe that this property does not hold in general for asymptotics in the ordinary sense (e.g., $f(x) = x^2 + e^{ix^3} \sim x^2$, but $f'(x)$ is not asymptotic to $2x$ in the classical sense). Show that the same differentiation property holds for the O landau symbol.

Exercise 2.9. Show Proposition 2.6 for $\alpha = 0$ (**Hint**: Select a positive and non-increasing $\varphi \in \mathcal{S}[0,\infty)$ such that $\varphi(0) = 1$, then apply the monotone convergence theorem to exchange limit and integral in $\lim_{\lambda\to\infty} \int_0^{\infty} \varphi(u/\lambda)\mathrm{d}S(u)$).

Exercise 2.10. Show if $S \in L^1_{\text{loc}}[0,\infty)$ is so that $S(x) = O(x^{\alpha})$ as $x \to \infty$ for some $\alpha > -1$, then $S(\lambda u) = O(\lambda^{\alpha})$ as $\lambda \to \infty$ in the distributional sense.

2.2 A First Example: Littlewood's Theorem

In this section we complete the argument that was started in Section 1.4 and prove Littlewood's theorem, Theorem 1.2. We need the following lemma, whose proof can be tracked down to that of Tauber's original theorem [18, p. 149], [39].

Lemma 2.11. *Let $\{c_n\}_{n=0}^{\infty}$ be a sequence of complex numbers. If $c_n = O(1/n)$, then,*

$$\sum_{n=0}^{\infty} c_n e^{-ny} - \sum_{n\le\frac{1}{y}} c_n = O(1), \quad y > 0. \tag{2.11}$$

Proof. Choose M such that $n\,|c_n| \leq M$, for every n. Then,

$$\left|\sum_{n=0}^{\infty} c_n e^{-ny} - \sum_{n\leq \frac{1}{y}} c_n\right| \leq \sum_{n\leq \frac{1}{y}} |c_n| \left(1-e^{-ny}\right) + \sum_{\frac{1}{y}<n} |c_n|\, e^{-ny} \leq My \sum_{n\leq\frac{1}{y}} 1 + My \sum_{\frac{1}{y}<n} e^{-ny}$$
$$\leq M + My \int_{\frac{1}{y}}^{\infty} e^{-yu}\mathrm{d}u = 2M.$$

□

Lemma 2.11 is the last ingredient we need for the proof of Littlewood's theorem. So, assume that $c_n = O(1/n)$ and

$$\lim_{y\to 0^+} \sum_{n=0}^{\infty} c_n e^{-ny} = \beta. \tag{2.12}$$

Define $f(u) := \sum_{n=0}^{\infty} c_n \delta(u-n)$. Then, as we already observed in Section 1.4,

$$\lim_{\lambda\to\infty} \langle \lambda f(\lambda u), \varphi(u)\rangle = \beta \,\langle \delta(u), \varphi(u)\rangle \quad \text{for each } \varphi \in \mathfrak{D},$$

where $\varphi \in \mathfrak{D}$ is the linear span of functions of the form $e^{-su} \in \mathcal{S}[0,\infty)$, $s > 0$. Such a set is dense in $\mathcal{S}[0,\infty)$ (cf. Corollary 1.15). We want to apply Theorem 1.6 to $f_\lambda(u) = \lambda f(\lambda u)$ in order to conclude (2.9), the rest would follow at once from Proposition 2.7. So, all we have to verify that $\lambda f_\lambda(u) = O(1)$ in $\mathcal{S}'[0,\infty)$. Define $S(x) = \sum_{n<x} c_n$, by Lemma 2.11, we have $S(x) = O(1)$, but this ordinary order relation implies the distributional one $S(\lambda u) = O(1)$ (cf. Exercise 2.10). Differentiating (cf. Exercise 2.8), we get $\lambda S'(\lambda u) = \lambda f(\lambda u) = O(1)$, and Littlewood's theorem has been fully established.

2.3 Tauberian Theorems for Laplace Transforms

2.3.1 Distributional Tauberian Theorem. First version

We now give a Tauberian theorem for Laplace transforms of distributions. The proof goes in the same lines as that given in Section 2.2 for Littlewood's theorem.

Theorem 2.12. *Let $f \in \mathcal{S}'[0,\infty)$. Suppose that*

$$\mathcal{L}\{f;\sigma\} \sim \frac{C}{\sigma^{\alpha+1}} \quad \text{as } \sigma \to 0^+. \tag{2.13}$$

Then, the Tauberian condition

$$f(\lambda u) = O(\lambda^\alpha) \quad \text{as } \lambda \to \infty, \tag{2.14}$$

implies that

$$f(\lambda u) \sim C\lambda^\alpha \frac{u_+^\alpha}{\Gamma(\alpha+1)} \quad \text{as } \lambda\to\infty. \tag{2.15}$$

Recall that when $\alpha = -k$ is a negative integer then the distribution in the right hand side of (2.15) is interpreted as $C\lambda^{-k}\delta^{(k-1)}(u)$ (cf. (1.16) in Example 1.9). Observe that the converse of Theorem 2.12 is trivial: (2.15) always implies (2.13) and (2.14).

Proof. Consider the net $f_\lambda(u) = \lambda^{-\alpha} f(\lambda u)$. It is enough to verify that conditions (i) and (ii) of Theorem 1.6 are satisfied for this net (here $E = \mathcal{S}[0,\infty)$). In this case, condition (ii) is actually the same as (2.14). Condition (i) is satisfied with the dense set (cf. Corollary) $\mathfrak{D}$ given by the linear span of functions of the form $e^{-\tau u}$, $\tau > 0$; indeed, because of (2.13),

$$\lim_{\lambda\to\infty} \frac{1}{\lambda^\alpha} \left\langle f(\lambda u), e^{-\tau u} \right\rangle = \lim_{\lambda\to\infty} \frac{1}{\lambda^{\alpha+1}} \left\langle f(u), e^{-\frac{\tau}{\lambda}u} \right\rangle = \tau^{-\alpha-1} \lim_{\lambda\to\infty} \left(\frac{\tau}{\lambda}\right)^{\alpha+1} \left\langle f(u), e^{-\frac{\tau}{\lambda}u} \right\rangle$$
$$= \frac{C}{\tau^{\alpha+1}} = \left\langle \frac{C u_+^\alpha}{\Gamma(\alpha+1)}, e^{-\tau u} \right\rangle .$$

□

2.3.2 Hardy-Littlewood Theorems for Stieltjes Integrals

Theorem 2.12 easy yields the following generalization of Theorem 2.1 to Stieltjes integrals.

Theorem 2.13. *Let S be a non-decreasing function such that $S(x) = 0$ for all $x \le 0$. Assume that $\alpha \ge 0$. Then,*

$$\mathcal{L}\{\mathrm{d}S;\sigma\} = \int_0^\infty e^{-\sigma u}\mathrm{d}S(u) \sim \frac{C}{\sigma^\alpha} \quad \text{as } \sigma \to 0^+ \tag{2.16}$$

if and only if

$$S(x) \sim C' x^\alpha \quad \text{as } x\to\infty, \quad \text{where } C' = \frac{C}{\Gamma(\alpha+1)} . \tag{2.17}$$

Proof. We only need to show that (2.16) implies (2.17). Notice first that integration by parts in (2.16) yields

$$\mathcal{L}\{S;\sigma\} = \frac{1}{\sigma}\int_0^\infty e^{-\sigma u}\mathrm{d}S(u) \sim \frac{C}{\sigma^{\alpha+1}} \quad \text{as } \sigma \to 0^+$$

Next, we show that $S(x) = O(x^\alpha)$,

$$S(x) = \int_0^x \mathrm{d}S(u) \le e\int_0^x e^{-\frac{u}{x}}\mathrm{d}S(u) \le e\int_0^\infty e^{-\frac{u}{x}}\mathrm{d}S(u) = O(x^\alpha).$$

It follows then that $S(\lambda u) = O(\lambda^\alpha)$ (cf. Exercise 2.10). Thus, the hypotheses of Theorem 2.12 are satisfied and we conclude $S(\lambda u) = C'\lambda^\alpha u_+^\alpha$. Since for decreasing functions distributional asymptotics are equivalent to ordinary ones (cf. Proposition 2.6), we obtain (2.17) at once. □

Let us now deduce Theorem 2.1 from Theorem 2.13. The case $\alpha > 0$ is very easy. Indeed, write $S(x) = \sum_{n<x}(c_n + Mn^{\alpha-1})$, the Tauberian condition (2.2) ensures that S is non-decreasing. On the other hand, (2.1) gives

$$\int_0^\infty e^{-\sigma u}\mathrm{d}S(u) = \sum_{n=0}^\infty c_n e^{-\sigma u} + M\sum_{n=0}^\infty n^{\alpha-1}e^{-\sigma u} \sim \frac{C + M\Gamma(\alpha)}{\sigma^\alpha} \quad \text{as } \sigma\to 0^+.$$

Theorem 2.13 now yields $S(x) \sim (C' + M/\alpha)x^\alpha$, but since $\sum_{n<x} n^{\alpha-1} \sim x^\alpha/\alpha$, we obtain

$$\sum_{n<x} c_n \sim C' x^\alpha,$$

as required. Observe that the remaining case corresponds to the one-sided version of Theorem 1.2.

The proof of the case $\alpha = 0$ may be obtained by reducing it to the case $\alpha = 2$ and using Proposition 2.7. Let us give a proof. We may assume that $c_0 = 0$. Write $S(x) = \sum_{n<x} c_n$, our assumptions are $\int_0^\infty e^{-\sigma u}\mathrm{d}S(u) \to \beta$ as $\sigma \to 0^+$ and that $nc_n \geq -M$. Furthermore, we set $S^{(-1)}(x) = \int_0^x S(u)\mathrm{d}u = \sum_{n=0}^\infty c_n(x-n)$ and $S^{(-2)}(x) = \int_0^x S^{(-1)}(u)\mathrm{d}u = 2^{-1}\sum_{n=0}^\infty c_n(x-n)^2$, so that $(S^{(-2)}(x))'' = S(x)$. Since $1 - t \leq e^{-t}$, we have that

$$\begin{aligned}\limsup_{x\to\infty} \frac{S^{(-1)}(x)}{x} &= \limsup_{x\to\infty} \sum_{n<x}\left(1-\frac{n}{x}\right)\left(c_n + \frac{M}{n}\right) - M\sum_{n<x}\left(1-\frac{n}{x}\right)\frac{1}{n} \\ &\leq \lim_{x\to\infty}\sum_{n=0}^{\infty} e^{-\frac{n}{x}}\left(c_n+\frac{M}{n}\right) - M\sum_{n<x}\left(1-\frac{n}{x}\right)\frac{1}{n} \\ &= \beta + M\lim_{x\to\infty}\frac{1}{x}\sum_{\frac{n}{x}<1}\left(e^{-\frac{n}{x}}-1+\frac{n}{x}\right)\frac{x}{n} + \frac{1}{x}\sum_{1\leq\frac{n}{x}} e^{-\frac{n}{x}}\frac{x}{n} \\ &= \beta + M\int_0^1 \frac{e^{-u}-1+u}{u}\,\mathrm{d}u + M\int_1^\infty \frac{e^{-u}}{u}\,\mathrm{d}u.\end{aligned}$$

So for some $K > 0$, $T(x) := Kx^2 - S^{(-2)}(x)$ has positive derivative $T'(x) = 2Kx - S^{(-1)}(x)$. It follows that T is increasing, and we also have

$$\mathcal{L}\left\{T'(u);\sigma\right\} = 2K\mathcal{L}\left\{u_+;\sigma\right\} - \mathcal{L}\left\{S^{(-1)}(u);\sigma\right\} = \frac{2K}{\sigma^2} - \frac{1}{\sigma^2}\mathcal{L}\left\{\mathrm{d}S;\sigma\right\} \sim \frac{(2K-\beta)}{\sigma^2}.$$

Hence, by applying Theorem 2.13 to T, we conclude $T(x) \sim (K - \beta/2)x^2$, or which is the same, $S^{(-2)}(x) \sim (\beta/2)x^2$. In particular, $S^{(-2)}(\lambda u) \sim (\beta/2)\lambda^2 u_+^2$, differentiating three times, we obtain

$$S'(\lambda u) = \sum_{n=0}^{\infty} c_n \delta(\lambda u - n) \sim \beta\frac{\delta(u)}{\lambda} \quad \text{as } \lambda \to \infty.$$

Finally, the convergence $\sum_{n=0}^\infty c_n = \beta$ follows from Proposition 2.7.

2.4 Second Version of the Distributional Tauberian Theorem

In 1977 [11], Drozhzhinov and Zavialov showed the following Tauberian theorem for the Laplace transform of distributions. It is essentially equivalent to Theorem 2.12. Interestingly, it describes the distributional asymptotics by pure Laplace transform information. While Theorem 2.12 makes only use of real values of the Laplace transform, in Drozhzhinov-Zavialov theorem the Tauberian hypothesis (2.14) is replaced by a condition on the complex angular behavior of the Laplace transform. A multidimensional version will be discussed in Section 4.2.

