ANNALES DE L'I. H. P., SECTION B

J. GALAMBOS

Distribution of arithmetical functions. A survey

Annales de l'I. H. P., section B, tome 6, nº 4 (1970), p. 281-305.

<http://www.numdam.org/item?id=AIHPB_1970__6_4_281_0>

© Gauthier-Villars, 1970, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section B » (http://www.elsevier.com/locate/anihpb), implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/legal.php). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Henri Poincaré,
Vol. VI, n° 4, 1970, p. 281-305.

Section B :
Calcul des Probabilités et Statistique.

Distribution of arithmetical functions. A survey

by

J. GALAMBOS
Department of Mathematics, University of Ghana, Legon (*).

1. SUMMARY

Though the object of the present paper is to give a survey on the development concerning the distribution of values of additive arithmetical functions, some mention will be made of arbitrary arithmetical functions, too. Since by the method of characteristic functions, the distribution problem of additive number theoretical functions reduces to the existence of the limit of the mean of multiplicative functions, it seems natural to include in such a survey the results of the latter topic. I shall however not discuss this problem in its generality, only the well known Delange theorem will be stated with remarks on two different lines of generalizations of it. I wish to emphasize the probabilistic character of the problem and this paper was prepared for an audience whose interest was assumed to be probability theory.

Out of the references three kinds of works are left out intentionally:

(*i*) Books on number theory having a minor section on this topic.

(*ii*) Survey papers touching our problem but hardly contains more than the formulation of the problem.

(*iii*) Some of those papers which were published by the same author on the same line before 1962; in regard to these papers the reader is referred to the very comprehensive monograph by J. Kubilius (1962).

I intend to go through the whole development of this subject and hence there will be a considerable overlap with the very popular works of M. Kac (1959) and J. Kubilius (1962). My intention is however different from

(*) « Now at temple University, Philadelphia, Pa 19122 ».

theirs, and in any case, the results of the past ten years may justify to have a new look at the earlier results.

After the introduction of the number theoretical concepts and the listing of probabilistic results needed for the understanding of the references made in the sequel, the paper is split into the following sections: 4. The laws of large numbers; 5. The Erdos-Wintner theorem; 6. The central limit problem; 7. Asymptotic expansion of the distribution function of arithmetical functions. In all sections it will be observed that the results remain valid if the arithmetical functions in question are considered on a wide class of sequences of integers (others than the successive ones) and also for functions defined on algebraic number fields.

2. THE FORMULATION OF THE PROBLEM

Any function defined on the integers 1, 2, ... is called an arithmetical (number theoretical) function. Throughout the paper, $2=p_1<p_2<\ldots$ will denote the successive prime numbers. The fundamental theorem of arithmetic says that there is a unique representation of any integer n in the form

$$n = \prod_{j=1}^{+\infty} p_j^{s_j(n)} \tag{1 a}$$

with $s_j(n) \geqslant 0$ integers. Clearly, only a finite number of $s_j(n) \neq 0$; (1 *a*) can be written as

$$n = \prod_{p_j \mid n} p_j^{s_j(n)} \tag{1 b}$$

where $p_j \mid n$ signifies that p_j is a divisor of n. In (1 *b*) we always assume that $p_j \neq p_t$ if $j \neq t$.

DEFINITION 1

An arithmetical function $f(n)$ is called additive if for co-prime u and v

$$f(uv) = f(u) + f(v) \tag{2}$$

By (1 *b*) and by induction we have that if $f(n)$ is additive then

$$f(n) = \sum_{p_j^{s_j(n)} \| n} f\left(p_j^{s_j(n)}\right) \tag{3}$$

$p_j^{s_j(n)} \| n$ signifying, that $p_j^{s_j(n)} \mid n$ but $p_j^{s_j(n)+1} \nmid n$ (does not divide n).

DEFINITION 2

An additive arithmetical function $f(n)$ is called strongly additive, if

$$f(p^\alpha) = f(p), \ \alpha \geqslant 1 \text{ integer}, \ p \text{ prime}. \tag{4}$$

For strongly additive functions $f(n)$, (3) reduces to

$$f(n) = \sum_{p_j | n} f(p_j) \tag{5}$$

Note, that the sequence $\{ f(p_j) \}$ and (5) uniquely determine the strongly additive function $f(n)$.

DEFINITION 3

An arithmetical function $g(n)$ is called multiplicative, if for co-prime u and v

$$g(uv) = g(u)g(v) \tag{6}$$

and strongly multiplicative, if (6) holds and

$$g(p^\alpha) = g(p), \ \alpha \geqslant 1 \text{ integer}, \ p \text{ prime}. \tag{7}$$

By induction and (1 b) we have that a strongly multiplicative function $g(n)$ satisfies the relation

$$g(n) = \prod_{p_j | n} g(p_j) \tag{8}$$

Again, the sequence $\{ g(p_j) \}$ and (8) uniquely determine the strongly multiplicative $g(n)$.

In the sequel, unless otherwise stated, we assume that an additive function is real-valued and multiplicative functions are complex valued.

There is a strong relation between additive and multiplicative functions. Obviously, if $f(n)$ is (strongly) additive, then the function $\exp(Zf(n))$ is (strongly) multiplicative for any fixed complex number Z. On the other hand, let $g(n) \neq 0$ be a (strongly) multiplicative function. Then there is a unique solution of the relation

$$g(p) = e^{f(p)}, \ -\pi < \operatorname{Im} f(p) \leqslant +\pi, \tag{9}$$

and thus $g(n) = \exp(f(n))$ where $f(n)$ is a complex valued (strongly) additive function. If $g(n)$ may take 0 also, then consider the sequence $\{ p_{j_t} \}$ of primes for which $g(p_{j_t}) \neq 0$. In this case the relation (9) determines a (strongly) additive function $f(n)$ on the multiplicative semi-group generated by $\{ p_{j_t} \}$ and $f(n)$ is undetermined for other integers. This remark will be exploited later on.

Consider the following special arithmetical functions

$$\varepsilon_j(m) = \begin{cases} 1 & \text{if} \quad p_j \mid m \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

By (5) and (10), if $f(n)$ is strongly additive,

$$f(n) = \sum_{j=1}^{+\infty} f(p_j)\varepsilon_j(n) \tag{11}$$

Clearly, the right hand side is always a sum of a finite number of terms and especially, for any $n \leqslant \mathrm{N}$,

$$f(n) = \sum_{j=1}^{\mathrm{N}} f(p_j)\varepsilon_j(n) \tag{11 a}$$

From (8) and (10) we have, that for any strongly multiplicative function $g(n)$,

$$g(n) = \prod_{j=1}^{+\infty} g(p_j)\varepsilon_j(n) = \prod_{j=1}^{\mathrm{N}} g(p_j)\varepsilon_j(n) \tag{12}$$

where $n \leqslant \mathrm{N}$ is arbitrary.

DEFINITION 4

The arithmetical function $f(n)$ is said to have the asymptotical distribution $\mathrm{F}(x)$ if

$$\lim_{\mathrm{N}=+\infty} \frac{1}{\mathrm{N}} \sum_{\substack{n \leqslant \mathrm{N} \\ f(n) < x}} 1 = \mathrm{F}(x) \tag{13}$$

for all continuity points of $\mathrm{F}(x)$.

This is a special case of a general concept called asymptotic density.

