An alternative trigonometric integral representation of the Zeta function on the critical line

Dr. Klaus Braun
March 14, 2015
update of note 34: April 1, 2015

www.riemann-hypothesis.de

The Hilbert transform $f_H(x)$ of the Gauss-Weierstrass function $f(x):=e^{-\pi x^2}$ (with its relationships to the Gaussian, the Fresnel and the Dawson integral functions, [AbM], [LeN]) is given by ([Grl] 3.952, 9.212)

$$f_H(x)=2\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi=2xf(x)_1F_1(\frac{1}{2};\frac{3}{2},\pi x^2)=2x_1F_1(1;\frac{3}{2},-\pi x^2)$$

whereby

$$f_H\notin L_1(0,\infty)\,,\quad f_H(x)\in L_2(-\infty,\infty)\,.$$

The corresponding Mellin transform of $f_H(x)$ in the critical stripe is given by

$$M[f_H](s)=\frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}\,.$$

Due to corresponding properties of the Hilbert transform the Gaussian function and its Hilbert transform are identical in a weak $L_2(-\infty,\infty)-$ sense. It holds

$$\|f\|=\|f_H\|\quad,\quad f_H(0)=\hat{f}_H(0)=0\quad,\quad xf_H(x)\text{ is non-decreasing for } x\geq 0\,.$$

The Riemann duality equation is given by

$$\xi(s):=\zeta(s)(1-s)\cdot M[xf'(x)](s)=\xi(1-s)\quad.$$

The weak equivalence of $f(x), f_H(x)$ ensures equivalence of both corresponding Poisson summation formulas, i.e. replacing

$$f(x)\to f_H(x)\qquad\text{resp.}\qquad e^{-x}={}_1F_1(a;a,-x)\to\sqrt{x}\,{}_1F_1(1;\frac{3}{2},-x)\approx x^{-1/2}\ \text{ for } x\to\infty$$

provides an alternative Theta function relationship ([EdH] 1.7) in the form

$$1+2\psi(x^2):=1+2\sum_1^\infty f(nx)=\frac{1}{x}(1+2\psi(\frac{1}{x^2}))\to\psi_H(x^2):=\sum_1^\infty f_H(nx)=\frac{1}{x}\psi_H(\frac{1}{x^2})\,.$$

This enables an alternative representation $\Xi^*(t)$ of the Zeta function $\Xi(t)$ ([EdH] 1.8, [TiE] 2.16, 10), defined by

$$\Xi(t):=\xi(\frac{1}{2}+it)=8\int_1^\infty\sqrt{y}\cos(t\log y)d\left[y^3\psi'(y^2)\right]$$

in the form

$$\Xi^*(t):=\xi^*(\frac{1}{2}+it):=2\int_0^\infty x^s\psi_H(x^2)\frac{dx}{x}=4\int_1^\infty\sqrt{y}\cos(t\log y)\psi_H(y^2)\frac{dy}{y}=4\int_0^\infty e^{z/2}\psi_H(e^{2z})\cos(tz)dz\,.$$

Its related series representation is given by

$$\Xi^*(t)=\xi^*(s)=\sum_0^\infty\frac{a_{2n}}{(2n)!}(s-\frac{1}{2})^{2n}=\sum_0^\infty(-1)^n\frac{a_{2n}}{(2n)!}t^{2n}$$

with

$$a_{2n}:=4\int_1^\infty y^{1/2}\psi_H(y^2)(\log y)^{2n}\frac{dy}{y}\quad,$$

whereby the first term of the $\psi_H(y^2)-$series predominates for y large.

The Hilbert transform of the Theta function
and related Polya criteria to prove the RH

For the notation and the content of this section we refer to [EdH] and [PoG]. The Gauss-Weierstrass function

$$f(x):=e^{-\pi x^2}$$

is used to define the Theta function

$$\vartheta(x):=\sum_{-\infty}^{-\infty}e^{-n^2\pi x}=\frac{1}{\sqrt{x}}\sum_{-\infty}^{-\infty}e^{-\frac{n^2\pi}{x}} \quad .$$