Theorem 2.14 (Drozhzhinov and Zavialov, 1977)**.** *Necessary and sufficient conditions for $f \in \mathcal{S}'[0,\infty)$ to have the distributional asymptotic behavior*

$$f(\lambda u) \sim C\lambda^\alpha \frac{u_+^\alpha}{\Gamma(\alpha+1)} \quad \text{as } \lambda \to \infty \tag{2.18}$$

are:

$$\mathcal{L}\left\{f;\sigma\right\} \sim \frac{C}{\sigma^{\alpha+1}} \quad \text{as } \sigma \to 0^+, \tag{2.19}$$

and the existence of constants $k \in \mathbb{N}$ and $M, r_0 > 0$ such that

$$r^{\alpha+1}\left|\mathcal{L}\left\{f; re^{i\vartheta}\right\}\right| \leq \frac{M}{(\cos\vartheta)^k} \quad \text{for } r < r_0 \text{ and } \vartheta \in (-\pi/2, \pi/2)\,. \tag{2.20}$$

The condition (2.20) tells that $|s|^{\alpha+1}\,\mathcal{L}\{f;s\}$ remains bounded as $s\to 0^+$ over cones in $\Re e\,s>0$. As the angle of the cone becomes wider $|s|^{\alpha+1}\,\mathcal{L}\{f;s\}$ may be rather large, but we keep control of it through a bound which is proportional to a negative power of the angle between the cone and the imaginary axis, i.e., $\pi/2-|\vartheta|\sim\cos\vartheta$ (as $\vartheta\to\pm\pi/2$).

Theorem 2.14 follows at once on combining Theorem 2.14 with the ensuing proposition.

Proposition 2.15. *Let $f\in\mathcal{S}'[0,\infty)$, then the following properties are equivalent:*

(i) f satisfies (2.14), namely, the net $\{\lambda^{-\alpha}f(\lambda u)\}_{\lambda\in[1,\infty)}$ is weakly bounded in $\mathcal{S}'[0,\infty)$.

(ii) There exist constants $d,\nu\in\mathbb{N}$ and $K>0$ such that

$$r^{\alpha+1}\left|\mathcal{L}\{f;rs\}\right|\le K\frac{(1+|s|)^d}{\sigma^\nu}\quad \text{for } r,\sigma\in(0,1]\ (\Re e\ s=\sigma). \tag{2.21}$$

(iii) For each $\varphi\in\mathcal{D}(\mathbb{R})$, $\langle f(\lambda u),\varphi(u)\rangle=O(\lambda^\alpha)$ as $\lambda\to\infty$.

(iv) The estimate (2.20) holds for some $M,r_0>0$ and $k\in\mathbb{N}$.

Proof. Clearly, (ii) $\Rightarrow$ (iv); it is then enough to show (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii).

(i) $\Rightarrow$ (ii). By the Banach-Steinhaus theorem, there is $\nu\in\mathbb{N}$ and $M>0$ such that

$$\frac{1}{\lambda^{\alpha+1}}\left|\left\langle f(u),\varphi\left(\frac{u}{\lambda}\right)\right\rangle\right|\le M\sup_{0\le j\le\nu,\,u\in[0,\infty)}(1+u)^\nu\left|\varphi^{(j)}(u)\right|,\quad \text{for all } \varphi\in\mathcal{S}[0,\infty) \text{ and } \lambda\in[1,\infty).$$

Setting $\lambda=1/r$ and $\varphi(u)=e^{-su}$ in the above inequality, we obtain (2.21) with $d=2\nu$ and $K=M\sup_{u\in[0,\infty)}(1+u)^\nu e^{-u}$ (see the estimates in the proof of Proposition 1.17).

(ii) $\Rightarrow$ (iii). Let $\varphi\in\mathcal{D}(\mathbb{R})$. Set $F(s)=\mathcal{L}\{f;s\}$ and $G(s)=\mathcal{L}\{\varphi;s\}$. By Exercise 1.22, $G(s)$ is an entire function and $G(\sigma+it)$ belongs to $\mathcal{S}(\mathbb{R})$ for each fixed $\sigma\in\mathbb{R}$. We use Proposition 1.14 and the formula $\langle h(u),\psi(u)\rangle=(2\pi)^{-1}\left\langle\hat{h}(t),\hat{\psi}(-t)\right\rangle$ in the following calculation,

$$\begin{aligned}\langle f(\lambda u),\varphi(u)\rangle&=\left\langle e^{-u}f(\lambda u),e^{u}\varphi(u)\right\rangle=\frac{1}{2\pi}\left\langle\mathcal{F}\left\{e^{-u}f(\lambda u);t\right\},\mathcal{F}\left\{e^{u}\varphi(u);-t\right\}\right\rangle\\&=\frac{1}{2\pi\lambda}\left\langle\mathcal{F}\left\{e^{-u/\lambda}f(u);t/\lambda\right\},G(-1-it)\right\rangle\\&=\frac{1}{2\pi\lambda}\int_{-\infty}^{\infty}\mathcal{L}\left\{f;\frac{1+ti}{\lambda}\right\}G(-1-it)\mathrm{d}t.\end{aligned}$$

Thus, (2.21) implies, for $\lambda\in[0,\infty)$,

$$\left|\langle f(\lambda u),\varphi(u)\rangle\right|\le\lambda^\alpha\frac{K}{2\pi}\int_{-\infty}^{\infty}(2+|t|)^d\left|G(-1-it)\right|\mathrm{d}t.$$

(iv) $\Rightarrow$ (ii). Assume (2.20). By Proposition 1.17, there exist $l,m,$ and $C>0$ such that (1.26) holds. We may assume that $k\ge\alpha+1$ and $m\ge\alpha+1$. We keep always $r,\sigma\in(0,1]$. Write $s=|s|\,e^{i\vartheta}$, if $|s|\,r<r_0$, we obtain from (2.20)

$$r^{\alpha+1}\left|\mathcal{L}\{f;rs\}\right|=|s|^{-\alpha-1}(|s|\,r)^{\alpha+1}\left|\mathcal{L}\left\{f;r\,|s|\,e^{i\vartheta}\right\}\right|\le|s|^{k-\alpha-1}\frac{M}{(|s|\cos\vartheta)^k}\le M\frac{(1+|s|)^{k-\alpha-1}}{\sigma^k}.$$

Suppose now that $|s|\,r\ge r_0$. Then, by (1.26),

$$r^{\alpha+1}\left|\mathcal{L}\{f;rs\}\right|=r^{\alpha+1-m}\frac{C(1+|s|)^l}{\sigma^m}\le\frac{C(1+|s|)^{l+m-\alpha-1}}{r_0^{m-\alpha-1}\sigma^m}.$$

Therefore, (2.21) is satisfied with $K = \max\left\{M, r_0^{\alpha+1-m}C\right\}$ and the exponents $\nu = \max\{k, m\}$ and $d = \max\{k-\alpha-1, l+m-\alpha-1\}$.

(iii) $\Rightarrow$ (i). Let $\mathcal{D}[-1,1] \subset \mathcal{S}(\mathbb{R})$ be the closed subspace consisting of test functions which vanish off $[-1,1]$. Then, $\{\lambda^{-\alpha} f(\lambda u)\}_{\lambda\in[1,\infty)}$ can be seen as a net of functional in $\mathcal{D}'[-1,1]$ which is weakly bounded. By the Banach Steinhaus theorem, there exists m such that

$$|\langle f(\lambda u), \varphi(u)\rangle| \le M\lambda^{\alpha} \left\|\varphi\right\|_m, \quad \forall \varphi \in \mathcal{D}[-1,1], \tag{2.22}$$

where the norm is given by (1.13). We assume that $m+\alpha+1>0$. Let

$$\mathcal{D}^m[-1,1] = \{\varphi \in C^m(\mathbb{R}) :\ \operatorname{supp}\varphi \subseteq [-1,1]\},$$

then $\mathcal{D}^m[-1,1]$ is a Banach space with norm $\| \ \|_m$ and $\mathcal{D}[-1,1] \subset \mathcal{D}^m[-1,1]$ is a dense subspace. Thus, (2.22) remains valid for all $\varphi \in \mathcal{D}^m[-1,1]$. If we now take $\varphi(u) = \eta(u)(1-u)_+^{m+1} \in \mathcal{D}^m[-1,1]$, where $\eta \in C^\infty(\mathbb{R})$ such that $\eta(u) = 0$ for $u \le -1$ and $\eta(u) = 1$ for $u \ge 0$, then

$$\begin{aligned} f^{(-m-2)}(x) &:= \frac{1}{(m+1)!}\left\langle f(u), \eta(u)(x-u)_+^{m+1}\right\rangle = \frac{x^{m+1}}{(m+1)!}\left\langle f(u), \varphi\left(\frac{u}{x}\right)\right\rangle \\ &= \frac{x^{m+2}}{(m+1)!}\left\langle f(xu), \varphi(u)\right\rangle = O(x^{\alpha+2+m}) \quad \text{as } x\to\infty, \end{aligned}$$

and $f^{(-m-2)}$ is a continuous function on $[0,\infty)$ which is defined as 0 on $(-\infty,0]$. Thus, $f^{(-m-2)}(\lambda u) = O(\lambda^{\alpha+2+m})$ in $\mathcal{S}'(\mathbb{R})$. We now verify that $f = (f^{(-m-2)})^{(m+2)}$, it would then follow that $f(\lambda u) = O(\lambda^\alpha)$ in $\mathcal{S}'(\mathbb{R})$. Indeed, if $\psi \in \mathcal{S}(\mathbb{R})$, then

$$\psi(u) = \frac{(-1)^m}{(m+1)!}\int_u^\infty \psi^{(m+2)}(t)(t-u)^{m+1}\mathrm{d}t.$$

Hence,

$$\begin{aligned} \langle f(u), \eta(u)\psi(u)\rangle &= \frac{(-1)^m}{(m+1)!}\left\langle f(u), \eta(u)\int_u^\infty \psi^{(m+2)}(t)(t-u)^{m+1}\mathrm{d}t\right\rangle \\ &= \frac{(-1)^m}{(m+1)!}\left\langle f(u), \eta(u)\int_0^\infty \psi^{(m+2)}(t)(t-u)_+^{m+1}\mathrm{d}t\right\rangle \\ &= \frac{(-1)^m}{(m+1)!}\int_0^\infty \left\langle f(u), \eta(u)\psi^{(m+2)}(t)(t-u)_+^{m+1}\right\rangle \mathrm{d}t \\ &= (-1)^{m+2}\left\langle f^{(-m-2)}(t), \psi^{(m+2)}(t)\right\rangle. \end{aligned}$$

□

Chapter 3

Distributional Wiener-Ikehara Tauberian Theorem

Theorem 1.4 may be formulated in terms of the Laplace transform. The following theorem will be referred in the sequel as the Wiener-Ikehara theorem.

Theorem 3.1. *Let S be a non-decreasing function supported in $[0,\infty)$. Suppose*

$$\mathcal{L}\left\{\mathrm{d}S;s\right\} = \int_0^\infty e^{-su}\mathrm{d}S(u) \quad \textit{is convergent for } \Re e\, s > 1. \tag{3.1}$$

If there exists a constant β such that the function

$$G(s) = \mathcal{L}\left\{\mathrm{d}S;s\right\} - \frac{\beta}{s-1} \tag{3.2}$$

has a continuous extension to $\Re e\, s = 1$, then

$$S(x) \sim \beta e^x, \quad x \to \infty. \tag{3.3}$$

In particular, if $G(s)$ admits an analytic continuation beyond the line $\Re e\, s = 1$, the hypothesis of Theorem 3.1 is satisfied.

Ikehara's theorem follows directly from Theorem 3.1 by considering $S(x) = \sum_{n<e^x} c_n$.

The boundary requirements for the function G may be relaxed. Wiener's original proof of Theorem 3.1 also applies to show that if G has locally L^1 boundary values, then (3.3) remains true. Such a version seems to be first stated in [3], where that form of the Wiener-Ikehara theorem was needed in order to deduce Beurling's prime number theorem (cf. Section 3.4) by Tauberian arguments. Locally L^1 boundary values means that there exists $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that for each $a > 0$

$$\lim_{\sigma\to 1^+} \int_{-a}^{a} |g(t) - G(\sigma + it)|\,\mathrm{d}t = 0.$$

A version of Theorem 3.3 with reminder can be found in [17].

Korevaar has recently taken the boundary requirements in the Wiener-Ikehara theorem to a minimum [27]. His theorem is in terms of local pseudo-function boundary behavior (cf. Section 3.1), which includes the local L^1 and continuous boundary behaviors. The purpose of this chapter is to present a proof of Korevaar's theorem and discuss its applications in the theory of Beurling's generalized primes. We present here a new approach to the proof, it is based essentially in the distributional methods developed in [41, 36, 44].

3.1 Pseudo-functions

We now proceed to define local pseudofunction boundary behavior, which is the key concept in Korevaar extension of the Wiener-Ikehara theorem.