DEFINITION 5

The sequence $\mathrm{A} = \{ a_1 < a_2 < \dots \}$ is said to have asymptotic density $\mathrm{D(A)}$ if

$$\mathrm{D(A)} = \lim_{\mathrm{N}=+\infty} \frac{1}{\mathrm{N}} \sum_{a_k \leqslant \mathrm{N}} 1 \tag{14}$$

exists.

A weaker assumption on A is to suppose to have logarithmic density.

DEFINITION 6

We say that the logarithmic density L(A) of A exists, if

$$\text{(15)} \qquad L(A) = \lim_{N=+\infty} \frac{1}{\log N} \sum_{a_k \leq N} 1/a_k$$

exists $(0 < a_1)$.

These definitions introduced by number theoreticians clearly show that the concepts involved are probabilistic. Namely, consider the probability space $S_N = (\Omega_N, \mathscr{A}_N, P_N)$, where $\Omega_N = \{1, 2, \ldots, N\}$, $\mathscr{A}_N$ the set of all subsets of Ω_N, and P_N generates the uniform distribution on $\mathscr{A}_N$, i. e. $P_N(\{i\}) = N^{-1}$, $i = 1, 2, \ldots, N$. Then any arithmetical function $f(n)$ restricted to Ω_N is a random variable on S_N, and

$$P_N(f(n) < x) = N^{-1} \sum_{\substack{n \leq N \\ f(n) < x}} 1$$

and for a sequence $A = \{a_1 < a_2 < \ldots\}$,

$$P_N(A\Omega_N) = \frac{1}{N} \sum_{a_k \leq N} 1$$

hence, (13) is equivalent to the relation

$$\text{(13 a)} \qquad \lim_{N=+\infty} P_N(f(n) < x) = F(x)$$

for all continuity points of $F(x)$. Also, if $D(A)$ exists then

$$\text{(14 a)} \qquad D(A) = \lim_{N=+\infty} P_N(A\Omega_N)$$

Note, that

$$P_N(\varepsilon_j(m) = 1) = P_N(\{p_j, 2p_j, \ldots, rp_j, \ldots\}) = \frac{1}{N}[N/p_j]$$

where $[x]$ denotes the integer part of x. Hence

$$\text{(16)} \qquad P_N(\varepsilon_j(m) = 1) = 1/p_j + 0(N^{-1})$$

and similarly

$$\text{(17)} \quad P(\varepsilon_{j_1} = 1, \varepsilon_{j_2} = 1, \ldots, \varepsilon_{j_t} = 1) = 1/p_{j_1}p_{j_2} \ldots p_{j_t} + 0(N^{-1})$$

(16) and (17) show that the random variables $\{ \varepsilon_k \}$ are almost independent on the probability space S_N, and hence by (11 a), any strongly additive arithmetical function is a finite sum of « almost independent » random variables if we restrict ourselves to the probability space S_N (what can be done by (13) if we are interested in the distribution problem only). We remark here that it would be convenient to consider strongly additive functions on the whole set of successive integers if D(A) defined in (14) were probability measure, since by (11), (14), (16) and (17) we have that the functions $\{ \varepsilon_j \}$ were independent random variables and $f(n)$ were a series of independent terms. This is however not the case, the set function D(A) is finitely additive only, and hence we have to deal with the dependent (but « almost » independent) terms $\{ \varepsilon_j \}$ on S_N.

We shall be able to investigate the distribution problem of the sum (11 a) assuming only that the terms $\{ \varepsilon_j \}$ satisfy a condition of « almost independence », partially expressed by the relations (16) and (17), i. e. we do not make reference to the actual elements of Ω_N. This approach allows us to obtain results concerning the distribution of strongly additive arithmetical functions considered on a wide class of sequences of integers. As an example I mention the following case: let $\Omega_N = \{ Q(1), Q(2), \ldots, Q(N) \}$, where $Q(x) > 0$ is polynomial of integral coefficients. Then, putting $\varrho(p_k)$ for the number of residue classes mod p_k satisfying the congruence

$$Q(x) \equiv 0 \bmod p_k,$$

the random variables $\varepsilon_j(m)$ defined in (10), where now $m = Q(x)$, satisfy the relations (16) and (17) p_k^{-1} being replaced by $\rho(p_k)p_k^{-1}$, i. e.

$$(18) \qquad P_N(\varepsilon_{j_1}(Q(x)) = 1, \ldots, \varepsilon_{j_t}(Q(x)) = 1) = \prod_{s=1}^{t} \frac{\rho(p_{j_s})}{p_{j_s}} + 0(N^{-1})$$

for $t = 1, 2, \ldots$ (see Hardy and Wright (1954, p. 96-97).

To conclude this section I list some additive and positive valued multiplicative functions: $U(n)$, the number of prime divisors of n (with multiplicity); $V(n)$, the number of distinct prime divisors of n; $\Delta(n) = U(n) - V(n)$; $\Phi(n)$, the so called Euler function, the number of integers between 1 and n prime to n; $d(n)$, the number of divisors of n, $\sigma(n)$, the sum of divisors of n; $a(n)$, the number of distinct Abelian groups of order n.

3. PROBABILISTIC LEMMAS

Let $S = (\Omega, \mathscr{A}, P)$ be a probability space. For any random variable X on S, E(X) denotes the expectation of X and

$$\varphi(t) = \mathrm{E}[\exp(\mathrm{it}\, \mathrm{X})]$$

is called the characteristic function of X.

LEMMA 1 (P. LÉVY)

Let $\varphi_1, \varphi_2, \ldots$ be the characteristic functions of the random variables $X_1, X_2, \ldots$ on S. Assume that $\varphi_n \to \varphi$ continuous at $t = 0$, then $P(X_n < x)$ has a limit, $F(x)$, φ is a characteristic function and uniquely determines $F(x)$.

LEMMA 2 (P. LÉVY (1931))

Let $X_1, X_2, \ldots$ be independent random variables on S and assume that ΣX_i converges. Let d_k be the largest *jump* of the distribution function of X_k. The distribution function of ΣX_i is continuous at every x if and only if

$$\prod_{k=1}^{+\infty} d_k = 0.$$

Further, if all terms X_i have purely discontinuous distribution, then the distribution of ΣX_i is either continuous at every x or purely discontinuous.

LEMMA 3 (FRÉCHET and SHOHAT (1931))

Let $X_1, X_2, \ldots$ be a sequence of random variables and assume that $\lim_{n=+\infty} E(X_n^k) = m_k$ exist for all k. Assume further, that $\{ m_k \}$ uniquely determines the distribution function $F(x)$. Then $\lim_n P(X_n < x) = F(x)$.

Especially, if

$$m_k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} x^k e^{-x^2/2} dx,$$

then

$$F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt$$

LEMMA 4 (LIAPOUNOV)

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, $|X_n| \leqslant e$, and putting

$$V_k^2 = E([X_k - E(X_k)]^2),$$

$s_n^2 = \sum_{k=1}^{n} V_k^2$ is assumed to tend to $+\infty$. Then

$$\frac{1}{s_n} \sum_{k=1}^{n} [X_k - E(X_k)]$$

is asymptotically normally distributed.

LEMMA 5 (KOLMOGOROV (1928))

Let $X_1, X_2, \ldots$ be independent random variables and put

$$X_k^* = \begin{cases} X_k & \text{if } |X_k| \leqslant 1 \\ 1 & \text{otherwise} \end{cases}$$

The series ΣX_k converges (with probability 1) if and only if the three series

$$\sum_{k=1}^{\infty} E(X_k^*); \quad \sum_{k=1}^{\infty} V(X_k^*), \quad \sum_{k=1}^{\infty} P(X_k \neq X_k^*)$$

converge.

