AN INTRODUCTION TO TAUBERIAN THEOREMS

TIMOTHY TRUDGIAN

1. Introduction

Consider the Taylor-expansion

$$\log(1+x) = x - \frac{x^2}{2} + \ldots + (-1)^n \frac{x^n}{n} + \ldots, \tag{1}$$

which can be derived by elementary means to be valid for $|x| < 1$. Due to the singularity at $x = 1$, where the logarithm is undefined[1] this series has a radius of convergence equal to one. One might well ask whether it is possible to deduce that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \ldots$ from the above equation. The answer is 'yes, but not yet'.

Substituting $x = 1$ in the above formula uses the fact that $\lim_{x\to c} f(x) = f(c)$, which is equivalent to the function $f(x)$ being continuous at c. Do we know this to be true? Well, we know that, since this is a power series with radius of convergence 1, the series is uniformly convergent for in any region $0 \le |x| \le \rho < 1$ for any $\rho < 1$. We also know that a uniformly convergent series converges to a continuous function. So unfortunately we do not know that $\log(1 + x)$ is continuous at $x = 1$ and so we cannot proceed as we might have hoped - indeed we need the following

Abel's Theorem. *Suppose that $\sum a_n x^n$ has radius of convergence equal to unity, and that $\sum a_n \to s$. Then*

$$\sum a_n x^n \to s,$$

where the convergence is uniform, hence

$$\lim_{x\to 1} \sum a_n x^n = s.$$

So, in the case of the logarithm, the sum of the coefficients $\{-\frac{1}{n}\}$ converges and hence we can say with confidence that $\log 2 = 1 - \frac{1}{2} + \frac{1}{3} + \ldots$.

2. Tauber's Theorem

Abel's Theorem says

$$\sum a_n \to s \qquad \Longrightarrow \qquad \lim_{x\to 1} \sum a_n x^n = s;$$

the converse of this result is false in general. Take $f(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1}{x+1}$, and this geometric series is valid whenever $|x| < 1$. Then $f(1) = \frac{1}{2}$ but $\sum_{n=0}^{\infty}(-1)^n$ is divergent. Tauber's Theorem provides a partial solution to this converse problem

Tauber's Theorem. *Suppose that $f(x) \to s$ as $x \to 1$ **and** that $a_n = o\left(\frac{1}{n}\right)$. Then $\sum a_n \to s$.*

For the proof we need the following auxiliary result which is left as an exercise

Exercise. *If $b_n \to 0$ as $n \to \infty$, then*

$$\frac{b_0 + b_1 + \ldots + b_n}{n+1} \to 0.$$

Littlewood [2] was able to relax the condition on the rate of growth of the coefficients to prove Tauber's Theorem when $a_n = O\left(\frac{1}{n}\right)$. In the seminar we will discuss a specific problem of Hadwiger and Agnew [1] as well as more general 'Tauberian theorems'.

References

[1] R. P. Agnew. Tauberian relations among partial sums, Riesz transforms, and Abel transforms of series. *Journal reine ange. Math.*, 193:94–118, 1954.

[2] J. E. Littlewood. The converse of Abel's theorem on power series. *Proc. Lond. Math. Soc.*, pages 434–448, 1910.

[1]Note that, blasphemous as it is to write such things, that the left hand side then diverges to $-\log\infty$ as $x \to -1$, and the right hand side does the same - a result written by Euler in the 18th century.