A tempered distribution $g \in \mathcal{S}'(\mathbb{R})$ is called a *pseudo-function* if $\hat{g} \in C_0(\mathbb{R})$, that is, $\hat{g}$ is a continuous function such that $\lim_{x\to\pm\infty} \hat{g}(x) = 0$.

The distribution $g \in \mathcal{D}'(\mathbb{R})$ is said to be *locally a pseudo-function* if it coincides with a pseudo-function on each finite open interval.

The property of being locally a pseudo-function admits a characterization [27] in terms of a generalized "Riemann-Lebesgue lemma"; indeed, we have,

Lemma 3.2 (Riemann-Lebesgue lemma for local pseudo-functions)**.** *A tempered distribution $g \in \mathcal{S}(\mathbb{R})$ is locally a pseudo-function if and only if for each $\phi \in \mathcal{D}(\mathbb{R})$*

$$\lim_{|h|\to\infty} \left\langle g(t), e^{iht}\phi(t) \right\rangle = 0. \tag{3.4}$$

Proof. Assume that g is locally a pseudo-function. It explicitly means that for any (a, b) there exists a pseudo-function f such that $\langle g, \phi\rangle = \langle f, \phi\rangle$ for all $\phi \in \mathcal{D}(\mathbb{R})$ with support in (a, b). Then, for any such a ϕ, we have

$$\left\langle g(t), e^{iht}\phi(t) \right\rangle = \left\langle f(t), e^{iht}\phi(t) \right\rangle = \left\langle \hat{f}(u), \mathcal{F}^{-1}\left\{ e^{iht}\phi(t); u \right\} \right\rangle = \frac{1}{2\pi}\left\langle \hat{f}(u), \hat{\phi}(-u-h) \right\rangle.$$

Since, by definition, $\hat{f}$ is a continuous function that vanishes at $\pm\infty$, we have that

$$\lim_{|h|\to\infty} \left\langle g(t), e^{iht}\phi(t) \right\rangle = \frac{1}{2\pi} \lim_{|h|\to\infty} \int_{-\infty}^{\infty} \hat{f}(u-h)\hat{\phi}(-u) = 0.$$

Since the interval was arbitrary, we conclude that (3.4) holds for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Conversely, let $(-b, b)$ be an arbitrary interval. Choose $\phi \in \mathcal{D}(\mathbb{R})$ such that $\operatorname{supp} \phi \subseteq (-2b, 2b)$ and $\phi(t) = 1$ for all $t \in [-b, b]$. Set $f = \phi g$. Then, clearly, $g = f$ on the interval $(-b, b)$. We show that f is a pseudo-function. Since f is compactly supported (it is supported in $[-2b, 2b]$), its Fourier transform is given by the following continuous function (cf. [47]),

$$\hat{f}(u) = \left\langle f(t), e^{-iut} \right\rangle = \left\langle g(t), e^{-iut}\phi(t) \right\rangle.$$

Hence, by (3.4), $\hat{f} \in C_0(\mathbb{R})$ and thus f is a pseudo-function. □

It is then clear that if $g \in L^1_{\text{loc}}(\mathbb{R})$, then it is locally a pseudo-function, due to the classical Riemann-Lebesgue lemma. Thus, any continuous function is also locally a pseudo-function.

Let $G(s)$ be analytic on $\Re e\ s > 1$. We shall say that G *has local pseudo-function boundary behavior* on the line $\Re e\ s = 1$ if it has distributional boundary values [7] in such a line, namely,

$$\lim_{\sigma\to\alpha^+} \int_{-\infty}^{\infty} G(\sigma + it)\phi(t)\mathrm{d}t = \langle g(t), \phi(t)\rangle, \quad \forall\phi \in \mathcal{D}(\mathbb{R}),$$

and the boundary distribution g is locally a pseudo-function. Let us emphasize once again that if G admits an analytic continuation beyond $\Re e\ s = 1$, or just a continuous extension to $\Re e\ s = 1$, then G has local pseudo-function boundary behavior on $\Re e\ s = 1$. More generally, local pseudo-function boundary behavior on $\Re e\ s = 1$ is also guaranteed if G has L^1_{loc} boundary values on such a line.

Exercise 3.3. Show that if g is locally a pseudo-function and $\psi \in C^\infty(\mathbb{R})$, then ψg is also a local pseudo-function.

3.2 A Tauberian Theorem for S-limits

In Chapter 2, we studied the asymptotic behavior of distributions by looking at the behavior of their dilates. We now take a different point of view and study their translates.

Let $f \in \mathcal{S}'(\mathbb{R})$, a limit of the form

$$\lim_{h\to\infty} f(u+h) = \beta \quad \text{in } \mathcal{S}'(\mathbb{R}), \tag{3.5}$$

means that for each test function from $\varphi \in \mathcal{D}(\mathbb{R})$ the following limit holds,

$$\lim_{h\to\infty} \langle f(u+h), \varphi(u)\rangle = \beta \int_{-\infty}^{\infty} \varphi(u)\mathrm{d}u. \tag{3.6}$$

Relation (3.5) is an example of the so called *S-asymptotics* of generalized functions (it stands for shift-asymptotics). The notion of S-asymptotics is due to Pilipović and Stanković [30, 31, 32]. We can also study error terms by introducing $S-$asymptotic boundedness, let ρ be a positive function, then we write

$$f(u+h) = O(\rho(h)) \quad \text{as } h \to \infty \ \text{ in } \mathcal{S}'(\mathbb{R}) \tag{3.7}$$

if for each $\varphi \in \mathcal{S}(\mathbb{R})$ we have $\langle f(u+h), \varphi(u)\rangle = O(\rho(h))$, for large values of h. The little o symbol and $S-$asymptotics as $h \to -\infty$ are defined in a similar way. With this notation we might write (3.5) as $f(u+h) = \beta + o(1)$ as $h \to \infty$ in $\mathcal{S}'(\mathbb{R})$.

Example 3.4. Suppose $f \in L^1_{\mathrm{loc}}[0,\infty)$. If $\lim_{x\to\infty} f(x) = \beta$, then

$$\lim_{h\to\infty} f(u+h) = \beta \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Indeed, for each test function,

$$\lim_{h\to\infty} \langle f(u+h), \varphi(u)\rangle = \lim_{h\to\infty} \int_0^{\infty} f(u)\varphi(u-h)\mathrm{d}u = \lim_{h\to\infty} \int_{-h}^{\infty} f(u+h)\varphi(u)\mathrm{d}u = \beta \int_{-\infty}^{\infty} \varphi(u)\mathrm{d}u.$$

Recall the Heaviside function H is given by the characteristic function of $[0,\infty)$, i.e., $\langle H(u), \varphi(u)\rangle = \int_0^{\infty} \varphi(u)\mathrm{d}u$. We obtain in particular that it has the S-limit

$$\lim_{h\to\infty} H(u+h) = 1 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

We now provide a simple but useful Tauberian theorem for the S-limit (3.5). It is in terms of the Fourier transform and local pseudo-functions.

Theorem 3.5. *Let $f \in \mathcal{S}'(\mathbb{R})$. Assume that*

$$g = \hat{f} - \beta\hat{H} \quad \textit{is locally a pseudo-function.} \tag{3.8}$$

If

$$f(u+h) = O(1) \quad \textit{as } h \to \infty \ \textit{ in } \mathcal{S}'(\mathbb{R}), \tag{3.9}$$

then f has the S-limit (3.5).

Proof. Because of Example 3.3, we may assume that $\beta = 0$, so that $g = \hat{f}$ in (3.8). Let $\phi \in \mathcal{D}(\mathbb{R})$. Then, by the Riemann-Lebesgue lemma (Lemma 3.2),

$$\lim_{h\to\infty} \left\langle f(t+h), \hat{\phi}(t) \right\rangle = \lim_{h\to\infty} \left\langle e^{ihu} \hat{f}(u), \phi(u) \right\rangle = 0.$$

So, (3.6) holds with $\beta = 0$ for each $\varphi = \hat{\phi} \in \mathcal{F}(\mathcal{D}(\mathbb{R}))$, the image under Fourier transform of $\mathcal{D}(\mathbb{R})$. But since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$ and the Fourier transform is continuous, we have that $\mathcal{F}(\mathcal{D}(\mathbb{R}))$ is also dense in $\mathcal{S}(\mathbb{R})$. Finally, the Tauberian hypothesis (3.9) allows us to use Theorem 1.6 and conclude $\lim_{h\to\infty} f(u+h) = 0$ in $\mathcal{S}'(\mathbb{R})$. □

If f is a function and it has the S-limit (3.8), it does not follow in general that f has a limit at infinity in the ordinary sense.

Example 3.6. The function $f(x) = e^{ix^2}$ does not have a limit at infinity, however,

$$\lim_{h\to\infty} f(u+h) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}).$$

Indeed, if $\varphi \in \mathcal{S}(\mathbb{R})$, observe that $\Phi(x) = \varphi(x) e^{ix^2}$ belongs to $L^1(\mathbb{R})$, then, by the classical Riemann-Lebesgue lemma, as $h \to \infty$

$$\langle f(u+h), \varphi(u) \rangle = \int_{-\infty}^{\infty} e^{ih^2} e^{i2hu} e^{iu^2} \varphi(u) \mathrm{d}u = e^{ih^2} \int_{-\infty}^{\infty} e^{i2hu} \Phi(u) \mathrm{d}u = e^{ih^2} o(1) = o(1).$$

The next theorem gives a sufficient condition that allows us to come back from S-limits to ordinary ones.

Proposition 3.7. *Let T be a function such that $T(x) = 0$ for $x \in (-\infty, 0)$ and*

$$\lim_{h\to\infty} T(u+h) = \beta \quad \text{in } \mathcal{S}'(\mathbb{R}). \tag{3.10}$$

Suppose that there is $\alpha \geq 0$ such that $\mathrm{e}^{\alpha x} T(x)$ is non-decreasing. Then,

$$\lim_{x\to\infty} T(x) = \beta, \tag{3.11}$$

in the ordinary sense.

Proof. Let $S(x) = e^{\alpha x} T(x)$. Since $\varphi \in \mathcal{D}(\mathbb{R})$ implies that $e^{\alpha x}\varphi \in \mathcal{D}(\mathbb{R})$, the S-limit (3.10) yields

$$\int_{-h}^{\infty} S(u+h)\varphi(u)\mathrm{d}u = e^{\alpha h} \int_{-h}^{\infty} T(u+h) e^{\alpha u} \varphi(u) \mathrm{d}u \sim \beta e^{\alpha h} \int_{-\infty}^{\infty} e^{\alpha u} \varphi(u) \mathrm{d}u, \quad \forall \varphi \in \mathcal{D}(\mathbb{R}). \tag{3.12}$$

Let $\varepsilon > 0$ be arbitrary. Choose in (3.12) a non-negative test function φ such that $\operatorname{supp} \varphi \subseteq [0, \varepsilon]$ and $\int_0^\varepsilon \varphi(u)\mathrm{d}u = 1$. Using the fact that S is non-decreasing and (3.12), we obtain

$$\begin{aligned}\limsup_{h\to\infty} T(h) &= \limsup_{h\to\infty} e^{-\alpha h} S(h) \int_0^\varepsilon \varphi(u)\mathrm{d}u \leq \lim_{h\to\infty} e^{-\alpha h} \int_0^\varepsilon S(u+h)\varphi(u)\mathrm{d}u = \beta \int_0^\varepsilon e^{\alpha u}\varphi(u)\mathrm{d}u \\ &\leq \beta e^{\alpha\varepsilon},\end{aligned}$$

taking $\varepsilon \to 0^+$, we have shown $\limsup_{h\to\infty} T(h) \leq \beta$. Similarly, choosing in (3.12) a non-negative φ such that $\operatorname{supp} \varphi \subseteq [-\varepsilon, 0]$ and $\int_{-\varepsilon}^0 \varphi(u)\mathrm{d}u = 1$, one obtains $\liminf_{h\to\infty} T(h) \geq \beta$. This shows that (3.11) is satisfied. □

3.3 Distributional Wiener-Ikehara Tauberian Theorem

We are ready to state and prove Korevaar's extension of the Wiener-Ikehara Tauberian theorem.