LEMMA 6 (ESSEEN (1945))

If $F(x)$ and $G(x)$ are two distribution functions, $G'(x)$ exists for all x and $|G'(x)| \leqslant A$, $\varphi_F(u)$ and $\varphi_G(u)$ denote the characteristic functions of F and G respectively, and if

$$\int_{-T}^{+T} |\varphi_F(u) - \varphi_G(u)| \, u^{-1} du < \varepsilon$$

then for $-\infty < x < +\infty$

$$|F(x) - G(x)| < K(\varepsilon + A/T)$$

where K is a constant not depending on any of x, ε, A, T.

LEMMA 7 (KUBIK (1959))

Let $X_1, X_2, \ldots$ be independent random variables taking two values only. Let further A_n and B_n be constants such that

$$B_n \to +\infty, \ \max_{k \leqslant n} P(|X_k| B_n^{-2} > \varepsilon) \to 0 (n \to +\infty)$$

Assume that

$$\frac{1}{B_n}\sum_{1}^{n} X_k - A_n$$

has a limit distribution admitting finite variance and that

$$\frac{1}{B_n^2}\sum_{1}^{n} V(X_k) \to \text{ variance of the limit distribution.}$$

Then this limit distribution always belongs to the class F(A, C, a, b) of distributions the characteristic function $\varphi(u)$ of which is determined by the integral

$$\log \varphi(u) = \int_{-\infty}^{+\infty} (e^{iux} - 1 - iux)x^{-2}dK(x),$$

where

$$K(x) = \begin{cases} 0 & \text{if } \; x < A \\ a(1 - x^2A^{-2}) & \text{if } \; A \leqslant x < 0 \\ bx^2C^{-2} + 1 - b & \text{if } \; 0 < x \leqslant C \\ 1 & \text{otherwise} \end{cases}$$

Finally, I shall make reference to the Chebisev inequality: for any random variable X having finite expectation E and variance V,

$$P(|X - E| > \varepsilon\sigma) \leqslant \frac{1}{\varepsilon^2}$$

where $\varepsilon > 0$ is any constant and $\sigma = \sqrt{V}$.

4. THE LAW OF LARGE NUMBERS

Since any arithmetical function $f(n)$ is a random variable on the probability space $S_N = (\Omega_N, \mathscr{A}_N, P_N)$, $\Omega_N = \{1, \ldots, N\}$, $\mathscr{A}_N$ the set of all subsets of Ω_N and $P_N(\{i\}) = 1/N$, the Chebisev inequality provides a good tool to investigate the « avarage behaviour » of $f(n)$. Here

$$E_N(f(n)) = \frac{1}{N}\sum_{1}^{N} f(n), \quad V_N(f(n)) = \frac{1}{N}\sum_{n=1}^{N} f^2(n) - E_N^2(f(n)).$$

The first result of this kind obtained by purely number theoretical methods was discovered by Hardy and Ramanujan (1917) for which Turan (1934) gave a proof still without reference to probability theory but by the same argument what is usual to obtain the Chebiev inequality itself. The theorem says that for almost all $n \leqslant N$, the number of prime divisors of n is asymptotically log log N, or in our terminology, the exact result is

$$E_N(U(n)) \sim \log\log N, \quad V_N(U(n)) \sim \log\log N$$

and hence by the Chebisev inequality,

$$P_N(|U(n) - \log\log N| \geqslant \log\log^{3/4} N) \leqslant \frac{1}{\sqrt{\log\log N}}$$

Turan (1936) generalized the Hardy-Ramanujan theorem for a wider class of additive functions and finally Kubilius (1956) proved the following inequality: for any complex valued additive arithmetical function

$$\frac{1}{N}\sum_{n=1}^{N}\left| f(n) - \sum_{p\leqslant N}\frac{f(p)}{p}\right|^2 \leqslant A\sum_{p\leqslant N}\frac{|f(p)|^2}{p} \tag{19}$$

with a suitable constant A, not depending on $f(n)$, or in other words

$$E_N[|f(n) - E_N(f(n))|^2] \leqslant A\sum_{p\leqslant N}\frac{|f(p)|^2}{p} \tag{20}$$

and hence, putting

$$A_N = \sum_{p\leqslant N}\frac{f(p)}{p}, \quad B_N^2 = \sum_{p\leqslant N}\frac{f^2(p)}{p} \tag{21}$$

by the Chebisev inequality we have that for any additive arithmetical function $f(n)$,

$$P_N(|f(n) - A_N| \geqslant \varepsilon B_N) \leqslant \frac{1}{\varepsilon^2} \tag{22}$$

(22) gives an interesting result only, if $B_N \to +\infty$, and if $B_N^2 = 0(A_N)$, and then it says that for almost all $n \leqslant N$, $f(n)$ is asymptotically equal to A_N. In the preceding calculations I used the relations that

$$E_N(f(n)) = \sum_{p\leqslant N}\frac{f(p)}{p} + 0\left(\frac{1}{N}\sum_{p\leqslant N} 1\right) = \sum_{p\leqslant N}\frac{f(p)}{p} + \circ(1)$$

what follows immediately from (11 *a*) and (16). Note that (22) was obtained by making use of (11 *a*), (16) and (19) only, i. e. it was not essential that $f(n)$ is an additive arithmetical function defined on the integers 1, 2, ..., N; and hence we obtained

THEOREM 1

Let $S_N = (\Omega_N, \mathscr{A}_N, P_N)$ be a sequence of probability spaces, and let the random variables $\varepsilon_{1,N}, \ldots, \varepsilon_{n,N}$, $n = n(N)$ taking the values 1 or 0 only, be defined on Ω_N. Assume that there is a sequence $q_1, q_2, \ldots$ such that

$$P_N(\varepsilon_{k,N} = 1) = q_k + 0(N^{-1}) \tag{23}$$

further, let $\{ c_k \}$ be a sequence of real numbers such that

$$E_N\left[\left(\sum_{k=1}^{n} c_k\varepsilon_{k,N} - \sum_{k=1}^{n} c_k q_k\right)^2\right] \leqslant A\sum_{k=1}^{n} c_k^2 q_k \tag{24}$$

(A constant not depending on $\{ c_k \}$), and putting

$$f_{n,N} = \sum_{k=1}^{n} c_k\varepsilon_{k,N}, \quad A_N = \sum_{k=1}^{n} c_k q_k, \quad B_N^2 = \sum_{k=1}^{n} c_k^2 q_k \tag{25}$$

if

$$n = \circ(N), \quad B_N \to +\infty, \tag{26}$$

then

$$P_N(| f_{n,N} - A_N | \geqslant \alpha(N)B_N) \to 0$$

where $\alpha(N)$ tends to $+\infty$ arbitrary slowly.

