International Journal of Progressive Sciences and Technologies (IJPSAT)
ISSN: 2509-0119.

http://ijpsat.ijsht-journals.org
Vol. 13 No. 2 March 2019, pp. 233-232

Quaternionic Quantum Wave Equation and Applications

Mahgoub Salih

Department of Physics, Faculty of Arts and Science at al-Muznab, Al Qassim University,KSA

P.O. Box 931,Al Muznab 51931, Saudi Arabia.

Abstract - The perception of duality nature in physics is extended to covers the idea of superposition of field and its dual field by assuming there is resultant field emerges as a superposition when a field and its dual field are combined. Quaternion formulation is used in this model. This generalization enables in constructing a general plane wave to generalize, Klein Gordon equation, Helmholtz's equation. It can be applied to electromagnetic and Meissner effect.

Keywords - Generalized Klein Gordon Equation, Helmholtz's, Quaternion, Meissner Effect

I. INTRODUCTION

The principle of wave particle duality expounded by Louis de Broglie is one of the foundations of quantum mechanics states that, all physical entities have a dual character. They are both waves and particles [1-2]. This phenomenon has been verified not only for elementary particles, but also for compound particles. This duality is experimentally verified by Davidson and Germier in 1928 [3]. For macroscopic objects, because of their extremely short wavelengths, wave behaviors cannot be detected obviously [4]. In year 2000 Carver Mead proposed that the duality can be replaced by wave functions, and attributes the apparent particle-like behavior to quantization effects and eigenstates [5]. Consequently, a search for wave representation of particles becomes inventible. As we know, waves and particles are fundamentally different objects, waves extend over the whole space, but particles are localized at particular positions in space. Thus, with these distinguishable properties we can choose the wave. Anyway, an equation governing this wave-function is needed to provide a full explanation of the mechanics of such a particle. This equation cannot be derived from a classical equation. It had to be hypothesized and its validity has to be checked against the experiment. Since a particle cannot be described by a single wave but by multitudes of waves (wavelets), then the solution of the intended equation should incorporate this fact.

The idea that I proposed in this work is to join between the duality principles and matter wave functions by defining wave function constructing as a superposition of field and its dual field. To this aim I used quaternion multiplication to find out two equations, one vector and the other is scalar.

As Quaternions are mathematical construct, that are generalization of complex numbers, introduced by Irish mathematician Sir William Rowan Hamilton in 1843[6]. They consist of four components. Quaternions are closed under multiplication. Because of their interesting properties one can use them to write the physical laws in a compact way.

The multiplication rule for two quaternions

$\tilde{A} = (a_0, \vec{A})$ and $\tilde{B} = (b_0, \vec{B})$ is given by

$$\tilde{A}\tilde{B} = \left(a_0 b_0 - \vec{A}\cdot\vec{B}, a_0\vec{B} + \vec{A}b_0 + \vec{A}\times\vec{B}\right) \quad (1)$$

Therefore, the ordinary continuity equation can be written in a quaternionic continuity equation by defining the operator $\tilde{\nabla} = (\bar{\nabla}, \vec{\nabla})$ and current $\tilde{J} = (\bar{J}, \vec{J})$,

$$\tilde{\nabla}\tilde{J} = \left(\bar{\nabla}\bar{J} - \vec{\nabla}\cdot\vec{J}\ ,\ \bar{\nabla}\vec{J} + \vec{\nabla}\bar{J} + \vec{\nabla}\times\vec{J}\right) \quad (2)$$

But $\tilde{\vec{\nabla}}\tilde{\vec{J}}$ is Lorentz invariant. That means $\tilde{\vec{\nabla}}\tilde{\vec{J}} = 0$.

Let us with

$$\left(\tilde{\nabla}\tilde{J} - \vec{\nabla}\cdot\vec{J}\ ,\ \tilde{\nabla}\vec{J} + \vec{\nabla}\tilde{J} + \vec{\nabla}\times\vec{J}\right) = 0 \qquad (3)$$

Under duality transformation equation (3) gives

$$i\vec{\nabla} \to \tilde{\nabla}\ ,\ i\tilde{\nabla} \to -\vec{\nabla}\ ,\ i\vec{J} \to \tilde{J}\ ,$$

$$i\tilde{J} \to -\vec{J} \text{ and } i\left(\vec{\nabla}\times\vec{J}\right) \to \tilde{\nabla}\times\tilde{J} \qquad (4)$$

One finds $\left(-\tilde{\nabla}\vec{J} - \vec{\nabla}\tilde{J}\ ,\ \tilde{\nabla}\tilde{J} - \vec{\nabla}\cdot\vec{J} + \tilde{\nabla}\times\tilde{J}\right) = 0$