Theorem 3.8. *Let S be a non-decreasing function supported in $[0,\infty)$. Suppose*

$$\mathcal{L}\{\mathrm{d}S;s\} = \int_0^\infty e^{-su}\mathrm{d}S(u) \quad \textit{is convergent for } \Re e\, s > 1. \tag{3.13}$$

If there exists a constant β such that the function

$$G(s) = \mathcal{L}\{\mathrm{d}S;s\} - \frac{\beta}{s-1} \tag{3.14}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$, then

$$S(x) \sim \beta e^x \quad \textit{as } x \to \infty. \tag{3.15}$$

We first transform the assumptions (3.13) and (3.14). Set $T(x) = e^{-x}S(x)$. We may assume that $S(0) = 0$. Then

$$\begin{aligned}
\mathcal{L}\{T;s\} &= \int_0^\infty e^{(-s-1)u}S(u)\mathrm{d}u = \frac{1}{1+s}\int_0^\infty e^{(-s-1)u}\mathrm{d}S(u) \\
&= \frac{1}{1+s}\mathcal{L}\{\mathrm{d}S;s+1\} = \frac{1}{1+s}\left(G(s+)+\frac{\beta}{s}\right) \\
&= \frac{\beta}{s} + \frac{1}{1+s}\left(G(s+1)+\frac{\beta}{s}-\frac{\beta(s+1)}{s}\right) = \frac{\beta}{s} + \frac{G(s+1)}{1+s} + \frac{\beta}{s+1}, \quad \Re e\, s > 0.
\end{aligned}$$

Observe that $\mathcal{L}\{H;s\} = \int_0^\infty e^{-su}\mathrm{d}u = 1/s$, here, as usual, H is the Heaviside function. Since $(it+1)^{-1}$ is smooth, we immediately see (cf. Exercise 3.4) that

$$\mathcal{L}\{T;s\} - \frac{\beta}{s} = \mathcal{L}\{T-\beta H;s\} \tag{3.16}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 0$.

Proof. Suppose momentaneously that we were able to show that

$$T(x) = e^{-x}S(x) = O(1) \quad \text{as } x \to \infty. \tag{3.17}$$

Then, $T \in \mathcal{S}'(\mathbb{R})$ and by taking boundary values on $\Re e\, s = 0$ in (3.16), cf. Proposition 1.14, we would obtain that $\hat{T} - \hat{H}$ is locally a pseudo-function. Thus, Theorem 3.5 and (3.17) would yield that $\lim_{h\to\infty} T(u+h) = \beta$ in $\mathcal{S}'(\mathbb{R})$, and hence, by Proposition 3.7, we would obtain (3.11) which is exactly the same as (3.15). Therefore, the proof will be complete after establishing (3.17). Let then g be the boundary local pseudo-function of (3.16). Recall that means that

$$\lim_{\sigma\to 0^+}\int_{-\infty}^\infty \mathcal{L}\{T-\beta H;\sigma+it\}\phi(t)\mathrm{d}t = \langle g(t),\phi(t)\rangle, \quad \forall \phi \in \mathcal{D}(\mathbb{R}).$$

Since $\mathcal{L}\{T-\beta H;\sigma+it\} = \mathcal{F}\{T(u)-\beta H(u))e^{-\sigma u};t\}$, we obtain that for each $\phi \in \mathcal{D}(\mathbb{R})$

$$\begin{aligned}
\lim_{\sigma\to 0^+}\int_0^\infty T(u)e^{-\sigma u}\hat{\phi}(u)\mathrm{d}u &= \beta\int_0^\infty \hat{\phi}(u)\mathrm{d}u + \lim_{\sigma\to 0^+}\int_0^\infty (T(u)-\beta H(u))e^{-\sigma u}\hat{\phi}(u)\mathrm{d}u \\
&= \beta\int_0^\infty \hat{\phi}(u)\mathrm{d}u + \lim_{\sigma\to 0^+}\int_{-\infty}^\infty \mathcal{L}\{T-\beta H;\sigma+it\}\phi(t)\mathrm{d}t \\
&= \beta\int_0^\infty \hat{\phi}(u)\mathrm{d}u + \langle g(t),\phi(t)\rangle .
\end{aligned}$$

By the monotone convergence theorem, we obtain that, for every $\phi \in \mathcal{D}(\mathbb{R})$ with $\hat{\phi} \geq 0$,

$$\int_0^\infty T(u)\hat{\phi}(u)\mathrm{d}u = \beta \int_0^\infty \hat{\phi}(u)\mathrm{d}u + \langle g(t), \phi(t)\rangle .$$

Now, choose one of such a ϕ with $\hat{\phi}(0) > 0$ and observe that $e^{iht}\phi(t)$ is also compactly supported and has Fourier transform $\hat{\phi}(u-h)$, a non-negative function. Replacing ϕ by $e^{iht}\phi(t)$ in the above equation, we obtain,

$$\int_0^\infty T(u)\hat{\phi}(u-h)\mathrm{d}u = \int_{-h}^\infty T(u+h)\hat{\phi}(u)\mathrm{d}u = \beta \int_{-h}^\infty \hat{\phi}(u)\mathrm{d}u + \left\langle g(t), e^{iht}\phi(t)\right\rangle = O(1)+o(1) = O(1).$$

Using the fact that S is non-decreasing, we have that for u and h positive $e^{-u}T(h) = e^{-u-h}S(h) \leq e^{-u-h}S(h+u) = T(u+h)$. Finally, setting $C = \int_0^\infty e^{-u}\hat{\phi}(u)\mathrm{d}u > 0$,

$$T(h) = C^{-1}\int_0^\infty e^{-u}T(h)\hat{\phi}(u)\mathrm{d}u \leq C^{-1}\int_0^\infty T(u+h)\hat{\phi}(u)\mathrm{d}u = O(1).$$

□

We state a version of Theorem 3.8 for Dirichlet series, it is suitable for several applications.

Corollary 3.9. *Let $\{n_k\}_{k=1}^\infty$ be a non-decreasing sequence of positive real numbers tending to infinity. Assume that the Dirichlet series*

$$F(s) := \sum_{k=1}^\infty \frac{c_k}{n_k^s} \quad \textit{is convergent for } \Re e\ s > 1,$$

where the coefficients satisfy the Tauberian condition $c_n \geq 0$. If there exists a constant β such that

$$F(z) - \frac{\beta}{z-1}$$

has local pseudo-function boundary behavior on the line $\Re e\ z = 1$, then

$$\sum_{n_k < x} c_n \sim \beta x \quad \textit{as } x \to \infty.$$

Proof. Set

$$S(x) = \sum_{n_k < e^x} c_k = \sum_{\log n_k < x} c_k,$$

then $S'(u) = \sum_{k=0}^\infty c_k \delta(u - \log n_k)$, and hence

$$\mathcal{L}\{\mathrm{d}S; s\} = \left\langle \sum_{k=0}^\infty c_k\delta(u-\log n_k), e^{-su}\right\rangle = \sum_{k=1}^\infty c_k e^{-s\log n_k} = \sum_{k=1}^\infty \frac{c_k}{n_k^s}.$$

Thus, we can apply Theorem 3.8 and conclude that $S(x) \sim \beta e^x$, i.e., $\sum_{n_k<x} c_k \sim \beta x$. □

Remark 3.10. The converse of Theorem 3.8 is true, that is, if (3.15) is satisfied, then (3.13) holds and the analytic function G given by (3.14) has local pseudo-function boundary behavior on the line $\Re e\ s = 1$. Therefore, Theorem 3.8 is optimal in the sense that the boundary requirements in the Wiener-Ikehara theorem cannot be weaker than local pseudo-function behavior.

In order to show the converse of Theorem 3.8, set again $T(x) = e^{-x}S(x)$ and assume that $T(x) \to \beta$ as $x \to \infty$. Thus, $\lim_{|x|\to\infty}(T(x) - \beta H(x)) = 0$. The convergence of (3.13) follows at once. The same calculation performed in the proof of Theorem 3.8 shows that G given by (3.14) has local pseudo-function boundary behavior on $\Re e\, s = 1$ if and only if (3.16) has such a boundary behavior on $\Re e\, s = 0$. But the boundary value of (3.16) is precisely $\hat{T} - \beta\hat{H}$, hence, we must show that the later distribution is locally a pseudofunction. Let $\phi \in \mathcal{D}(\mathbb{R})$. Then, as $|h| \to \infty$,

$$\left\langle \hat{T}(t) - \beta\hat{H}(t), e^{iht}\phi(t) \right\rangle = \left\langle T(u) - \beta H(u), \hat{\phi}(u-h) \right\rangle = \int_{-h}^{\infty} (T(u+h) - \beta H(u+h))\hat{\phi}(u)\mathrm{d}u \to 0.$$

3.4 Applications to Prime Number Theory

In this section we discuss applications to prime number theory in the context of generalized number systems. The main idea is to replace the set of ordinary prime numbers by an arbitrary sequence of positive real numbers, called below generalized prime numbers. In the same way that the natural numbers are constructed out of multiplications of ordinary prime numbers, a generalized number system has the generalized primes as multiplicative building blocks. One then asks up to which extend some properties of the natural numbers remain true for generalized number systems. Observe that if one of such properties remains valid in this context, it would then be independent from the additive structure of the natural numbers. For instance, the prime number theorem is independent of the additive structure of the natural numbers, as seen below.

Let $1 < p_1 \leq p_2, \dots$ be a non-decreasing sequence of real numbers tending to infinity. Following Beurling [4], we shall call such a sequence $P = \{p_k\}_{k=1}^{\infty}$ a set of *generalized prime numbers*. We arrange the set of all possible products of generalized primes in a non-decreasing sequence $1 < n_1 \leq n_2, \dots$, where every n_k is repeated as many times as it can be represented by $p_{\nu_1}^{\alpha_1} p_{\nu_2}^{\alpha_2} \dots p_{\nu_m}^{\alpha_m}$ with $\nu_j < \nu_{j+1}$. The sequence $\{n_k\}_{k=1}^{\infty}$ is called the set of *generalized integers*.

The function π denotes the counting function of the generalized prime numbers,

$$\pi(x) = \pi_P(x) = \sum_{p_k \leq x} 1, \tag{3.18}$$

while the function N denotes the counting function of the generalized integers,

$$N(x) = N_P(x) = \sum_{n_k \leq x} 1. \tag{3.19}$$

Beurling's problem is then to find conditions over the function N which ensure the validity of the prime number theorem (in short: PNT), i.e.,

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as } x \to \infty. \tag{3.20}$$

In his seminal work [4], Beurling proved that the condition of following theorem suffices for the PNT to hold.

Theorem 3.11 (Beurling's PNT, 1937)**.** *Suppose there exist constants $a > 0$ and $\gamma > 3/2$ such that*

$$N(x) = ax + O\left(\frac{x}{\log^{\gamma} x}\right) \quad \text{as } x \to \infty, \tag{3.21}$$

Then the prime number theorem (3.20) holds.

Observe that Beurling's prime number theorem naturally includes the classical prime number theorem, because the counting function of the natural numbers is actually $N(x) = [x] = x + O(1)$ and thus fulfills Beurling's condition (3.21) with $a = 1$ for any $\gamma > 0$, and in particular for any $\gamma > 3/2$.

If $\gamma = 3/2$, then the PNT need not to hold, as showed by Diamond who exhibited an explicit example of generalized primes not satisfying the PNT. The interested reader can find Diamond's example in [9].

We shall use Corollary 3.9 to give a proof of Beurling's prime number theorem (cf. Section 3.4.3), we follow closely the exposition from [3]. In fact, the PNT holds under milder conditions than (3.21), we discuss those more general prime number theorems in Section 3.4.4 (without proofs).

Throughout this section, the sequence $P = \{p_k\}_{k=1}^{\infty}$ stands for a fixed set of generalized prime numbers with generalized integers $\{n_k\}_{k=1}^{\infty}$. The functions N and π are given by (3.19) and (3.18), respectively. The letter s stands for complex numbers $s = \sigma + it$.

3.4.1 Functions Related to Generalized Primes

We denote by $\Lambda = \Lambda_P$ the *von Mangoldt function* of P, defined on the set of generalized integers as

$$\Lambda(n_k) = \begin{cases} \log p_j, & \text{if } n_k = p_j^m, \\ 0, & \text{otherwise.} \end{cases} \tag{3.22}$$

The *Chebyshev function* of P is defined by

$$\psi(x) = \psi_P(x) = \sum_{p_k^m \leq x} \log p_k = \sum_{n_k \leq x} \Lambda(n_k). \tag{3.23}$$

It is very well known since the time of Chebyshev that the PNT is equivalent to the statement

$$\psi(x) \sim x. \tag{3.24}$$

Indeed, this is a consequence of the following lemma. Notice that we do not need to impose any condition on the growth of N.