Applying theorem 1, we always assume that Ω_N is a finite set, $\mathscr{A}_N$ the set of all subsets of Ω_N and P_N generates the uniform distribution on $\mathscr{A}_N$. Let first Ω_N be the set $\{ Q(1), \ldots, Q(N) \}$ where $Q(x) > 0$ is an irreducible polynomial of integral coefficients. Then (18) says that (23) is satisfied with $q_k = \rho(p_k)p_k^{-1}$, and (24) was proved by Uzdavinys (1959, 1962) and Barban (1960) hence theorem 1 is applicable to the additive arithmetical function

$$f_{n,N}(Q(x)) = \sum_{p_k \leqslant N} f(p_k)\varepsilon_{k,N}(Q(x)). \tag{27}$$

(in order to satisfy the restriction (26) on the number of terms), but in this case it is very easy to show that in probability

$$f(Q(x)) - f_{n,N}(Q(x)) \to 0. \tag{28}$$

Another corollary to theorem 1 is obtained if we take Ω_N as the set of consecutive integer ideals arranged according to the usual norm in a number field over the field of rational numbers. (23) is well known (with $q_k = r^{-1}(p_k)$, where $r(p_k)$ is the norm of the kth prime ideal) and (24) was proved by Danilov (1965). (26) is also valid, and well known. The advantage of the probabilistic approach to the number theoretical one is clearly seen already from these examples what will be made even more obvious in the coming sections.

5. THE ERDÖS-WINTNER THEOREM

The problem of giving necessary and sufficient conditions for the existence of the asymptotic distribution of an additive arithmetical function was settled by Erdös and Wintner (1939) and reads as follows.

THEOREM 2 (ERDÖS and WINTNER (1939))

For the existence of the asymptotic distribution of the additive arithmetical function $f(n)$ it is necessary and sufficient that the series

$$\sum_{|f(p)|<1} \frac{f(p)}{p}, \quad \sum_{|f(p)|<1} \frac{f^2(p)}{p}, \quad \sum_{\substack{p \\ |f(p)|\geq 1}} \frac{1}{p} \tag{29}$$

converge.

The sufficiency part of this theorem was obtained by Erdös (1938) generalizing results of Behrend (1931-1932), Davenport (1933) and Schoenberg (1936). Behrend and Davenport investigated the asymptotic distribution of $\sigma(n)$, the sum of the divisors of n, while Schoenberg obtained a general theorem giving as sufficient condition for the existence of the asymptotic distribution of $f(n)$ the convergence of the three series (29) assuming the absolute convergence of the first one. The case of Schoenberg is easy, the general case however was obtained by deep number theoretical tools. Notice that the conditions (29) are the same as those in the Kolmogorov three series theorem (lemma 5), but theorem 2 can not be deduced from lemma 5 directly because, as we have remarked, the density as a set function is only finitely additive. Several attempts have been made to re-obtain the sufficiency part of theorem 2 (in this section this will be referred to as « the Erdös theorem ») by purely probabilistic arguments. One line of attempts can be described as follows: map the set of integers Z into a set Ω, and construct a σ-field $\mathscr{A}$ of subsets of Ω and a measure P on $\mathscr{A}$. If this procedure maps the arithmetical progressions into elements of $\mathscr{A}$, and the P-measure of these sets is the density of the arithmetical progressions then the arithmetical functions $\{ \varepsilon_j \}$ defined in (10) will be rendom variables on $S = (\Omega, \mathscr{A}, P)$ and

$$P(\varepsilon_j = 1) = 1/p_j$$

and hence any additive arithmetical function $f(n)$ satisfying (29) will be a random variable on S (by (11) and lemma 5). The hardest part is then to take care in the construction and mapping to guarantee that in this case $P(f(n) < x) = D(f(n) < x)$. This is the point where all attempts have broken down, and hence only weaker results than the Erdös theorem were obtained by Novoselov (1960), Billingsley (1964) and Paul (1964). Paul succeded by this procedure to show that (29) guarantees that

$$P(f(n) < x) = L(f(n) < x)$$

what is again weaker than the Erdös theorem. Novoselev (1964), after several publications on this line, came very close to settling the problem of reducing the statistical properties of additive arithmetical functions to that of sums of independent terms, his tools are however a bit complicated especially for the number theoreticians and in the last step mentioned above he still needs quite a lot of computations of number theoretical character. At the same time by his approach interesting new results can be obtained.

Another (successful) approach will be described after some remarks on the several proofs of a theorem of Delange (1961) from which the Erdös theorem can be re-obtained. By lemma 1, the Erdös theorem can be proved by turning to characteristic functions, i. e. to investigate

$$E_N[\exp(\mathrm{it}\, f(n))]$$

for $f(n)$ additive. Since the function

$$g(n) = \exp(\mathrm{it}\, f(n)) \tag{30}$$

is multiplicative and $|g(n)| \leqslant 1$, the Erdös theorem is contained in the following

THEOREM 3 (DELANGE (1961))

Let $g(n)$ be a (complex valued) strongly multiplicative arithmetical function with $|g(n)| \leqslant 1$. Then

$$\lim_{N=+\infty} \frac{1}{N} \sum_{n=1}^{N} g(n) = A \tag{31}$$

exists and is different from 0 if and only if the series

$$\sum_{p} \frac{g(p) - 1}{p} \tag{32}$$

converges. In this case

$$A = \prod_p \left[1 + \frac{g(p) - 1}{p}\right]. \tag{33}$$

Applying theorem 3 with $g(n)$ in (30), we get the sufficiency part of theorem 2 by the fact that the convergence of (32) implies the convergence of all the series of (29).

Delange (1962) gave another proof of theorem 3, both proofs being of analytical character. Rényi (1965) gave a very simple proof for the sufficiency part of theorem 3 which at the same time gives a very simple proof for the Erdös theorem. By developing this proof of Rényi I gave a purely probabilistic proof for the Erdös theorem. Before formulating this theorem and approach of mine, I wish to give the references to important results on multiplicative functions. By dropping the conditions of $|g(n) \leqslant 1$ and $A \neq 0$, the Delange theorem was generalized by Wirsing (1967) and Halasz (1968) in very extensive but fairly difficult papers. To earlier results the reader is referred to Rényi (1965, II) and Delange (1962). Another generalization of the Delange theorem is given in Galambos (1970), where

$$\lim_{N=+\infty} \frac{1}{N} \sum_{i=1}^{N} g(m_i)$$

is investigated for a general class of integers m_i with $|g(m)| \leqslant 1$, by a probabilistic approach.

As remarked above, I gave a purely probabilistic proof for the Erdös theorem. This approach resulted in more than re-proving the Erdös theorem, what will be seen later on. In order to formulate my theorem let us introduce a concept:

Definition 7

Let $S_N = (\Omega_N, \mathscr{A}_N, P_N)$ be a sequence of probability spaces. We say that the random variables $\varepsilon_{k,N}$, $k = 1, \ldots, n = n(N)$, taking two values 0 or 1, form a class K, if there exists a sequence $0 \leqslant q_k \leqslant 1$, such that

$$P_N(\varepsilon_{k,N} = 1) = q_k + 0(N^{-1}), \tag{K1}$$

there exists an

$$M = M(N) \to \infty$$

with N, such that for any $1 \leqslant i_1 < i_2 \ldots < i_t \leqslant M$,

(K2) $$P_N(\varepsilon_{i_1,N} = 1, \ldots, \varepsilon_{i_t,N} = 1) = q_{i_1}, \ldots, q_{i_t} + r(M, N; i_1, \ldots, i_t)$$
$$|r(M, N; i_1, \ldots, i_t)| \leqslant R(M, N) \to 0 \text{ as } N \to +\infty.$$

and there is a set Γ of real sequences $\{ c_k \}$ such that

(K3) $$V_N\left(\sum_{k=1}^{n} c_k \varepsilon_{k,N}\right) \overset{\text{def}}{=} E_N\left[\left(\sum_{k=1}^{n} c_k(\varepsilon_{k,N} - q_k)\right)^2\right] \leqslant A \sum_{k=1}^{n} c_k^2 q_k$$

with a constant A not depending on $\{ c_k \}$ and N.