This shows that the term $\tilde{\nabla}\times\tilde{J}$ is scalar .Therefore, the quaternion set missed this term. So we can write the quaternion multiplication as

$$\tilde{\vec{\nabla}}\tilde{\vec{J}} = \left(\tilde{\nabla}\tilde{J} - \vec{\nabla}\cdot\vec{J} + \tilde{\nabla}\times\tilde{J}\ ,\ \tilde{\nabla}\vec{J} + \vec{\nabla}\tilde{J} + \vec{\nabla}\times\vec{J}\right) \qquad (5)$$

From above we write two equations

$$\vec{\nabla}\times\vec{J} = -\tilde{\nabla}\vec{J} - \vec{\nabla}\tilde{J} \qquad (6)$$

$$\tilde{\nabla}\times\tilde{J} = \vec{\nabla}\cdot\vec{J} - \tilde{\nabla}\tilde{J} \qquad (7)$$

If we define $\tilde{\vec{\nabla}} = \left(\frac{i}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ and $\tilde{\vec{J}} = \left(i\rho c, \vec{J}\right)$, equations (6) and (7) give the well-known continuity equations.

Now we can recognize equations (6) and (7) are complex equations, describe the action and reaction of field and its dual field. Therefore, effective curl- is the sum of equations (6) and (7), and conservation of energy stated that the curl- is zero. So we can write

$$\tilde{\nabla}\times\tilde{J} + \vec{\nabla}\times\vec{J} = \vec{\nabla}\cdot\vec{J} - \tilde{\nabla}\tilde{J} - \left(\tilde{\nabla}\vec{J} + \vec{\nabla}\tilde{J}\right) = 0 \qquad (8)$$

Implies

$$\left(\vec{\nabla}\cdot\vec{J} - \tilde{\nabla}\tilde{J}\right)_F = \left(\tilde{\nabla}\vec{J} + \vec{\nabla}\tilde{J}\right)_{DF} \qquad (9)$$

Where DF refers to dual field and F to the field.

Applying transformation (4) to equation (9) we find, $\vec{\nabla}\cdot\vec{J} = 0$, which means the modification of quaternion product generates solenoidal field, that means equation (9) is a generalization of continuity equation in quaternion form.

According to above if we have a field $\vec{F}$ in space time and dual field $\tilde{F}$, the observer in field region records field's action and the reaction of the dual field as a result of rotation in the field separately. For this, the observer writes two equations

$$\vec{\nabla}\times\vec{J} = 0\ ;\quad \vec{\nabla}\cdot\vec{J} - \tilde{\nabla}\tilde{J} = 0 \qquad (10\text{-a})$$

But in dual field, observer records

$$\tilde{\nabla}\times\tilde{J} = 0\ ;\quad -\tilde{\nabla}\vec{J} - \vec{\nabla}\tilde{J} = 0 \qquad (10\text{-b})$$

As found by Arbab [7] they did not distinguish between the field and dual field and missed dual field -curl term in attempting to generalize continuity equations.

Generalized wave equation:

By using equations (6) and (7) we get

$$\vec{\nabla}\times\left(\tilde{\nabla}\times\tilde{F}\right) = -\tilde{\nabla}\left(\vec{\nabla}\cdot\vec{F} - \tilde{\nabla}\tilde{F}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{F} - \tilde{\nabla}\tilde{F}\right) = 0 \quad (11)$$

$$\tilde{\nabla}\times\left(\vec{\nabla}\times\vec{F}\right) = \vec{\nabla}\cdot\left(-\tilde{\nabla}\vec{F} - \vec{\nabla}\tilde{F}\right) - \tilde{\nabla}\left(-\tilde{\nabla}\vec{F} - \vec{\nabla}\tilde{F}\right) = 0 \quad (12)$$

Subtract equation (12) from (11) and equate vector parts and scalar parts to zero, we find

$$\left[\vec{\nabla}, \tilde{\nabla}\right]\vec{F} + \left(\vec{\nabla}^2 + \tilde{\nabla}^2\right)\tilde{F} = 0 \qquad (13)$$

$$\left[\vec{\nabla}, \tilde{\nabla}\right]\tilde{F} - \left(\vec{\nabla}^2 + \tilde{\nabla}^2\right)\vec{F} = 0 \qquad (14)$$

Equation (13) is scalar and equation (14) is vector.