With Gauss' definition of the Gamma function ([EdH] 1.3)

$$\Pi(\frac{s}{2}):=\Gamma(1+\frac{s}{2})=\int_0^\infty x^{s/2}e^{-x}dx \quad \text{for} \quad s>-2$$

the Mellin transformation in the form (see also the function space $G_1^2(0,\infty)$ in [NaC])

$$F(s):=\Pi(\frac{s}{2})\pi^{-s/2}=\int_0^\infty x^s[xf'(x)]\frac{dx}{x}=\int_0^\infty x^s df$$

gives the Riemann duality equation

$$\xi(s):=\zeta(s)(s-1)\int_0^\infty x^s df=\xi(1-s) \quad , \quad s\in C \quad ,$$

whereby

$$\zeta(s)\Gamma(\frac{s}{2})\pi^{-s/2}=\int_0^\infty \psi(x)x^{s/2}\frac{dx}{x}, \quad \mathrm{Re}(s)>1 \quad .$$

Putting

$$\psi(x):=\sum_1^\infty f(n\sqrt{x})=\sum_1^\infty e^{-n^2\pi x} \quad , \quad \omega(t):=\frac{1}{4}\sum_{-\infty}^{-\infty}e^{-n^2\pi e^{4t}}=\frac{1}{4}e^t(1+2\psi(e^{4t}))=\omega(-t)$$

$$\Phi_R(t):=\omega''(t)-\omega(t)$$

then the Riemann integral representation of the $\xi-$function is given by

$$(*) \qquad \xi(\frac{z}{2})=\xi_R(\frac{z}{2})=\frac{1}{2}-\int_0^\infty\left[(2\omega(t)-\frac{1}{2}e^t)*\left(1+z^2\right)\right]\cos(zt)dt=2\int_0^\infty\Phi_R(t)\cos(zt)dt \quad .$$

It holds ([PoG1], ([TiE] 10.1)

$$\Phi_R(t)=4\sum_1^\infty(2n^4\pi^2e^{9t}-3n^2\pi e^{5t})e^{-n^2\pi e^{4t}}$$

and

$$(**) \qquad \Phi_R(t)\approx 4\pi^2e^{\frac{9t}{2}-\pi e^{2t}}=:\Phi_?(t) \quad \text{for} \quad t\to\infty$$

and $\Phi_R(t)$ is even.

From this it follows

$$\Phi^*(t) \approx 4\pi^2 (e^{\frac{9t}{2}} + e^{-\frac{9t}{2}}) e^{-\pi(e^{2t}+e^{-2t})} =: \Phi_P(t) \quad \text{for } t \to \pm\infty \text{ .}$$

If one would replace the function $\Phi_R(t)$ by $\Phi_?(t)$ would then the entire function on the left hand side of (*) has only real zeros? Unfortunately the answer to this question is negative, as the defined function has an infinite number of imaginary zero ([GPo1], ([TiE] 10.1). Alternatively, G. Polya built the function $\Phi_P(t)$.

In [PoG] entire functions of higher genus than 1, which are functions of genus 0 or 1, polynomials and special functions of genus 2, are considered in the form

$$F(s) = 2\int_0^\infty \Psi(t)\cos(st)dt$$

For the function $\Psi(t)$ the following properties are valid/ required:

i) $\Psi(t)$ is defined, not identical zero and is real for $t \geq 0$

ii) $\Psi(t)$ is infinitely differentiable

iii) $\lim_{t\to\infty} \frac{1}{t}\log\left|\Psi^{(n)}(t)\right| = -\infty$ for $n = 0,1,2,3,...$ (and therefore $F(s)$ is an entire function!)

iv) $F(s)$ is of higher genus than 1 .

G. Polya ([PoG1]) derived his Zeta fake function building on the function $\Phi_P(t)$, which has all its zeros on the critical line, based on the

Theorem of J. Jensen: The entire function $F(s)$ has then and only then only real zeros if the polynomial

$$F(D)s^n = \int_0^\infty \Psi(t)\left[e^{itD} + e^{-itD}\right]s^n dt = \int_0^\infty \Psi(t)\left[(s+it)^n + (s-it)^n\right]dt$$

has only real zeros for $n = 1,2,3,...$.

From [PoG1] we recall:

Lemma 1: Let the function $F(u)$ be analytical and real valued for all non-negative, real values of u . Further let

$$\lim_{u\to\infty} u^2 F^{(n)}(u) = 0 \quad \text{for } n = 0,1,2,3,... .$$

If $F(u)$ is not an even function, then the function

$$G(x) = \int_0^\infty F(u)\cos(xu)du$$

is different from zero for sufficiently large values of x .