By the method of characteristic functions, we can easily prove

THEOREM 4 (GALAMBOS (to appear))

Let $\varepsilon_{k,N}$, $k = 1, \ldots, n = n(N)$, be a class K on S_N, and let $\{ c_k \}$ be a sequence of real numbers for which (K3) is satisfied. Then the limit distribution of

$$f_{n,N} = \sum_{k=1}^{n} c_k \varepsilon_{k,N}$$

exists, as $N \to +\infty$, if

(34) $$n = \circ(N)$$

and if the series

(35) $$\sum_{|c_k| < 1} c_k q_k, \quad \sum_{|c_k| < 1} c_k^2 q_k, \quad \sum_{|c_k| \geqslant 1} q_k$$

converge.

By (16), (17) and by the Kubilius inequality (19) we have that the random variables defined in (10) form a class K ((K3) being satisfied for any sequence of real numbers) and hence theorem 4 implies the Erdös theorem. Theorem 4 is much more than the Erdös theorem, namely, choosing Ω_N of the probability space S_N as any finite set of (not necessarily successive) integers, the power of theorem 4 becomes that it reduces the distribution problem of additive functions considered on this sequence of integers to checking whether the random variables corresponding to those defined in (10) form a class K. The properties (K1) and (K2) are usually simple, and (K3), the generalization of the Kubilius inequality, is always much simpler than the usual tools applied by number theoreticians, in most cases very sharp sieve results. As interpretations I mention some results obtained by number theoretical tools and which follow immediately from theorem 4.

Let $f(n)$ be an additive arithmetical function and $Q(x) > 0$ a polynomial of integral coefficients. Considering the probability space S_N with $\Omega_N = \{ Q(1), \ldots, Q(N) \}$, $\mathscr{A}_N$ the set of all subsets of Ω_N and P_N generating the uniform distribution on $\mathscr{A}_N$, the random variables

$$\varepsilon_{k,N}(Q(j)) = \begin{cases} 1 & \text{if } p_k \mid Q(j) \\ 0 & \text{otherwise} \end{cases} \tag{36}$$

form a class K, (K3) being satisfied for any sequence $\{ c_k \}$ tending to 0 by results of Uzdavinys (1959, 1962) and Barban (1960) and hence by (28) we have that the existence of the limit distribution of $f(Q(x))$ is guaranteed by (35) whenever $f(p_k) \to 0$ as $k \to +\infty$. This was recently proved by Katai (1969) and Uzdavinys (1967) obtained this theorem under a slight assumption on $Q(x)$, but at the same time he showed that for $f(n)$ with $f(p_k) \to 0, k \to +\infty$, the conditions (35) are also necessary. Both Katai and Uzdavinys use sharp versions of sieve theorems.

Theorem 4 yields immediately sufficient conditions for the existence of the limit distribution of additive functions $f(n)$ defined on number fields. Simply take $\Omega_N = \{ a_1, \ldots, a_N \}$, a_i integer ideals, and hence whenever the random variables $\varepsilon_{k,N}$, p_k being the prime ideals arranged in increasing order with respect to the usual norm, form a class K, theorem 4 is applicable directly with $q_k = r^{-1}(p_k)$, $r(.)$ being the norm. This line of research has a very extensive literature, without attempting to be complete, I mention the papers Juskis (1964), de Kroon (1965), Grigelionis (1962) and Rieger (1962, I), and the last chapter of Kubilius (1962).

To conclude this section I mention the problem of determining the possible limit laws of additive arithmetical functions. If (29) is satisfied, the characteristic function of the limit law is

$$\varphi(t) = \prod_p \left(1 + \frac{e^{itf(p)} - 1}{p} \right)$$

what by lemma 2, is the characteristic function of a distribution function $F(x)$ continuous at every x or discrete (note that $d_k = 1 - 1/p_k$ if $f(p_k) \neq 0$ and 1 if $f(p_k) = 0$). It is continuous at every x if and only if

$$\sum_{f(p) \neq 0} \frac{1}{p} = +\infty.$$

Erdős (1939) showed, by giving examples, that it is possible that the limit be singular and also that it be entire, but so far no neccessary and sufficient condition is known to decide which is the case.

6. THE CENTRAL LIMIT PROBLEM

For an additive arithmetical function $f(n)$ we consider the normed function

$$f^*(n) = \frac{f(n) - A_N}{B_N} \tag{37}$$

where

$$A_N = \sum_{p \leq N} \frac{f(p)}{p}, \quad B_N^2 = \sum_{p \leq N} \frac{f^2(p)}{p} \tag{38}$$

In this section we follow the historical development of this subject.

The first result was obtained by Erdös and Kac (1940) who proved that for $f(n) = V(n)$, the number of distinct prime divisors of n,

$$\frac{V(n) - \log\log N}{\sqrt{\log\log N}} \tag{39}$$

is asymptotically normally distributed, more exactly, the number of integers $n \leq N$, for which the expression (39) is less than x is asymptotically N times the standard normal distribution function $\Phi(x)$. The result is again of probabilistic character but obtained by number theoretical tools. This theorem will be refered to as the Erdös-Kac theorem. Le Veque (1949) gave a new proof for the Erdös-Kac theorem by making use of the fact established by Landau that

$$P_N(V(n) = k) \sim \frac{N}{\log N} \frac{(\log\log N)^{k-1}}{(k-1)!}$$

i. e. that $V(n)$ is asymptotically a « Poisson variate » with parameter $\log\log N$. Erdös and Kac (1940) generalized the theorem to additive functions when $f(p) = 0(1)$, and finally Shapiro (1956) and Kubilius (1955) proved a theorem solving the problem in its most generality:

THEOREM 5 (KUBILIUS (1955), SHAPIRO (1956))

Suppose that for the strongly additive $f(n)$

$$\frac{1}{B_N^2} \sum_{\substack{p \leq N \\ |f(p)| > \varepsilon B_N}} \frac{f^2(p)}{p} \to 0, \quad B_N \to +\infty \tag{40}$$

then

$$\frac{1}{N} \sum_{\substack{n \leqslant N \\ f^*(n) < x}} 1 \to \Phi(x) \tag{41}$$

Later Kubilius (see his book, 1962) showed that for a wide class of additive functions the condition (40) is also necessary for the validity of (41). The condition (40) corresponds to the Lindeberg condition in the central limit problem of sums of independent random variables and therefore again it was felt necessary to find probabilistic approaches to theorem 5 or at least to the Erdös-Kac theorem. The first results of this kind were given by Delange (1953, 1956), Halberstam (1955) and Delange and Halberstam (1957) who proved the Erdös-Kac theorem by the method of moments (lemma 3); in evaluating the moments, however, still much reference had to be made to deep number theoretical tools. Theorem 5 was proved by the method of moments by Vilkas (1957). A wonderful probabilistic proof, by the method of moments again, for the Erdös-Kac theorem, was recently given by Billingsley (1969) who compared the moments of $f(n)$ with those of sums of independent random variables having the same distribution as ε_j of (10) and then applied the Ljapunov theorem and the method of moments (lemmas 3 and 4). The proof is very simple and short. It seems that Billingsley's method can be applied to sums of random variables having interdependence similar to that expressed by the concept of class K and then we would have the advantage that again the additive functions could be investigated on a wide class of sequences of integers by a single approach and at the same time this approach would be extremly simple. This seems to be worthy to work out in detail. For another approach see Novoselov (1964).