Equations (13) and (14) can be written as

$$\left[\vec{\nabla}, \tilde{\nabla}\right] = -\left(\vec{\nabla}^2 + \tilde{\nabla}^2\right)_{DF} \qquad (15)$$

$$\left[\vec{\nabla}, \tilde{\nabla}\right]_{DF} = \left(\vec{\nabla}^2 + \tilde{\nabla}^2\right) \qquad (16)$$

If the operators are commute, equation (15) and (16) give the wave equation if we define $\tilde{\nabla} = \frac{i}{c}\frac{\partial}{\partial t}$.

This is a new notion of wave equation as a result of commutation relation of operator and its dual operator. Therefore, equation (15) and (16) are generalization of wave equation. Otherwise, we will find a general form of Klein Gordon equation.

Wave of reality:

We try to construct a wave to describe the reality. We mean by reality, the system that contains field and its dual field together as a supper position of two waves. To this end we develop wave function of plane wave and construct dual plane wave as

$$\psi = Ae^{-i a(\alpha t-\beta r)} \quad ; \quad \tilde{\psi} = \tilde{A}e^{a(\bar{\alpha}t-\bar{\beta}r)} \qquad (17)$$

And the resultant wave is

$$\Psi = \psi_0 e^{-a\{(i\alpha-\bar{\alpha})t+(\bar{\beta}-i\beta)r\}} \qquad (18)$$

Where

$$a = \frac{mc^2}{\hbar\omega_0 k_0} \ , \bar{\alpha} = k\underline{\hat{\omega}} \ , \alpha = k\hat{\omega} \ , \ \beta = \frac{\omega\hat{K}}{c} \quad , \quad \bar{\beta} = \frac{\omega\underline{\hat{K}}}{c} \qquad (19)$$

Where, $\hat{K}$ the wave number operator, it related to linear momentum operator, $\hat{P} = \hbar\underline{\hat{K}}$, where $\hat{K}$ the spatial operator and $\underline{\hat{K}}$ its adjoint operator. And $\hat{\omega}$ the frequency operator, it can be related to energy operator, $\hat{E} = \hbar\underline{\hat{\omega}}$,where $\hat{\omega}$ the temporal operator and $\underline{\hat{\omega}}$ its adjoint operator.

Now we can employ the resultant wave (18) in equations (15) and (16) to get the final wave equation as

$$(\vec{\nabla}^2 + \tilde{\nabla}^2)\Psi - \frac{1}{c}[\hat{K}, \hat{\omega}]\Psi = 0 \qquad (20)$$

The commutator bracket gives

$$[\hat{K}, \hat{\omega}]\Psi = 2\frac{a^2\omega k}{c}(\underline{\hat{K}} - i\hat{K})(\underline{\hat{\omega}} - i\hat{\omega})\Psi - a\left\{\frac{\partial k}{\partial r}(\underline{\hat{\omega}} - i\hat{\omega}) + \frac{\partial \omega}{c\partial t}(\underline{\hat{K}} - i\hat{K})\right\}\Psi \qquad (21)$$

If the eigen value of the operator $(\underline{\hat{\omega}} - i\hat{\omega})\Psi = \omega_0\Psi$ and $(\underline{\hat{K}} - i\hat{K})\Psi = k_0\Psi$, when we write equation (20) with Ψ we find

$$-\frac{\hbar^2}{2m}(\vec{\nabla}^2 + \tilde{\nabla}^2)\Psi + mc^2\left(\frac{\omega k}{\omega_0 k_0}\right)\Psi = \frac{\hbar c}{2}\left\{\frac{1}{k_0}\frac{\partial k}{\partial r} + \frac{1}{\omega_0 c}\frac{\partial \omega}{\partial t}\right\}\Psi \qquad (22)$$

We can write it in vector form as

$$-\frac{\hbar^2}{2m}(\vec{\nabla}^2 + \tilde{\nabla}^2)\vec{\Psi} + mc^2\left(\frac{\vec{\omega}\cdot\vec{k}}{\omega_0 k_0}\right)\vec{\Psi} = \frac{\hbar c}{2}\left\{\frac{1}{k_0}\vec{\nabla}\cdot\vec{k} + \frac{1}{\omega_0 c}\frac{\partial \omega}{\partial t}\right\}\vec{\Psi} \qquad (23)$$

As it has seen from equation (23), the wave nature arises from the distribution of the wave parameters, mass, frequency and wave number, beside the wave itself. Therefore, equation (23) describes a field with wave particle dual nature. Equation (23) is generalization of Klein Gordon equation.