Lemma 3.12. *There exists a number $M > 0$, which depends only on p_1, such that*

$$\frac{\psi(x)}{x} \leq \frac{\pi(x) \log x}{x} \leq \frac{\psi(x)}{x} + \left(\max_{1 \leq u \leq x} \frac{\psi(u)}{u} \right) \frac{M}{\log x} \ . \tag{3.25}$$

Proof. We establish first the lower inequality in (3.25). Observe that for given p_k there are precisely $[\log x / \log p_k]$ integers m which satisfy $p_k^m \leq x$. Thus, the very first expression in (3.23) reads as $\psi(x) = \sum_{p_k \leq x} [\log x / \log p_k] \log p_k$, and so

$$\psi(x) = \sum_{p_k \leq x} \left[\frac{\log x}{\log p_k} \right] \log p_k \leq \sum_{p_k \leq x} \frac{\log x}{\log p_k} \log p_k = \log x \sum_{p_k \leq x} 1 = \pi(x) \log x.$$

For the upper estimate,

$$\pi(x) = \sum_{p_k \leq x} 1 \leq \sum_{p_k \leq x} 1 + \sum_{\substack{p_k^m \leq x \\ 1 \leq m}} \frac{\log p_k}{\log p_k^m} = \sum_{p_k^m \leq x} \frac{\log p_k}{\log p_k^m} = \int_1^x \frac{\mathrm{d}\psi(u)}{\log u} = \frac{\psi(x)}{\log x} + \int_{p_1}^x \frac{\psi(u)}{u \log^2 u} \mathrm{d}u;$$

now,

$$\int_{p_1}^{x} \frac{\psi(u)}{u\log^2 u}\mathrm{d}u \le \left(\max_{1\le u\le x}\frac{\psi(u)}{u}\right)\int_{p_1}^{x}\frac{\mathrm{d}u}{\log^2 u} = \left(\max_{1\le u\le x}\frac{\psi(u)}{u}\right)\int_{p_1}^{x}\frac{\mathrm{d}u}{\log^2 u},$$

and

$$\int_{p_1}^{x}\frac{\mathrm{d}u}{\log^2 u} = \left(\int_{p_1}^{\sqrt{p_1 x}} + \int_{\sqrt{p_1 x}}^{x}\right)\frac{\mathrm{d}u}{\log^2 u} \le \frac{\sqrt{p_1 x}}{\log^2 p_1} + \frac{4x}{\log^2 x} \le M\frac{x}{\log^2 x}, \quad \text{for } x \ge p_1,$$

for some constant depending only on p_1. □

Corollary 3.13. *The PNT (3.20) is equivalent to (3.24).*

Proof. Indeed, if either asymptotic relation holds, Lemma 3.12 implies that both $x^{-1}\psi(x)$ and $x^{-1}\pi(x)\log x$ are bounded. Moreover, (3.25) yields, with $K = \max_{1\le u} u^{-1}\psi(u)$,

$$0 \le \frac{\pi(x)\log x}{x} - \frac{\psi(x)}{x} \le \frac{KM}{\log x}, \quad \text{for } x \ge 1,$$

namely,

$$\frac{\pi(x)\log x}{x} = \frac{\psi(x)}{x} + O\left(\frac{1}{\log x}\right),$$

which proves the equivalence. □

Our approach to the PNT will be to show (3.24). For it, we shall make use the *zeta function* of P, defined as the analytic function

$$\zeta(s) = \zeta_P(s) = \sum_{k=1}^{\infty}\frac{1}{n_k^s}, \quad \Re e\ s > 1. \tag{3.26}$$

The convergence of (3.26) on $\Re e\, s > 1$ is easily ensured for example if $N(x) = O(x)$; in such a case

$$\sum_{k=1}^{\infty}\frac{1}{n_k^\sigma} = \int_1^\infty u^{-\sigma}\mathrm{d}N(u) = \sigma\int_1^\infty \frac{N(u)}{u}u^{-\sigma}\mathrm{d}u = O(1)\int_1^\infty u^{-\sigma}\mathrm{d}u, \quad \text{converges for } \sigma > 1.$$

The condition $N(x) = O(x)$ is ensured if for instance (3.21) holds for some $\gamma > 0$.

Many properties of the zeta function can be derived by its Euler product representation.

Proposition 3.14. *Assume that $N(x) = O(x)$. Then*

$$\zeta(s) = \prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^s}\right)^{-1}, \quad \Re e\ s > 1.$$

In particular $\zeta(s)$ does not vanish on $\Re e\ s > 1$.

Proof. Formally,

$$\prod_{k=1}^{\infty}\left(1 - \frac{1}{p_k^s}\right)^{-1} = \prod_{k=1}^{\infty}\left(1 + \frac{1}{p_k^s} + \frac{1}{p_k^{2s}} + \frac{1}{p_k^{3s}} + \dots\right) = \sum_{k=1}^{\infty}\frac{1}{n_k^s}.$$

Recall that if $\sum_{k=1}^{\infty} |a_k| < \infty$, then the product $\prod_{k=1}^{\infty}(1 + a_k)$ converges. Hence, the convergence of the Euler product for $\sigma > 1$ ($s = \sigma + it$)

$$\prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right)^{-1} = \prod_{k=1}^{\infty}\left(\frac{p_k^s}{p_k^s-1}\right) = \prod_{k=1}^{\infty}\left(1+\frac{p_k^{-s}}{1-p_k^{-s}}\right)$$

follows from that of

$$\sum_{k=0}^{\infty}\left|\frac{p_k^{-s}}{1-p_k^{-s}}\right| = \sum_{k=0}^{\infty}\left|\frac{1}{1-p_k^{-s}}\right|\frac{1}{p_k^{\sigma}} \leq \sum_{k=0}^{\infty}\frac{1}{1-p_k^{-\sigma}}\frac{1}{p_k^{\sigma}} \leq \frac{1}{1-p_1^{-\sigma}}\sum_{k=0}^{\infty}\frac{1}{p_k^{\sigma}} \leq \frac{1}{1-p_1^{-\sigma}}\sum_{k=0}^{\infty}\frac{1}{n_k^{\sigma}} < \infty.$$

□

The connection between the Chebyshev function and the zeta function is given by the next proposition.

Proposition 3.15. *Suppose that $N(x) = O(x)$, then*

$$\sum_{k=1}^{\infty}\frac{\Lambda(n_k)}{n_k^s} = -\frac{\zeta'(s)}{\zeta(s)}, \quad \Re e\ s > 1. \tag{3.27}$$

Proof. The Dirichlet series converges because $\psi(x) = \sum_{n_k \leq x}\Lambda(n_k) \leq \log x \sum_{n_k\leq x} 1 = O(x\log x)$. First notice that

$$\sum_{k=1}^{\infty}\frac{\Lambda(n_k)}{n_k^s} = \sum_{p_k^m}\frac{\log p_k}{p_k^{ms}} = \sum_{k=1}^{\infty}\log p_k\sum_{m=1}^{\infty}\frac{1}{p_k^{ms}} = \sum_{k=1}^{\infty}\log p_k\frac{p_k^{-s}}{1-p_k^{-s}}\ .$$

The equality (3.27) follows now by logarithmic differentiation of the Euler product,

$$-\frac{\zeta'(s)}{\zeta(s)} = -\left(\log\zeta(s)\right)' = \left(\log\prod_{k=1}^{\infty}\left(1-\frac{1}{p_k^s}\right)\right)' = \sum_{k=1}^{\infty}\left(\log\left(1-\frac{1}{p_k^s}\right)\right)' = \sum_{k=1}^{\infty}\log p_k\frac{p_k^{-s}}{1-p_k^{-s}}\ .$$

□

In view of Corollary 3.9, Corollary 3.13, and the formula (3.27), we have that PNT would follow at once if we are able to show that the analytic function

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \tag{3.28}$$

has local pseudo-function boundary behavior on the line $\Re e\, s = 1$. We will follow such a strategy to show Theorem 3.11 in Section 3.4.3, after studying more properties of the zeta function in Section 3.4.2. The key ingredient to show the local pseudo-function boundary behavior of (3.28) is the non-vanishing property of $\zeta(s)$ on $\Re e\ s = 1$.

Exercise 3.16. Show that if $N(x) = O(x)$, then the following representation for the zeta function holds:

$$\zeta(s) = \exp\left(\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{j}p_k^{-js}\right)$$

(**Hint**: Use the Euler product and the Taylor series for $\log(1-z)$).

Exercise 3.17. Show that

$$\zeta(s) = s\int_1^{\infty}\frac{N(x)}{x^{s+1}}\mathrm{d}x$$

(**Hint**: Integrate by parts $\zeta(s) = \int_1^{\infty} x^{-s}\mathrm{d}N(x)$).

3.4.2 Properties of the Zeta Function

We shall now study properties of $\zeta(s)$ on $\Re e\ s = 1$. We begin by assuming that the counting function of the generalized integers satisfies

$$N(x) = ax + O\left(\frac{x}{\log^\gamma x}\right) \quad \text{as } x \to \infty, \tag{3.29}$$

for some $a > 0$ and $\gamma > 1$.

Proposition 3.18. *Let N satisfy (3.29) for some $a > 0$ and $\gamma > 1$. Then $\zeta(s) - a/(s-1)$ extends to a continuous function on $\Re e\ s \geq 1$. Consequently, $t\zeta(1+it)$ is continuous over the whole real line and so $\zeta(1+it)$ is continuous in $\mathbb{R} \setminus \{0\}$.*

Proof. Let

$$T(u) = \frac{N(e^u)}{e^u} - aH(u) = O(u^{-\gamma}).$$

where H is the Heaviside function. Then $T \in L^1(\mathbb{R})$. Recall that the Fourier transforms of L^1 functions are always continuous functions. We then have

$$\begin{aligned}
\zeta(s) - \frac{a}{s-1} &= \int_1^\infty x^{-s}\mathrm{d}N(x) - \frac{a}{s-1} = s\int_1^\infty \frac{N(x)}{x^s}\frac{\mathrm{d}x}{x} - \frac{a}{s-1} \\
&= s\int_0^\infty e^{-su}N(e^u)\mathrm{d}u - \frac{a}{s-1} \\
&= s\int_0^\infty e^{-(s-1)u}e^{-u}N(e^u)\mathrm{d}u - \frac{a}{s-1} \\
&= s\mathcal{L}\left\{T(u) + aH(u); s-1\right\} - \frac{a}{s-1} \\
&= s\mathcal{L}\left\{T; s-1\right\} + \frac{as}{s-1} - \frac{a}{s-1} = s\mathcal{L}\left\{T; s-1\right\} + a
\end{aligned}$$

Writing $s = \sigma + it$ and taking $\sigma \to 1^+$, we convince ourselves that $\zeta(s) - a/(s-1)$ has boundary distribution $it\hat{T}(t) + a$, a continuous function. □

The ensuing lemma is the first step toward the non-vanishing property of ζ on $\Re e\, s = 1$, in the case $\gamma > 3/2$.

Lemma 3.19. *Let N satisfy (3.29) with $a > 0$ and $1 < \gamma < 2$. For each $t_0 \neq 0$ there exists $C = C_{t_0} > 0$ such that for $1 < \sigma < 2$*

$$|\zeta(\sigma + it_0) - \zeta(1 + it_0)| < C(\sigma - 1)^{\gamma - 1}. \tag{3.30}$$

Proof. We work with $Z(s) = \zeta(s)/s$. It is easy to see that (3.30) holds if

$$|Z(\sigma + it_0) - Z(1 + it_0)| < D(\sigma - 1)^{\gamma - 1}. \tag{3.31}$$

The claim is a simple consequence of

$$|\zeta(s_1) - \zeta(s_2)| = |s_1 Z(s_1) - s_2 Z(s_2)| \leq |s_1(Z(s_1) - Z(s_2))| + |(s_1 - s_2)Z(s_2)|\,.$$

By Exercise 3.17,

$$\begin{aligned}
\frac{\zeta(s_1)}{s_1} - \frac{\zeta(s_2)}{s_2} &= \int_1^\infty (x^{-s_1} - x^{-s_2})\frac{N(x)}{x}\mathrm{d}x = \int_1^\infty (x^{-s_1} - x^{-s_2})\frac{N(x) - ax}{x}\mathrm{d}x + \frac{a}{s_1 - 1} - \frac{a}{s_2 - 1} \\
&= \int_1^\infty (x^{-s_1} - x^{-s_2})\frac{N(x) - ax}{x}\mathrm{d}x + \frac{a(s_2 - s_1)}{(s_1 - 1)(s_2 - 1)}.
\end{aligned}$$

Now, set $s_2 = \sigma + it_0$ and $s_1 = 1 + it_0$. We clearly have that

$$\left|\frac{a(s_1 - s_2)}{(s_1-1)(s_2-1)}\right| \leq \frac{a(\sigma-1)}{|t_0|\,|\Im m\,(s_2-1)|} = \frac{a(\sigma-1)}{t_0^2} := D_1(\sigma-1).$$

On the other hand, by (3.29), for some constant K,

$$\begin{aligned}
\left|\int_1^\infty (x^{-s_1} - x^{-s_2})\frac{N(x)-ax}{x}\mathrm{d}x\right| &= \left|\int_1^\infty x^{-it_0}(x^{-\sigma} - x^{-1})\frac{N(x)-ax}{x}\mathrm{d}x\right| \\
&\leq \int_1^\infty (x^{-\sigma} - x^{-1})\left|\frac{N(x)-ax}{x}\right|\mathrm{d}x \\
&\leq K\int_1^\infty (x^{-1} - x^{-\sigma})\log^{-\gamma} x\,\mathrm{d}x \\
&= K\int_0^\infty \frac{1-e^{-(\sigma-1)u}}{u^\gamma}\mathrm{d}u \\
&= K(\sigma-1)^{1-\gamma}\int_0^\infty \frac{1-e^{-u}}{u^\gamma}\mathrm{d}u := D_2(\sigma-1)^{1-\gamma}.
\end{aligned}$$

Thus, (3.31) holds with $D = \max\{D_1, D_2\}$. □

We are now in the position to show the non-vanishing of $\zeta(s)$ on $\Re e\ s = 1$, $s \neq 1$, for the case $\gamma > 3/2$. The proof is in essence the classical argument of Hadamard [27, p. 63].