Questions extending the Erdös-Kac theorem in the direction that to give conditions when for an additive function $f(n)$, $f^*(n)$ is asymptotically normal if n runs over a given sequence of integers were widely investigated. Barban (1960), Uzdavinys (1959, 1962), Halberstam (1955) obtained results when the sequence in question is the set of values of a polynomial of integral coefficients. Tanaka (1955-1960) reproved Halberstam's theorem with the aim to give no reference to probability theory. For more general sequences the asymptotic normality of $f^*(n)$ was investigated by Rieger (1962, II), Barban (1966) and Elliott (1967-1968). The asymptotic normality of $f^*(n)$ when $f(n)$ is an additive function defined on algebraic number fields was discussed in de Kroon (1965), Grigelionis (1962), Juskis (1964), Rieger (1962, I).

In order to formulate the more general result, when the limit law of $f^*(n)$ is not necessarily normal we have to introduce the concept of class H of Kubilius.

DEFINITION 8

The strongly additive function $f(n)$ belongs to the class H of Kubilius, if $B_N \to +\infty$, and if there exists an increasing unbounded function $r(N)$ such that

$$\log r(N)/\log N \to 0, \quad B_{r(N)}/B_N \to 1.$$

For a class H, Kubilius (1956) obtained the following

THEOREM 6 (KUBILIUS (1956))

In order that for $f(n) \varepsilon$ class H, $f^*(n)$ have a limit law with variance 1, it is necessary and sufficient that there exist a distribution function $K(u)$ with variance unity and such that

$$\frac{1}{B_N^2} \sum_{\substack{p \leqslant N \\ f(p) < uB_N}} \frac{f^2(p)}{p} \to K(u) \qquad (N \to +\infty)$$

for all continuity points of $K(u)$. $K(u)$ uniquely determines the limit distribution.

This theorem implies theorem 5 and stronger in that sense also that the condition is also necessary. The proof makes use of sieve methods when he shows that the function truncated at $r(N)$ has the same distribution as $f(n)$ itself.

By the Kubik theorem (lemma 7), Kubilius (1962) found all the possible limit laws of $f^*(n)$ for $f(n)$ from a class H.

By the method of moments, Misjavicjus (1965) gave a new proof for theorem 6.

Theorem 6 was generalized for $f^*(n)$, n running over the sequence $\{ p_k - l \}$ with $l \neq 0$ fixed, by Barban (1961) and Barban, Vinogradov, Levin (1965). A similar question is investigated in Levin, Fainleib (1966). The distribution problem of $f(n) - C_N$, C_N any sequence of real numbers, was investigated by Kubilius (1965) and he obtained a theorem similar to that of Erdös and Wintner. For a class K it was generalized by Galambos *(to appear)*.

7. ASYMPTOTIC EXPANSIONS OF LIMIT LAWS

Rényi and Turàn (1958) worked out a method usual in analytical number theory by which they proved the Erdos-Kac theorem and which resulted in settling a conjecture of Leveque concerning the error term in this theorem. They obtained, that, putting $\alpha(N, x)$ for the number of $n \leqslant N$, for which $U^*(n) < x$,

$$\frac{\alpha(N, x)}{N} = \Phi(x) + 0\left(\frac{1}{\sqrt{\log \log N}}\right) \tag{42}$$

the error term being best possible. The method is applicable to more general situations and subsequent generalizations were obtained by the same method in Kátai and Gyapjas (1964), Rieger (1961) and Galambos (1968), extending (42) to a general class of functions.

(42) was generalized by Kubilius (1960) and Delange (1962) in the way that they gave a series expansion for $\alpha(N, x)/N$, being the same as the Cramer expansion for sums of independent random variables.

These proofs are number theoretical and there is very little hope to replace them by probabilistic ones.

In algebraic number theory the method was used by Grigelionis (1960) and Juskis (1965) and obtained a result similar to that of Kubilius and Delange.

The Rényi-Turan method by making use of Dirichlet series, gives an asymptotic formula for the characteristic function of $U^*(n)$ and then by the Esseen theorem (lemma 6) the error term can be dealt with.

Rényi (1963) applied this analytical method to re-prove the Erdös-Wintner theorem, here however he could not obtain a good error term. In the case of the Erdös-Wintner theorem much less is known concerning the speed of convergence. Extensively only $\Phi(n)$, the number of relative prime numbers not exceeding n, $\Delta(n) = V(n) - U(n)$, $V(n)$ being the number of all prime factors of n (and then generalized to arbitrary integer valued additive functions) were investigated.

$\Phi(n)$ is multiplicative, positive, and hence $\log \Phi(n)$ is additive. It was among the first results to show that $\Phi(n)/n$ has asymptotic distribution (Schoenberg, 1936)), let it be $v(X)$. Very recently Iljasov (1968) showed that for any F continuous on (0, 1),

$$\lim \frac{1}{N} \sum_{n=1}^{N} F(\Phi(n)/n) = A(F)$$

exists, and

$$\frac{1}{N}\sum_{n=1}^{N} F(\Phi(n)/n) - A(F) = 0((\log N)^{-2})$$

which again by the characteristic functions and Esseen's theorem gives a good estimate for the speed of convergence. Earlier Tjan (1966) and Fainleib (1967) obtained an error term, in this regard, of 0(1/log log N).

From Schoenberg (1936)'s theorem it follows that $\Delta(n)$ has asymptotic distribution. Rényi (1955) discovered a neat formula for the generating function of its asymptotic distribution, what Kubilius (1962) generalized to arbitrary additive arithmetical functions taking positive integers. From these generating functions the terms of the distribution in question can be evaluated and asymptotic formulas can be obtained. For the case of $\Delta(n)$, Robinson (1966) gave an exact expression for the density of $\{ \Delta(n) = r \}$. Let $\alpha(N, \Delta, k)$ denote the number $m \leqslant N$ for which $\Delta(n)=k$. Kubilius (1962) obtained that

$$\alpha(N, \Delta, k)/N = d_k + 0(\log \log N/\log N) \tag{43}$$

where d_k is the density of $\{ \Delta(n) = k \}$. Delange (1965, 1968) and Kátai (1966) improved the error term here to $0\ ((\log\log N)^{k-1}/(\log N)\sqrt{N})$. Delange (1968) generalizes (43) in another direction, too, namely he obtains good error terms when $\Delta(n)$ is considered on a general sequence of integers (instead of the successive ones). Kubilius (1962) generalized (43) to arbitrary integer valued additive functions having asymptotic distribution which was further improved by Kubilius (1968). See also Fainleib (1967), Sathe (1953-1954), Selberg (1959) and Rieger (1965).

In concluding I remark that several papers investigated the joint distribution of arithmetical functions. For results prior to the appearance of the monograph by Kubilius (1962), see the corresponding chapters of Kubilius (1962, 1964). For more recent results see Uzdavinys (1962), Levin and Fainleib (1967), Kátai (1969, II) and Galambos (1970).

REFERENCES

BARBAN, M. B.