We can see that, equation (23) is invariant under the following sets of transformations,

$$\begin{matrix} \omega \leftrightarrow \pm ck \\ r \leftrightarrow \pm ct \\ \frac{\partial}{\partial r} \leftrightarrow \pm\frac{1}{c}\frac{\partial}{\partial t} \end{matrix} \quad \text{and} \quad \begin{matrix} \omega \leftrightarrow \pm ick \\ r \leftrightarrow \pm ict \\ \frac{\partial}{\partial r} \leftrightarrow \pm\frac{i}{c}\frac{\partial}{\partial t} \end{matrix} \qquad (24)$$

If k and ω are constants and , $\frac{\omega k}{\omega_0 k_0} = \frac{1}{2}$, we find

$$-\frac{\hbar^2}{2m}(\vec{\nabla}^2 + \tilde{\nabla}^2)\Psi + \frac{1}{2}mc^2\,\Psi = 0 \qquad (25)$$

Equation (25) is the ordinary Klein Gordon equation.

If $\tilde{\nabla}^2\Psi = 0$, we find

$$\frac{\partial^2\Psi}{\partial t^2} = \left\{k\frac{\partial}{\partial t}(\underline{\hat{\omega}} - i\hat{\omega}) + (\underline{\hat{\omega}} - i\hat{\omega})\frac{\partial k}{\partial t} + ak^2(\underline{\hat{\omega}} - i\hat{\omega})^2\right\}a\Psi = 0 \qquad (26)$$

If the operator $(\underline{\hat{\omega}} - i\hat{\omega})$ is constant with time and $(\underline{\hat{\omega}} - i\hat{\omega})\Psi = \omega_0\Psi$, above equation gives

$$\frac{1}{k^2}\frac{\partial k}{\partial t} = -a\omega_0 \qquad (27)$$

We find by integration and defining $\omega = \frac{constant}{T}$

$$\frac{k}{k_0} - \frac{\hbar\omega}{mc^2} = \pm C \qquad (28)$$

Where C is constant. Mass can be given as

$$m = \frac{\hbar\omega}{c^2}\left(\frac{k_0}{k \pm Ck_0}\right) \qquad (29)$$

This means the famous equation $mc^2 = \hbar\omega$ depends on the ratio $\frac{k_0}{k \pm Ck_0}$.

The phase velocity v_p also can be shown from equation (28)

$$v_p = \frac{\omega}{k} = v_{p0} \pm v_p(\lambda) \qquad (30)$$

Where, $v_{p0} = \frac{mc^2}{\hbar k_0}$; $v_p(\lambda) = C\frac{mc^2}{\hbar k}$

This implies that at any given time the quantum state is a mixture state. It is thus when $k \to \infty$ $v_p \to v_{p0}$.

Employing equation (29) $(C = 0)$ with equation (23) we get

$$-\vec{\nabla}^2\,\Psi + \left(\frac{\omega k_0}{ck}\right)^2\,\Psi = 0 \qquad (31)$$

if $\omega = \pm ck$,equation (31) gives

$$\vec{\nabla}^2\,\Psi = {k_0}^2\,\Psi \qquad (32)$$

Equation (32) with the wave Ψ gives $k_0 = \frac{mc}{\hbar}$, this is Meissner effect where the penetration length is equal to Compton wavelength of the massive photons inside the superconductors [8]

If $r = \pm ct$, we find

$$\frac{\partial^2\Psi}{\partial t^2} - \omega_0{}^2\,\Psi = 0 \qquad (33)$$

If we use imaginary frequency as $\omega = \pm ick$, we find

$$\vec{\nabla}^2\,\Psi + k_0{}^2\,\Psi = 0 \qquad (34)$$

And

$$\frac{\partial^2\Psi}{\partial t^2} + \omega_0{}^2\,\Psi = 0 \qquad (35)$$

We now have Helmholtz's equation for the spatial variable equation (34), and a second-order ordinary differential equation in time, equation (35). The solution in time will be a linear combination of sine and cosine functions, with angular frequency of ω, while the form of the solution in space will depends on the boundary conditions. Alternatively, integral transforms, such as the Laplace or Fourier transform, are often used to transform a hyperbolic PDE into a form of the Helmholtz equation.

If we go a further step in electromagnetic wave equation and study LC circuit, and make transformation in equation (23) as $k \to E$, $\omega \to B$ this equivalent to quaternion defines as $\tilde{\vec{F}} = (icB, \vec{E})$ and $mc^2 \to LI_0{}^2$.

Where, B the magnetic scalar and $\vec{E}$ the electric field vector. The stored energy in the coil is $LI_0{}^2$ .Therefore, we find

$$-\frac{\kappa^2}{2L}\left(\frac{q_P{}^2}{I_0}\right)^2(\vec{\nabla}^2 + \tilde{\nabla}^2)\Psi + LI_0{}^2\left(\frac{\vec{E}\cdot\vec{B}}{E_0B_0}\right)\Psi = \frac{\hbar c}{2}\left\{\frac{1}{E_0}\vec{\nabla}\cdot\vec{E} + \frac{1}{cB_0}\frac{\partial B}{\partial t}\right\}\Psi \qquad (36)$$

Where, $q_P = \sqrt{\frac{\hbar c}{\kappa}}$ is Planck charge and $\kappa = \frac{1}{4\pi\epsilon_0}$.