Theorem 3.20. *Let N satisfy (3.29) with $a > 0$ and $\gamma > 3/2$. Then, $t\zeta(1+it) \neq 0$, for all $t \in \mathbb{R}$. Consequently, $1/((s-1)\zeta(s))$ converges locally and uniformly to a continuous function as $\Re e\ s \to 1^+$.*

Proof. Without lost of generality we assume that $3/2 < \gamma < 2$. We shall use that for any $m \in \mathbb{N}$ and $\theta \in \mathbb{R}$,

$$m + 1 + \sum_{j=0}^{m-1} 2(m-j)\cos((j+1)\theta) \geq 0. \tag{3.32}$$

The proof of (3.32) is left to the reader (cf. Exercise 3.21). By Exercise 3.16,

$$\begin{aligned}
|\zeta(\sigma + it_0)| &= \left|\exp\left(\sum_{k=1}^\infty\sum_{\nu=1}^\infty \frac{1}{\nu}p_k^{-\nu\sigma - i\nu t_0}\right)\right| \\
&= \left|\exp\left(\sum_{k=1}^\infty\sum_{\nu=1}^\infty \frac{1}{\nu}p_k^{-\nu\sigma}(\cos(\nu t_0\log p_k) - i\sin(\nu t_0 \log p_k))\right)\right| \\
&= \exp\left(\sum_{k=1}^\infty\sum_{\nu=1}^\infty \frac{1}{\nu}p_k^{-\nu\sigma}\cos(\nu t_0\log p_k)\right),
\end{aligned}$$

Thus, for any $m \in \mathbb{N}$ and $t_0 \in \mathbb{R}$

$$\begin{aligned}
&|\zeta(\sigma)|^{m+1}\,|\zeta(\sigma+it_0)|^{2m}\prod_{j=1}^m |\zeta(\sigma + i(j+1)t_0)|^{2m-2j} \\
&= \exp\left(\sum_{k=1}^\infty\sum_{\nu=1}^\infty \frac{1}{\nu}p_k^{-\nu\sigma}\left(m+1+\sum_{j=0}^m 2(m-j)\cos(\nu(j+1)t_0\log p_k)\right)\right) \geq e^0 = 1.
\end{aligned}$$

If we now fix $t_0 \neq 0$ and m, Proposition 3.18 and the above inequality imply the existence of $A = A_{m,t_0} > 0$ such that for $1 < \sigma < 2$

$$1 \leq \frac{A\left|\zeta(\sigma + it_0)\right|^{2m}}{(\sigma - 1)^{m+1}} ,$$

or which is the same

$$D(\sigma - 1)^{1/2+1/(2m)} \leq |\zeta(\sigma + it_0)| ,$$

with $D = A^{1/2m}$.

Suppose we had $\zeta(1 + it_0) = 0$. Choose m such that $1/2 + 1/(2m) < \gamma - 1$. By the inequality (3.30) in Lemma 3.19, we would have

$$D(\sigma - 1)^{1/2+1/(2m)} \leq |\zeta(\sigma + it_0)| < C(\sigma - 1)^{\gamma-1} ,$$

which is certainly absurd. Therefore, we must necessarily have $\zeta(1+it) \neq 0$, for all $t \in \mathbb{R} \setminus \{0\}$. □

Exercise 3.21. Show (3.32) (**Hint**: The left hand side is equal to $\left|\sum_{\nu=1}^{m+1} e^{i\nu\theta}\right|^2$).

3.4.3 Proof of Beurling's Prime Number Theorem

We can now show Theorem 3.11. Assume (3.21) with $a > 0$ and $\gamma > 3/2$. Because of Corollary 3.9 and the formula (3.27), it is enough to show that

$$-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1}$$

has local pseudo-function boundary behavior on the line $\Re e\ s = 1$. Now,

$$\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} &= \frac{-\zeta'(s)(s-1) - \zeta(s)}{(s-1)\zeta(s)} \\
&= -\frac{1}{\zeta(s)}\left(\zeta'(s) + \frac{a}{(s-1)^2}\right) - \frac{1}{(s-1)\zeta(s)}\left(\zeta(s) - \frac{a}{s-1}\right) \\
&= -\frac{1}{\zeta(s)}\frac{d}{ds}\left(\zeta(s) - \frac{a}{s-1}\right) - \frac{1}{(s-1)\zeta(s)}\left(\zeta(s) - \frac{a}{s-1}\right) \\
&:= -\frac{1}{\zeta(s)}G_1'(s) - G_2(s).
\end{aligned}$$

Because of Proposition 3.18 and Theorem 3.20, $G_2(s)$ has a continuous extension to $\Re e\ s = 1$; in particular, $G_2(s)$ has local pseudo-function boundary behavior on such a line. On the other hand, Theorem 3.20 implies that $1/(\zeta(s))$ has continuous extension to $\Re e\ s = 1$. Thus, if we show that $G_1'(s)$ has local L^1 boundary values, we would have that $G_1'(s)/\zeta(s)$ has local pseudo-function boundary behavior. Let us show the latter. First notice that

$$\begin{aligned}
G_1'(s) = \frac{d}{ds}\left(\zeta(s) - \frac{a}{s-1}\right) &= \frac{d}{ds}\left(s\int_1^\infty x^{-s}\frac{N(x) - ax}{x}\mathrm{d}x + \frac{as}{s-1} - \frac{a}{s-1}\right) \\
&= \frac{d}{ds}\left(s\int_1^\infty x^{-s}\frac{N(x) - ax}{x}\mathrm{d}x\right) \\
&= \int_1^\infty x^{-s}\frac{N(x) - ax}{x}\mathrm{d}x - s\int_1^\infty x^{-s}\log x\frac{N(x) - ax}{x}\mathrm{d}x \\
&= \int_0^\infty e^{-(s-1)u}(e^{-u}N(u) - a)\mathrm{d}u - s\int_0^\infty e^{-(s-1)u}(e^{-u}N(u) - a)u\mathrm{d}u
\end{aligned}$$

Set $T_1(u) = e^{-u}N(u) - aH(u) = O(u^{-\gamma})$ and $T_2(u) = (e^{-u}N(u) - a)u = O(u^{1-\gamma})$. Because of $\gamma > 3/2$, $T_1, T_2 \in L^2(\mathbb{R})$. Thus G_1' has boundary values $G_1'(1+it) = \hat{T}_1(t) - (1+it)\hat{T}_2(t) \in L^2_{\mathrm{loc}}(\mathbb{R}) \subset L^1_{\mathrm{loc}}(\mathbb{R})$, in particular it has local pseudo-function boundary behavior.

3.4.4 Newest Extensions of the Prime Number Theorem

The PNT holds under weaker assumptions than that of Theorem 3.11. We state two general theorems without proof, they include Beurling's theorem as a very particular case. The interested reader can consult the corresponding references.

The following theorem relax the hypothesis (3.21) to an L^2 condition. It is due to Kahane [23], who gave a positive answer to a conjecture of Bateman and Diamond [3].

Theorem 3.22 (Kahane, 1997)**.** *Suppose there is a positive constant a such that*

$$\int_1^\infty \left| \frac{\log x (N(x) - ax)}{x} \right|^2 \frac{\mathrm{d}x}{x} < \infty,$$

Then the prime number theorem (3.20) holds.

The next PNT is a recent one and it is due to the author and Schlage-Puchta [36]. It replaces (3.21) by an average Cesàro estimate. The value of m below is allowed to be arbitrarily large.

Theorem 3.23 (Schlage-Puchta and Vindas, 2010)**.** *Suppose there exist constants $a > 0$, $\gamma > 3/2$, and m such that*

$$\int_1^x \frac{N(t) - at}{t} \left(1 - \frac{t}{x}\right)^m \mathrm{d}t = O\left(\frac{x}{\log^\gamma x}\right) \quad \text{as } x \to \infty.$$

Then the prime number theorem (3.20) holds.

Chapter 4

Multidimensional Theory for the Laplace Transform

In this chapter we sketch how to extend the one-dimensional theory from Chapter 2 to the multidimensional Laplace transform. We also discuss applications to the study of asymptotic properties of fundamental solutions to convolution equations, in particular, causal solutions to hyperbolic operators with constant coefficients. The Schwartz space of test functions $\mathcal{S}(\mathbb{R}^n)$ is readily defined, its topology is provided by the countable family of norms

$$\|\varphi\|_k = \sup_{0\leq|m|\leq k,\, x\in\mathbb{R}} (1+|x|)^k \left|\partial^m\varphi(x)\right|, \quad k\in\mathbb{N}.$$

Its dual is the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. All the distributional operations introduced in Section 1.5 can be obviously carried out in the multidimensional case. For instance, the partial derivative $\partial^m f$ is defined as $\langle\partial^m f,\varphi\rangle = (-1)^{|m|}\langle f,\partial^m\varphi\rangle$, and so on.

The Fourier transform is given on test functions by

$$\hat{\varphi}(x) = \mathcal{F}\left\{\varphi(u);x\right\} = \int_{\mathbb{R}^n} e^{ix\cdot u}\varphi(u)\mathrm{d}u,$$

with inverse

$$\mathcal{F}^{-1}\left\{\varphi(x);u\right\} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{-iu\cdot x}\varphi(x)\mathrm{d}x,$$

and it is defined, as usual, by duality on $\mathcal{S}'(\mathbb{R}^n)$. Observe that this definition differs from that used in the previous chapters by a negative sign in the exponential, we have conveniently switched the definition for the sake of simplicity through the present chapter.

The reader can consult the monograph [48] for a complete exposition on Tauberian theorems for the multidimensional Laplace transform (all the material from this chapter has been taken from there). See also [13, 33] for recent advances in multidimensional Tauberian theory for distributions.

4.1 Multidimensional Laplace Transform

4.1.1 Cones in $\mathbb{R}^n$

A set $\Gamma\subset\mathbb{R}^n$ is called a cone (with vertex at the origin) if $u\in\Gamma$ implies $\lambda u\in\Gamma$ for all $\lambda>0$. We set $\operatorname{pr}\Gamma = \{u\in\Gamma:\ |u|=1\}$. We shall always assume that Γ is a closed convex cone.

The cone Γ is said to be acute if it has a supporting hyperplane $\Xi\subset\mathbb{R}^n$ passing through the origin such that $\Xi\cap\Gamma=\{0\}$. The hyperplane Ξ divides into $\mathbb{R}^n\setminus\Xi$ two connected components, the property of being a supporting hyperplane means $\Gamma \setminus \{0\}$ is contained in one of such components. Equivalently, since hyperplanes can be described by equations $\omega \cdot u = 0$, where ω is some fixed vector $|\omega| = 1$, the cone Γ is acute if there exists such an ω such that

$$\omega \cdot u > 0, \quad \forall u \in \Gamma \setminus \{0\}. \tag{4.1}$$

Its conjugate cone is denoted by Γ^*. It is defined as

$$\Gamma^* = \{y \in \mathbb{R}^n : \ y \cdot u \geq 0, \ \forall u \in \Gamma\}.$$

It is easy to show that Γ^* is itself a closed convex cone. The property of being acute cone can be characterized in terms of the conjugate cone.

Lemma 4.1. *A closed convex cone Γ is acute if and only if Γ^* has non-empty interior.*

Proof. If Γ is acute, then there exists ω with norm 1 such that (4.1) holds. In particular, $\omega \in \Gamma^*$ and (4.1) holds for all $u \in \operatorname{pr}\Gamma$, a compact subset of the unit sphere, then there exists an open subset of the unit sphere $\omega \in V$ such that $e \cdot u > 0$ for all $e \in V$ and $u \in \operatorname{pr}\Gamma$, by homogeneity of the cones we conclude that $\xi \cdot u > 0$ for all $u \in \Gamma$ and ξ in the open subset $\{\xi = \lambda e : \ e \in V \text{ and } \lambda > 0\} \subset \Gamma^*$. Thus, $\omega \in \operatorname{int}\Gamma^*$. Conversely, if $\omega \in \operatorname{int}\Gamma^*$, then (4.1) holds. □

From now on, we shall *always* assume that Γ is a closed convex acute cone. We define the open convex acute cone

$$C := C_\Gamma = \operatorname{int}\Gamma^*,$$

and the *tube domain*

$$T^C := \mathbb{R}^n + iC \subset \mathbb{C}^n.$$

Example 4.2. The following are examples of closed convex acute cones and their conjugate cones.