[*1*] Arithmetical functions on « rare » sets (Russian), *Trudy Insta. Mat., Akad. Nauk Uzbek. SSR*, t. **22**, 1961, p. 1-20.

[*2*] The large sieve method and its application to number theory (Russian), *Uspehi Mat. Nauk*, t. **21**, 1966, n° 1 (127), p. 51-102.

Barban, M. B., Vinogradov, A. I. and Levin B. V.

[*1*] Limit laws for functions of the class H of Kubilius J. P. which are defined on a set of « shifted » primes (Russian). *Litovsk Mat. Sb.*, t. **5**, 1965, p. 5-8.

Behrend, F.

[*1*] Über numeri abundantes, I. II, *Sitzungsber. Preuss. Akad. Wiss. phys. math. Kl*, 1932, 322-328 ; 1933, 280-293.

Billingsley, P.

[*1*] An application of Prohorov's theorem to probabilistic number theory, *Illinois J. Math.*, t. **8**, 1964, p. 697-704.

[*2*] On the central limit theorem for the prime divisor function, *Amer. Math. Monthly*, t. **76**, 1969, p. 132-139.

Danilov, A. N.

[*1*] On the sequences of values of additive arithmetical functions defined on the set of ideals of a field K of degree *n* over the field of rational numbers (Russian), *Leningrad Gosudarst. Ped. Inst. A. I. Gercen, ucenye zapiski*, t. **274**, 1965, p. 273-304.

Davenport, H.

[*1*] Über numeri abundantes, *Sitzungsber. Preuss. Akad. Wiss, Phys-math., Kl.*, t. **27**, 1933, p. 830-837.

Delange, H.

[*1*] Sur le nombre des diviseurs premiers de *n*, *C. R. Acad. Sci.*, t. **237**, 1953, p. 542-544.

[*2*] Sur un théorème d'Erdôs et Kac, *Acad. roy. Belg. Bull., El. Sci.*, t. **42**, 1956, p. 130-144.

[*3*] Sur les fonctions arithmétiques fortement additives, *C. R. Acad. Sci.*, t. **244**, 1957, p. 1307-1309 ; p. 1604-1606 ; p. 2122-2124.

[*4*] Sur les fonctions arithmétiques multiplicatives, *Ann. Sci. École Normale Supérieure*, t. **78**, 1961, p. 273-304.

[*5*] *Application de la méthode du crible à l'étude des valeurs moyennes de certaines fonctions arithmétiques*, Séminaire Delange-Pisot (mimeographed lecture notes), Paris, 1962.

[*6*] *Acta Arith.*, t. **7**, 1962, p. 191-215.

[*7*] Sur un théorème de Rényi, *Acta Arith.*, t. **11**, 1965, p, 241-252.

[*8*] Sur un théorème de Rényi II, *Acta Arith.*, t. **13**, 1968, p. 339-362.

Delange, H. and Halberstam H.

[*1*] A note on additive functions, *Pacific J. Math.*, t. **7**, 1957, p. 1551-1556.

Elliott, P. D. T. A.

[*1*] On certain additive functions, *Acta Arith.*, t. **12**, 1967, p. 365-384.

[*2*] On certain additive functions II, *Acta Arith.*, t. **14**, 1968, p. 51-64.

Erdös, P.

[*1*] On the density of some sequences of numbers III, *J. London Math. Soc.*, t. **13**, 1938, p. 119-127.

[*2*] On the smoothness of the asymptotic distribution of additive arithmetical functions, *Amer. J. Math.*, t. **61**, 1939, p. 722-725.

Erdös, P. and Kac, M.

[*1*] The Gaussian law of errors in the theory of additive number theoretic functions, *Amer. J. Math.*, t. **62**, 1940, p. 738-742.

Erdös, P. and Wintner, A.

[*1*] Additive arithmetical functions and statistical independence, *Amer. J. Math.*, t. **61**, 1939, p. 713-721.

Fainleib, A. S.

[*1*] Distribution of values of Euler's function (Russian), *Mat. Zametki*, t. **1**, 1967, p. 645-652.

[*2*] On some asymptotic formulas for sums of multiplicative functions and applications (Russian), *Lietuvos matematikos rinkinys*, t. **7**, 1967, p. 535-546.

GALAMBOS, J.

[1] On the distribution of number theoretical functions, *Annales Univ. Sci. Budapest, Sectio Math.*, t. **7**, 1964, p. 25-31.

[2] On the distribution of prime-independent additive number theoretical functions, *Arch. Math.*, t. **19**, 1968, p. 296-299.

[3] Limit distribution of sums of (dependent) random variables with applications to arithmetical functions. To appear in *Z. Wahrscheinlichkeitstheorie verw. Geb.*

[4] A probabilistic approach to mean values of multiplication functions, *J. London Math. Soc.* (2), t. **2**, 1970, p. 405-419.

GRIGELIONIS, B.

[1] The distribution of values of multiplicative arithmetical functions (Lithuanian), *Vilniaus Valst-Univ. Mokslo Darbai Mat. Fiz.*, t. **9**, 1960, p. 71-73.

[2] On the distribution of values of multiplicative functions (Russian), *Litovsk Mat. Sb.*, t. **2**, 1962, p. 107-125.

GYAPJAS, F. and KÁTAI, I.

[1] Zur Statistik der Primfaktoren der natürlichen Zahlen, *Annales Univ. Sci. Budapest, Sectio Math.*, t. **7**, 1964, p. 59-66.

HALÁSZ, G.

[1] Über die Mittelwerte multiplikativer zahlentheoretischer Funktionen, *Acta Math. Acad. Sci. Hung.*, t. **19**, 1968, p. 365-402.

HALBERSTAM, H.

[1] On the distribution of additive number theoretic functions I, *J. London Math. Soc.*, t. **30**, 1955, p. 43-53.

HARDY, G. H. and RAMANUJAN, S.

[1] The normal number of prime factors of a number *n*, *Quart J. Pure and appl. Math.*, t. **48**, 1917, p. 76-92.

HARDY G. H. and WRIGHT E. M.

[1] *An introduction to the theory of numbers*, Oxford Univ. Press, 3rd ed., 1954.

ILJASOV, I.

[1] The summation of composite functions of Euler's function (Russian), *Dokl. Akad. Nauk SSSR*, t. **178**, 1968, p. 529-532.

JUSKIS, Z.

[1] Limit theorems for additive functions defined on ordered semigroups with a regular norm (Russian), *Litovsk Mat. Sb.*, t. **4**, 1964, p. 565-603.

[2] An asymptotic decomposition of distribution laws for certain functions defined on ordered semigroups with regular normalization (Russian), *Litovsk Mat. Sb.*, t. **5**, 1965, p. 167-183.

KAC, M.

[1] Statistical independence in probability, analysis and number theory, *Monographs of Amer. Math. Soc.*, 1959.

KÁTAI, I.

[1] Egy megjegyzés D. DELANGE : « Sur un théorème de Rényi » cimû dolgozatához (Hungarian), MTA III, *Oszt. Mat. Fiz.* , *Közl*, t. **16**, 1966, p. 269-273.

[2] On the distribution of arithmetical functions on the set of prime plus one, *Compositio Math.*, 1969, to appear.

[3] On the distribution of arithmetical functions, *Acta Math. Acad. Sci. Hung.*, t. **20**, 1969, p. 69-87.

DE KROON, J. P. M.

[1] The asymptotic behaviour of additive functions in algebraic number theory, *Compositio Math.*, t. **17**, 1965, p. 207-261.