With condition

$$\vec{\nabla}\cdot\vec{E} + \frac{\partial B}{\partial t} = 0 \qquad (37)$$

Inside conductor there is no electric charges, so $\rho = 0$, equation (37) agrees with Arbab [9], where he predicted a physical scalar function Λ has a dimension of magnetic field, satisfies the wave equation $\vec{\nabla}\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = -\Lambda$, which yield $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} - \frac{\partial\Lambda}{\partial t}$.

And $cB_0 = E_0$; $\frac{\vec{E}\cdot\vec{B}}{E_0B_0} = \frac{1}{2}$ equation (36) reads

$$-(\vec{\nabla}^2 + \tilde{\nabla}^2)\Psi + \left(\frac{LI_0{}^2}{\kappa q_P{}^2}\right)^2\Psi = 0 \qquad (38)$$

Equation (38) is Klein Gordon equation for electromagnetic circuit. We find the characteristics length depends on magnetic energy.

$$l = \frac{\hbar c}{LI_0{}^2} \qquad (39)$$

Energy of equation (39) equals to grand unification energy when the length is equal to Planck length.

When the magnetic energy is large, the second term in equation (38) becomes very large, where the first term can be neglected, and wave function breaks down as the equation tends to, $\left(\frac{LI_0{}^2}{\kappa q_P{}^2}\right)^2\Psi = 0$. That is what was found in superconductor, a superconductor with little or no magnetic field within it is said to be in Meissner state. The Meissner state breaks down when the applied magnetic field is too large [10].

Now we can construct electromagnetic wave function due to above equations and the reality wave in (18) as,

$$\Psi_{EM} = \varphi_0 e^{-a\{(i\alpha - \bar{\alpha})t + (\bar{\beta} - i\beta)r\}} \qquad (40)$$

Where,

$$a = \frac{LI_0{}^2}{\hbar B_0 E_0} \;,\; \bar{\alpha} = E\underline{\hat{B}} \;,\; \alpha = E\hat{B} \;,\; \beta = \frac{B\hat{E}}{c} \;,\; \bar{\beta} = \frac{B\underline{\hat{E}}}{c} \qquad (41)$$

Where $\hat{E}$ the electric strength operator and $\underline{\hat{E}}$ its adjoint operator. $\hat{B}$ the magnetic strength operator and $\underline{\hat{B}}$ its adjoint operator.

II. CONCLUSION

We have developed duality principle using modified quaternion with dual field rotation as a source of scalar field equation (7), this step enabled in generalizing wave equation and wave function .we introduced a new notion of wave equation as a result of commutation relation of operator and its dual operator. Generalized Klein Gordon equation and Helmholtz's equation are recovered. Finally we applied equation (23) to electromagnetic to find Meissner effect.

References:

[1] L. de Broglie, C. R. Acad. Sci. Paris 177 (1923), 506, 548, 630.

[2] Walter Greiner (2001). Quantum Mechanics: An Introduction. Springer. ISBN 978-3-540-67458-0)

[3] C.J. Davisson, The diffraction of electrons by a crystal of nickel, Bell Syst. Tech. J. 7 (1928) 90.

[4] R. Eisberg & R. Resnick (1985). Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles (2nd ed.). John Wiley & Sons. pp. 59–60. ISBN 978-0471873730)

[5] Collective Electrodynamics: Quantum Foundations of Electromagnetism (2000) ISBN 0-262-13378-4)

[6] Sweetser, 2005 [Sweetser D.B. Doing physics with quaternions. MIT, 2005. Accessed online: http://world.std.com/sweetser/quaternions/ ps/book.pdf])

[7] Arbab et al (On the generalized continuity equation)(arXiv:1003.0071v1 [physics.gen-ph] 27 Feb 2010).

[8] S. O. Kasap. Principles of Electronic Materials and Devices. New York: McGraw Hill, 2006, pp. 729-736.

[9] A.I.Arbab(The consequences of complex Lorentz force andviolation of Lorenz gauge condition) arXiv:1302.0695v1 [physics.gen-ph] 4 Feb 2013.

[10] Superconductivity: The Meissner Effect, Persistent Currents and the Josephson Effects." MIT Department of Physics, Feb. 8, 2011.