- In dimension $n = 1$, we have only two possibilities:

$$\Gamma = [0, \infty) \text{ and } C = (0, \infty) \ \text{ or } \ \Gamma = (-\infty, 0] \text{ and } C = (-\infty, 0).$$

- Let $\mathbb{R}^n_+ = \{u \in \mathbb{R}^n : \ u_1 > 0, \dots, u_n > 0\}$, then $(\overline{\mathbb{R}^n_+})^* = \overline{\mathbb{R}^n_+}$.
- The future light cone in $\mathbb{R}^{n+1}$ is defined as $V^n_+ = \{(u_0, u) \in \mathbb{R}_+ \times \mathbb{R}^n : \ u_0 > |u|\}$. Then $\left(\overline{V^n_+}\right)^* = \overline{V^n_+}$.

4.1.2 Laplace Transform

We denote by $\mathcal{S}'_\Gamma \subset \mathcal{S}(\mathbb{R}^n)$ the subspace consisting of distributions supported by Γ. Given $f \in \mathcal{S}'_\Gamma$, its *Laplace transform* is defined as follows.

Observe first that if $z = x + iy$ belongs to the tube domain T^C, then $e^{iz\cdot u} = e^{ix\cdot u}e^{-y\cdot u}$ is not a test function from $\mathcal{S}(\mathbb{R}^n)$; however, when u is restricted to the cone Γ is indeed a rapidly decreasing function since $y \cdot u > 0$ in this case. We multiply $e^{iz\cdot u}$ by a suitable factor in order to obtain a test function. Let $\Gamma^\varepsilon := \Gamma + B(0, \varepsilon)$, where $B(0, \varepsilon)$ is the ball with radius ε and center at the origin; Γ^ε is then an ε-neighborhood of Γ. Let $\eta \in C^\infty(\mathbb{R}^n)$ be a functions such that

$$\eta(u) = 1 \ \forall u \in \Gamma^\varepsilon, \quad \eta(u) = 0 \ \forall u \notin \Gamma^{2\varepsilon}, \quad \text{and } \ |\partial^m \eta(u)| \leq K_\alpha \ \forall u \in \mathbb{R}^n \text{ and } m \in \mathbb{N}^n. \tag{4.2}$$

Such an η always exists (cf. Exercise 4.4). If $z \in T^C$, then one can show that $\eta(u)e^{iz\cdot u} \in \mathcal{S}(\mathbb{R}^n)$ (cf. [47, p. 101]). So, we define the Laplace transform as

$$\mathcal{L}\{f; z\} = \left\langle f(u), \eta(u)e^{iz\cdot u}\right\rangle, \quad z = x + iy \in T^{C_\Gamma};$$

it is a holomorphic function in the tube domain T^C. Clearly, the definition of the Laplace transform does not depend on the choice of η. Similar properties to those shown in Subsection 1.5.1 hold for the multidimensional Laplace transform, we refer to [47] for further details.

It should be pointed out that when $n = 1$ the Laplace transform that we just defined in this subsection differs from the one that we have been using in the previous chapters; nevertheless, they are related to each other through the change of variables $s = -iz$.

The Laplace transform is also given by

$$\mathcal{L}\{f; x + iy\} = \mathcal{F}_u\left\{f(u)\eta(u)e^{-y\cdot u}; x\right\}$$

so we have the inversion formula

$$f(u) = e^{y\cdot u}\mathcal{F}_x^{-1}\left\{\mathcal{L}\{x + iy\}; u\right\},$$

where y is any fixed point in C. In particular, because of this inversion formula, the Laplace transform is *one-to-one*.

Exercise 4.3. Show that $\mathcal{L}\{\partial^m\delta; z\} = (-iz)^m$, $m \in \mathbb{N}^n$. More generally, let P be a polynomial of degree k and consider the partial differential operator with constant coefficients

$$P(\partial) = \sum_{|m|\leq k} a_m\partial^m.$$

Show that $\mathcal{L}\{P(\partial)\delta; z\} = P(-iz)$.

Exercise 4.4. Show the existence of η satisfying (4.2) (**Hint**: Consider $\varphi \in \mathcal{D}(\mathbb{R}^n)$ a non-negative function such that $\operatorname{supp}\varphi \subseteq B(0, 1/2)$ and $\int_{\mathbb{R}^n}\varphi(u)\mathrm{d}u = 1$. Let $\varphi_\varepsilon(u) = \varepsilon^{-n}\varphi(u/\varepsilon)$. Show that

$$\eta(u) := \int_{\Gamma^{\frac{3\varepsilon}{2}}}\varphi_\varepsilon(u - \xi)\mathrm{d}\xi$$

has the desired properties).

4.1.3 The Convolution Algebra $\mathcal{S}'_\Gamma$

The space $\mathcal{S}'_\Gamma$ is an algebra when provided with the convolution operation [47], which we now define. Let $f, h \in \mathcal{S}'_\Gamma$. Find $\eta \in C^\infty$ satisfying (4.2), one can show that $\eta(u)\eta(\xi)\varphi(u + \xi) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ whenever $\varphi \in \mathcal{S}(\mathbb{R}^n)$ [47, p. 83]. The convolution $f * h$ is defined as

$$\langle f * h, \varphi\rangle = \langle f(u)h(\xi), \eta(u)\eta(\xi)\varphi(u + \xi)\rangle.$$

It can also be shown [47, p. 83] that $f * h \in \mathcal{S}'_\Gamma$. Clearly, this operation is commutative and associative; furthermore, the definition does not depend on the choice of η.

Exercise 4.5. (*Laplace transform of a convolution*). Let $f, h \in \mathcal{S}'_\Gamma$. Show that

$$\mathcal{L}\{f * h; z\} = \mathcal{L}\{f; z\}\mathcal{L}\{h; z\}.$$

Exercise 4.6. Show that $\partial^m(f * h) = (\partial^m f) * h = f * (\partial^m h)$ and in particular $\partial^m\delta * f = \partial^m f$.

4.2 Multidimensional Tauberian Theorems

We now state two Tauberian theorems for the multidimensional Laplace transform. Their proofs will be omitted, and we rather refer below to the corresponding literature where the proofs may be found.

We shall consider the distributional asymptotic behavior

$$f(\lambda u) \sim \lambda^{\alpha} g(u) \quad \text{as } \lambda \to \infty, \tag{4.3}$$

which should be interpreted in the sense of Section 2.1, i.e., as in (2.5). As in Proposition 2.4, one shows if $g \neq 0$, then it must be homogeneous of degree α.

We begin with a theorem of Drozhzhinov and Zavialov [12]. It is literally the multidimensional extension of Theorem 2.14.

Theorem 4.7 (Drozhzhinov and Zavialov, 1979)**.** *Let Γ be a closed convex acute cone and let $f \in \mathcal{S}'_{\Gamma}$. Necessary and sufficient conditions for f to have the asymptotic behavior (4.3) are the existence of a solid cone $C' \subset C$ (i.e., $\operatorname{int} C' \neq \emptyset$) such that*

$$\lim_{r \to 0^+} r^{\alpha+n} \mathcal{L}\{f; iry\} = G(iy), \quad \textit{for each } y \in C', \tag{4.4}$$

and the existence of $k \in \mathbb{N}$, $M > 0$ and a vector $\omega \in C$ such that

$$r^{n+\alpha} \left|\mathcal{L}\{f; r(x + i\sigma\omega)\}\right| < \frac{M}{\sigma^k}, \quad \textit{for all } r \in (0,1) \textit{ and } |x|^2 + \sigma^2 = 1. \tag{4.5}$$

In such a case, $G(z) = \mathcal{L}\{g; z\}$, $z \in T^C$, (4.4) holds for each $y \in C$, and an estimate of type (4.5) holds for any given $\omega \in C$.

The original statement [12] of Drozhzhinov and Zavialov theorem imposed additional assumptions over the cone Γ (see also [47, 48]). Those assumptions have been relaxed in Theorem 4.7. A proof of Theorem 4.7 can be found in [33, Sec. 8.6].

Let us discuss an example involving convolution.

Example 4.8. Suppose that f_1 and f_2 have the asymptotic behaviors

$$f_1(\lambda u) \sim \lambda^{\alpha_1} g_1(u) \quad \text{and} \quad f_2(\lambda u) \sim \lambda^{\alpha_2} g_2(u) \quad \text{as } \lambda \to \infty,$$

then,

$$(f_1 * f_2)(\lambda u) \sim \lambda^{\alpha_1+\alpha_2+n} (g_1 * g_2)(u) \quad \text{as } \lambda \to \infty.$$

Proof. In fact, set $F_j(z) = \mathcal{L}\{f_j, z\}$ and $G_j(z) = \mathcal{L}\{g_j, z\}$, $j = 1, 2$. Observe now that, by Exercise 4.5, $\mathcal{L}\{f_1 * f_2, z\} = F_1(z)F_2(z)$ and $\mathcal{L}\{g_1 * g_2, z\} = G_1(z)G_2(z)$. Applying Theorem 4.7 to both f_1 and f_2, we have

$$\lim_{r \to 0^+} r^{\alpha_j+n} F_j(iry) = G_j(iy) \quad \text{for each } y \in C,\ j = 1, 2,$$

and, given any fixed $\omega \in C$, the bounds

$$r^{n+\alpha_j} \left|F_j(r(x + i\omega))\right| < \frac{M_j}{\sigma^{k_j}} \quad \text{for all } r \in (0,1),\ \text{and } |x|^2 + \sigma^2 = 1,\ j = 1, 2.$$

Therefore, the hypotheses (4.4) and (4.5) are satisfied with $f = f_1 * f_2$, $G = G_1 G_2$, $\alpha = \alpha_1 + \alpha_2 + n$, $M = M_1 M_2$ and $k = k_1 + k_2$. Consequently, Theorem 4.7 yields the result. □

The condition (4.5) in Theorem 4.7 can be dropped for certain classes of distributions. We shall discuss two of such important results.

The next Tauberian theorem is due to Drozhzhinov, see [10, 48] for the proof. A function F, holomorphic on a domain $\Omega \subset \mathbb{C}^n$, is said to have *bounded argument* on Ω if $F(z) \neq 0$ for all $z \in \Omega$ and there is a constant M such that

$$|\arg F(z)| \leq M, \qquad \text{for all } z \in \Omega.$$

The function $\arg F(z)$ is assumed to vary continuously on Ω.

Theorem 4.9 (Drozhzhinov, 1982)**.** *Let $f \in \mathcal{S}'_\Gamma$ be such that its Laplace transform $\mathcal{L}\{f; z\}$ has bounded argument on T^C. Then, f has the asymptotic behavior (4.3) if and only if (4.4) is satisfied. In such a case $G(z) = \mathcal{L}\{g; z\}$.*

The following theorem is a multidimensional generalization of the Hardy-Littlewood theorem in the form of Theorem 2.13. It is due to Vladimirov [45, 47, 48]. A distribution is given by a nonnegative measure $f = \mu$ concentrated on the cone Γ, if its action on test functions is given by $\langle f, \varphi \rangle = \int_\Gamma \varphi(u) \mathrm{d}\mu(u)$.

Theorem 4.10 (Vladimirov, 1976)**.** *Let $f = \mu \in \mathcal{S}'_\Gamma$ be a nonnegative measure. Then, f has the asymptotic behavior (4.3) if and only if (4.4) is satisfied.*

Proof. In view of Theorem 4.7, we only need to show that (4.4) implies (4.5). Indeed, if $\omega \in C'$,

$$\begin{aligned} r^{\alpha+n} \left|\mathcal{L}\{f; r(x + i\sigma\omega)\}\right| &= r^{\alpha+n} \left| \int_\Gamma e^{irx\cdot u} e^{-\sigma r\omega\cdot u} \mathrm{d}\mu(u) \right| \\ &\leq r^{\alpha+n} \int_\Gamma e^{-\sigma r\omega\cdot u} \mathrm{d}\mu(u) \\ &= \sigma^{-\alpha-n} \left((\sigma r)^{\alpha+n} \mathcal{L}\{f; i\sigma r\omega)\} \right) \\ &\leq \frac{M}{\sigma^{\alpha+n}}, \quad \text{for all } \sigma, r \in (0,1), \end{aligned}$$

for some constant $M > 0$, as follows from (4.4). Theorem 4.10 has been established. □

Remark 4.11. Under the assumptions of Theorem 4.10, it follows from (4.3) that $g = \nu$ is also a nonnegative measure. For each $x \in \Gamma$, set

$$f^{(-1)}(x) = \int_{\Gamma\cap(x-\Gamma)} \mathrm{d}\mu(u) \text{ and } g^{(-1)}(x) = \int_{\Gamma\cap(x-\Gamma)} \mathrm{d}\nu(u),$$

their primitives with respect to the cone. It can be shown that both are continuous functions on $\operatorname{int}\Gamma$, and $g^{(-1)}$ is homogeneous of degree $\alpha + n$. Furthermore,

$$f^{(-1)}(x) \sim |x|^{\alpha+n} g^{(-1)}\left(\frac{x}{|x|}\right) \quad \text{as } |x|, \text{ uniformly for } x \text{ in compacts of } \Gamma.$$

The proof of this assertion is similar to that of Proposition 2.6. See [48, p. 60] for details. This supplementary information justifies our claim that Theorem 4.10 really generalizes the Hardy-littlewood Tauberian theorem (Theorem 2.13).