KUBILIUS, J.

[1] On the distribution of values of additive arithmetical functions (Russian), *Dokl. Akad. Nauk SSSR*, t. **100**, 1955, p. 623-626.

[2] Probabilistic methods in number theory (Russian), *Uspehi Mat. Nauk*, t. **11**, 1956, n° 2, p. 31-66.

[3] *On some problems in probabilistic number theory* (Russian), Proc. Sixth All-Union Conf. Theory prob. and Math. Statist, Vilnius, 1960, p. 57-68.

[4] *Probabilistic methods in number theory* (Russian), and ed., Vilnius, 1962.

[5] Probabilistic methods in the theory of numbers, *Amer. Math. Soc.*, Translations of Math. monographs Vol. 11, 1964, p. 1-182.

[6] On asymptotic laws of distribution of additive arithmetical functions (Russian), *Litovsk Mat. Sb.*, t. **5**, 1965, p. 261-273.

[7] *On local theorems for additive number theoretic functions, Abhandlungen aus Zahlentheorie und analysis* (Zur Erinnerung an E. Landau), Berlin, 1968, p. 175-191.

LEVEQUE, W. J.

[1] On the size of certain number theoretic functions, *Trans. Amer. Math. Soc.*, t. **66**, 1949, p. 440-463.

LEVIN, B. V. and FAINLEIB, A. S.

[1] Distribution of values of additive arithmetical functions (Russian), *Dokl. Akad. Nauk SSSR*, t. **171**, 1966, p. 281-284.

[2] Application of some integral equations to questions of number theory (Russian), *Uspehi Math. Nauk SSSR*, t. **22** (135), 1967, p. 119-197.

MISJAVICJUS, G.

[1] The use of the method of moments in probabilistic number theory (Russian), *Litovsk Mat. Sb*, t. **5**, 1965, p. 275-289.

NOVOSELOV, E. V.

[1] Topological theory of the divisors of integers (Russian), *Uc. Zap. Elabuzsk. Pedag. Inst.*, t. **8**, 1960, p. 3-23.

[2] A new method in probabilistic number theory (Russian), *Izv. Akad. Nauk SSSR*, Ser. Mat., t. **28**, 1964, p. 307-364.

PAUL, E. M.

[1] Density in the light of probability theory I-III, *Sankhya*, Ser. A, t. **24**, 1962, p. 103-114, 209-212; t. **25**, 1963, p. 273-280.

RÉNYI, A.

[1] On the density of certain sequences of integers, *Publ. Inst. Math. Acad. Serbe Sci.*, t. **8**, 1955, p. 157-162.

[2] On the distribution of values of additive number theoretical functions, *Publ. Math., Debrecen*, t. **10**, 1963, p. 264-273.

[3] A new proof of a theorem of Delange, *Publ. Math., Debrecen*, t. **12**, 1965, p. 323-329.

[4] Additive and multiplicative number theoretic functions, Lecture Notes, University of Michigan (Ann. Arbor), 1965, p. 1-123.

RÉNYI, A. and TURAN, P.

[1] On a theorem of Erdös-Kac, *Acta Arith.*, t. **4**, 1957, p. 71-84.

RIEGER, G. J.

[1] Zur Statistik der Primfaktoren der natürlichen Zahlen in arithmetischen Progressionen, *J. reine u. angew Math.*, t. **206**, 1961, p. 26-30.

[2] Über die Anzahl der Primfaketoren algebraischer Zahlen und das Gaussche Fehlergesetz, *Math. Nachr.*, t. **24**, 1962, p. 77-89.

[3] Über die mittlere Anzahl der Primfaktoren von *p*-1, *J. reine u. angew. Math.*, t. **210**, 1962, p. 113-120.

[4] Zum Teiler problem von Atle Selberg, *Math. Nachr.*, t. **30**, 1965, p. 181-192.

REMOND, P

[1] Évaluations asymptotiques dans certains semi-groupes, *C. R. Acad. Sci.*, t. **260**, 1965, p. 6250-6251.

ROBINSON, R. L.
[*1*] An estimate for the enumerative functions of certain sets of integers, *Proc. Amer. Math. Soc.*, t. **17**, 1966, p. 232-237.
SATHE, L. G.
[*1*] On a problem of Hardy on the distribution of integers having a given number of prime factors I-IV, *J. Indian Math. Soc.*, t. **17**, 1953, p. 63-82, p. 83-141 ; **18**, 1954, p. 27-42, p. 43-81.
SCHÖNBERG, I. J.
[*1*] On asymptotic distribution of arithmetical functions, *Trans. Amer. Math. Soc.*, t. **39**, 1936, p. 315-330.
SELBERG, A.
[*1*] Note on a paper by L. G. Sathe, *J. Indian Math. Soc.*, t. **18**, 1954, p. 83-87.
SHAPIRO, H. N.
[*1*] Distribution functions of additive arithmetic functions, *Proc. Nat. Acad. Sci. U. S. A.*, t. **42**, 1956, p. 426-430.
TANAKA, M.
[*1*] On the number of prime factors of integers. I, IV, *Japan. J. Math.*, t. **25**, 1955, p. 1-20 ; *J. Math. Soc. Japan.*, t. **9**, 1957, p. 171-191 ; *Japan. J. Math.*, t. **27**, 1958, p. 103-127 ; t. **30**, 1960, p. 55-83.
TJAN, M. M.
[*1*] On the question of the distribution of values of the Euler function $\rho(n)$ (Russian), *Litvosk Mat. Sb.*, t. **6**, 1966, p. 105-119.
TURAN, P.
[*1*] On a theorem of Hardy and Ramanujan, *J. London Math. Soc.*, t. **9**, 1934, p. 274-276.
[*2*] Über einige Verallgemeinerungen eines Satzes von Hardy and Ramanujan, *J. London Math. Soc.*, t. **11**, 1936, p. 125-133.
UZDAVINYS, R.
[*1*] On the distribution of values of additive arithmetical functions of polynomials (Russian), *Trudy Akad. Nauk Litov. SSR*, Ser. B, t. **18**, 1959, p. 9-29.
[*2*] Arithmetical functions on the set of values of integer valued polynomials (Russian), *Litvosk Mat. Sb.*, t. **2**, 1962, p. 253-280.
[*3*] An analogue theorem to that of Erdös and Wintner on the sequences of the values of integer valued polynomials (Russian), *Litovsk Mat. Sb.*, t. **7**, 1967, p. 329-338.
VILKAS, E.
[*1*] On the normal distribution of additive arithmetical functions, *Uc. Trudy Viln. Gosud. Inst.*, t. **16**, 1957, p. 23-31.
WIRSING, E.
[*1*] Das asymptotische Verhalten von Summen über multiplikative Funktionen II, *Acta Math. Acad. Sci. Hung.*, t. **18**, 1967, p. 411-467.

References to section 3.

KUBIK, L.
[*1*] The limiting distributions of cumulative sums of independent two-valued random variables, *Studia Math.*, t. **18**, 1959, p. 295-309.
LÉVY, P.
[*1*] Sur les séries dont les termes sont des variables éventuelles indépendantes, *Studia Math.*, t. **3**, 1931, p. 119-155.
LOÉVE, M.
[*1*] *Probability theory*, 3rd ed., Van Nostrand, Princeton, N. J., 1963 (lemmas 1, 3-6).

(Manuscrit reçu le 10 avril 1970).