4.3 Convolution Equations

In this subsection, given $f \in \mathcal{S}'_\Gamma$, we denote its Laplace transform simply by $\tilde{f}(z) = \mathcal{L}\{f; z\}$.

We are interested in convolution equations in the algebra $\mathcal{S}'_\Gamma$, namely, equations

$$K * h = f, \tag{4.6}$$

where K and f are given generalized functions from $\mathcal{S}'_\Gamma$ and $h \in \mathcal{S}'_\Gamma$ is the sought unknown solution.

Observe that (4.6) includes partial differential equations in $\mathcal{S}'_\Gamma$; indeed, if P is a polynomial, then $P(\partial)h = f$ is equivalent to (4.6) with $K = P(\partial)\delta$ (cf. Exercise 4.6).

We may derive a simple procedure to solve (4.6) if we take Laplace transform. By Exercise 4.5, (4.6) implies $\tilde{K}\tilde{h} = \tilde{f}$, thus $\tilde{h} = \tilde{f}/\tilde{K}$. Suppose further that we know that there exists $E \in \mathcal{S}'_\Gamma$ such that $\tilde{E} = 1/\tilde{K}$. Then, we would have $\tilde{h} = \tilde{E}\tilde{f}$ and the solution would be

$$h = E * f. \tag{4.7}$$

Observe also that, since $\tilde{K}\tilde{E} = 1 = \tilde{\delta}$, E is itself a solution of the convolution equation

$$K * E = \delta. \tag{4.8}$$

A distribution satisfying (4.8), if it exists, is called the *fundamental solution* of the convolution operator $K*$. When the fundamental solution exists, it is automatically unique, as follows from the injectivity of the Laplace transform and the fact that we must have $\tilde{E} = 1/\tilde{K}$, and furthermore, (4.7) admits a unique solution given by (4.7). For a partial differential equation, when it exists, $E \in \mathcal{S}'_\Gamma$ is called the tempered *causal* fundamental solution with respect to the cone Γ.

There are several criteria in terms of $\tilde{K}$ for existence of the fundamental solution of (4.8). Observe that a necessary condition is that $\tilde{K}(z) \neq 0$ for all $z \in T^C$, but the latter is not sufficient. The following theorem is due to Vladimirov, see [47, p. 174] for the proof.

Theorem 4.12. *Let $K \in \mathcal{S}'_\Gamma$. If $\tilde{K}(z) = \mathcal{L}\{K; z\}$ has bounded argument on T^C, then the fundamental solution E of (4.8) exists.*

Observe that if the hypotheses of Theorem 4.12 are satisfied, then $\tilde{E} = 1/\tilde{K}$ has also bounded argument on T^C. From Theorems 4.9 and 4.12, we obtain the following two corollaries on asymptotics of solutions to convolution equations.

Corollary 4.13. *Let $K \in \mathcal{S}'_\Gamma$ be such that $\tilde{K}$ has bounded argument on T^C and it has the asymptotic behavior*

$$K(\lambda u) \sim \lambda^\alpha K_0(u) \quad \text{as } \lambda \to \infty.$$

Then, E, the fundamental solution of (4.8) has asymptotics

$$E(\lambda u) \sim \lambda^{-\alpha-2n} E_0(u) \quad \text{as } \lambda \to \infty,$$

where E_0 is also of bounded argument and it is the fundamental solution of $K_0 * E_0 = \delta$.

Proof. We have that for z in compacts of T^C, $\lim_{r\to 0^+} r^{\alpha+n}\tilde{K}(rz) = \tilde{K}_0(z)$. Since the locally uniform limit of non-vanishing holomorphic functions is a non-vanishing holomorphic function, we obtain that $K_0(z) \neq 0$, $\forall z \in T^C$. By the same reason, the argument of K_0 remains bounded. Thus, E_0 exists by Theorem 4.12. Next, we already saw that E is of bounded argument, and actually for $y \in C$

$$\lim_{r\to 0^+} r^{-\alpha-2n+n}\tilde{E}(iry) = \lim_{r\to 0^+} \frac{1}{r^{\alpha+n}\tilde{K}(iry)} = \frac{1}{\tilde{K}_0(iry)} = \tilde{E}_0(iy).$$

It remains just to apply Theorem 4.9. □

Corollary 4.14. *Let the hypotheses of Corollary 4.13 hold. Assume further that* $f \in \mathcal{S}'_\Gamma$ *has the asymptotic behavior*

$$f(\lambda u) \sim \lambda^\beta f_0(u) \quad \text{as } \lambda \to \infty.$$

If h *is the solution of the convolution equation (4.6), then, it has asymptotics*

$$h(\lambda u) \sim \lambda^{\beta-\alpha-n} h_0(u) \quad \text{as } \lambda \to \infty,$$

where h_0 *is the solution of* $K_0 * h_0 = f_0$.

Proof. h is given by (4.8). Corollary 4.7 and Example 4.8 yield $h(\lambda u) \sim \lambda^{\beta-\alpha-n}(f_0 * E_0)(u)$, and actually $h_0 = f_0 * E_0$ is the solution to $K_0 * h_0 = f_0$. □

4.4 Applications to Hyperbolic Equations with Constant Coefficients

We now consider a differential operator of order k with constant coefficients

$$P(\partial) = \sum_{|m| \leq k} a_m \partial^m,$$

that is hyperbolic with respect to the cone C, in the sense that

$$P(-iz) \neq 0, \quad \forall z \in C. \tag{4.9}$$

It turns out that under this circumstances $P(\delta)$ has always a tempered fundamental solution $P(\delta)E = \delta$ that is causal with respect to the cone Γ, namely, $E \in \mathcal{S}'_\Gamma$. This result was first proved by Bogolyubov and Vladimirov [6] and resembles the celebrated Hörmander theorem on the division of tempered distributions by polynomials. It may also be deduced from the following lemma in combination with Theorem 4.12 (see [48, p. 192] for the proof of the lemma).

Lemma 4.15. *If* P *satisfies (4.9), then it has bounded argument on* T^C.

We now obtain the asymptotic behavior of the fundamental solution of $P(\partial)$. We write

$$P(-iz) = \sum_{j=l}^{k} P_j(-iz),$$

where each P_j is a homogeneous polynomial and $P_l \neq 0$.

Theorem 4.16. *Let* P *satisfy (4.9). If* $E \in \mathcal{S}'_\Gamma$ *is the solution of* $P(\partial)$, *then it has asymptotics*

$$E(\lambda u) \sim \lambda^{l-n} E_l(u) \quad \text{as } \lambda \to \infty, \tag{4.10}$$

where E_l *is the fundamental solution of the operator* $P_l(\partial)$, *i.e., it satisfies* $P_l(\partial)E_l = \delta$.

Proof. We have that, for $y \in C$

$$r^{-l-n+n} P(ry) = r^{-l} \sum_{j=l}^{k} P_j(ry) = \sum_{j=l}^{k} r^{j-l} P_j(y) \to P_l(y) \quad \text{as } r \to 0^+.$$

The rest follows from Corollary 4.13. □

It follows from Theorem 4.16 that $P_l(\partial)$ is hyperbolic ($P_l(-iz) \neq 0$, $\forall z \in T^C$).

In Theorem 4.16 we have exhibited only the first order term of the asymptotics of the fundamental solution, Wagner has found in [49] a complete asymptotic series for the fundamental solution. We have the following corollary of Theorem 4.16.

Corollary 4.17. *Let the hypotheses of Theorem 4.16 hold. Assume that $f \in \mathcal{S}'_\Gamma$ has the asymptotic behavior*

$$f(\lambda u) \sim \lambda^\beta f_0(u) \quad \text{as } \lambda \to \infty.$$

If h is the solution to $P(\partial)h = f$ in $\mathcal{S}'_\Gamma$, then, it has asymptotics

$$h(\lambda u) \sim \lambda^{\beta+l} h_0(u) \quad \text{as } \lambda \to \infty,$$

where $h_0 \in \mathcal{S}'_\Gamma$ is the solution of $P_l(\partial)h_0 = f_0$.

We end this chapter with some examples. As an illustration, we consider the future light cone in $\mathbb{R}^2$

$$\Gamma = \overline{V^1_+} = \left\{(u_0, u_1) \in \mathbb{R}^2 : \ u_0 \geq |u_1|\right\} \quad \text{and} \quad C = V^1_+ = \left\{(u_0, u_1) \in \mathbb{R}^2 : \ u_0 \geq |u_1|\right\},$$

and the hyperbolic operator

$$P(\partial) = \partial_0^2 - \partial_1^2 + c^2, \quad P((iz_0, iz_1)) = z_1^2 - z_0^2 + c^2, \quad (c \geq 0).$$

Example 4.18. *(The wave equation).* When $c = 0$, we obtain the wave operator $\partial_0^2 - \partial_1^2$. If we apply Theorem 4.16 ($l = n = 2$), we simply obtain that $E = E_2$, a homogeneous distribution of order zero. We calculate the causal fundamental solution of the wave equation via Laplace transform,

$$(\partial_0^2 - \partial_1^2)E = \delta.$$

The equation can be rewritten as $K * E = \delta$, where $K = P(\partial)\delta$ with $P(u_0, u_1) = u_0^2 - u_1^2$. By Exercise 4.3, $\tilde{K}(z_0) = P((-iz_0, -iz_1)) = z_1^2 - z_0^2$, then E must have Laplace transform $\tilde{E}(z_0, z_1) = (z_1^2 - z_0^2)^{-1}$. Since the Fourier transform is the boundary value of the Laplace transform, we have that for a fixed $(y_0, y_1) \in V^1_+$,

$$\hat{E}(x_0, x_1) = \lim_{\varepsilon \to 0^+} \tilde{E}(x_0 + i\varepsilon y_0, x_1 + \varepsilon y_1) = \lim_{\varepsilon \to 0^+} \frac{1}{(x_1 + x_0 + i\varepsilon(y_1 + y_0))(x_1 - x_0 + i\varepsilon(y_1 - y_0))},$$

in $\mathcal{S}'(\mathbb{R}^2)$. We now choose $(y_0, y_1) = (1, 0) \in V^1_+$, and since the Fourier transform is continuous

$$E(u_0, u_1) = \lim_{\varepsilon \to 0^+} \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{e^{-i(u_0 x_0 + u_1 x_0)}}{(x_1 + x_0 + i\varepsilon)(x_1 - x_0 - i\varepsilon)} \mathrm{d}x_0 \mathrm{d}x_1 \quad \text{in } \mathcal{S}'(\mathbb{R}^2).$$

Observe that "the integral" in the last formula even makes sense as the Fourier transform of an L^2 function. Changing variables by $t_1 = x_1 + x_0$ and $t_0 = x_0 - x_1$, we see that

$$E(u_0, u_1) = \frac{1}{2} \lim_{\varepsilon \to 0^+} -\mathcal{F}^{-1}_{t_0}\left\{(t_0 + i\varepsilon)^{-1}; u_0 + u_1\right\} \mathcal{F}^{-1}_{t_1}\left\{(t_1 + i\varepsilon)^{-1}; u_0 - u_1\right\},$$

where each of the inverse Fourier transforms are one-dimensional. Let H the one-dimensional Heaviside function, notice that

$$\mathcal{F}\left\{e^{-\varepsilon u} H(u); t\right\} = \int_0^\infty e^{-\varepsilon u} e^{itu} \mathrm{d}u = \frac{1}{i(t + i\varepsilon)}$$

and hence

$$\lim_{\varepsilon\to 0^+} \mathcal{F}^{-1}\left\{(t+i\varepsilon)^{-1}; u\right\} = i \lim_{\varepsilon\to 0^+} e^{-\varepsilon} H(u) = iH(u).$$

Therefore, $E(u_0, u_1) = -i^2 H(u_0+u_1)H(u_0-u_1)/2 = H(u_0+u_1)H(u_0-u_1)/2$. Finally, observe that $H(u_0+u_1)H(u_0-u_1) = H(u_0 - |u_1|)$ and hence,

$$E(u_0, u_1) = \frac{1}{2} H(u_0 - |u_1|),$$

the characteristic function of the future light cone V_+^1.

Example 4.19. *(The Klein-Gordon equation).* If $c > 0$, we have the Klein-Gordon operator,

$$P(\partial) = \partial_0^2 - \partial_1^2 + c^2 = P_2(\partial) + P_0(\partial).$$

The fundamental solution of $P_0(\partial)$ is simply $E_0 = c^{-2}\delta$. So if we apply Theorem 4.16 (here $l = 0, n = 2$), we conclude that the fundamental solution E of the Klein-Gordon operator has asymptotics

$$E(\lambda u) \sim \frac{\delta(u)}{c^2\lambda^2} \quad \text{as } \lambda \to \infty.$$

E can be explicitly calculated (cf. [46]), and actually

$$E(u) = \frac{1}{2} H(u_0 - |u_1|) J_0\left(c\sqrt{u_0^2 - u_1^2}\right),$$

where J_0 is the Bessel function and H is again the Heaviside function.

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