Lemma 2: Let a be a positive constant and let $G(z)$ be an entire function of genus 0 or 1 that for real z takes real values, has at least one real zero, and has only real zeros. Then the function $G(z+ia)+G(z-ia)$ has only real zeros.

From [EdH] 12.5 we recall ([PoG3])

Lemma 3: If Φ is a polynomial which has all its roots on the imaginary axis, or if Φ is an entire function which can be written in a suitable way as a limit of such polynomials, then if

$$\int_0^\infty x^{1-s}F(x)\frac{dx}{x}$$

has all its zeros on the critical line, so does

$$\int_0^\infty x^{1-s}F(x)\Phi(\log x)\frac{dx}{x}\cdot$$

and

Lemma 4: If a real function $F(x)$ satisfies, that

i) $xF(x)=F(\frac{1}{x})$

ii) $\sqrt{x}F(x)$ is non decreasing on the interval $[1,a]$,

then its Mellin transform has all its zeros on the critical line.

In [PoG3] the following general theorem about zeros of the Fourier transform of a real function is given:

Theorem 5: Let $a<b$ be finite real numbers and let $g(t)$ be a strictly positive continuous function on the open interval (a,b) and differentiable in (a,b) except possibly at finitely many points. Suppose that

$$\alpha\le-\frac{g'(t)}{g(t)}\le\beta$$

at every point of $a<b$ where $g(t)$ is differentiable. Then every zero of the integral

$$\hat{g}(z)=\int_a^b g(t)e^{zt}dt$$

lies in the open infinite strip

$$\alpha<\operatorname{Re}(z)<\beta$$

with the exception of the function

$$g(t)=e^{ct+d}$$

with c,d real constants.

After the change of variable $t \to \log x$ and approximating the integral over the half-line $(0,\infty)$ by integrals over finite intervals, one can obtain a theorem about zeros of Mellin transforms.

Theorem 6: Let $0 \le a < b \le \infty$ and let $g(x)$ be a strictly positive continuous function on (a,b) and differentiable there except possibly at the finitely many points. Suppose that

$$\alpha \le -x\frac{g'(x)}{g(x)} \le \beta$$

at every point of (a,b) where $g(x)$ is differentiable. Suppose further that the integral

$$F(s) = \int_a^b g(x)x^s \frac{dx}{x}$$

is convergent for $\alpha^* < \operatorname{Re}(s) < \beta^*$. Then all zeros ρ of $F(s)$ in this stripe satisfy

$$\alpha < \operatorname{Re}(\rho) < \beta .$$

The Polya criterion is for integration over a finite interval ([PoG3]). To extend it to an infinite interval it needs certain conditions [PoG3], [SeA1].

From [BeB1] we recall

Theorem 13.3.3: If $f(x)$ has a Fourier integral representation, $g(x)$ tends to zero as $x \to \infty$, and $xg'(x)$ belongs to $L^p(0,\infty)$, for some $p, 1 < p \le 2$, then

$$\sum_{n=1}^{\infty} g(n) - \int_0^{\infty} g(t)dt = \sum_{n=1}^{\infty} h(n) - \int_0^{\infty} h(t)dt$$

where

$$h(x) := 2\int_0^{\infty} g(t)\cos(2\pi xt)dt .$$

In the following we deal with the confluent hypergeometric (Kummer) function ([GrI] 9.2)

$$ {}_1F_1(a;c,z) := 1 + \sum_{k=1}^{\infty} \frac{a(a+1)(a+2)...(a+k-1)}{c(c+1)(c+2)...(c+k-1)} \frac{z^k}{k!} = \sum_{k\geq 0} \frac{(a)_k}{(c)_k} \frac{z^k}{k!} $$

with the Pochhammer symbol

$$ (b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)....(b+k-1) \; . $$

The Kummer function is related to the Whittaker functions $M_{\chi,\mu}(z)$ (which are regular near zero and valid for all finite values of z ([WhE] XVI)), defined by

$$ M_{\lambda,\mu}(x) := x^{\mu+\frac{1}{2}} e^{-\frac{x}{2}} {}_1F_1(\mu-\lambda+\frac{1}{2};2\mu+1;x) \; , \; M_{\lambda,-\mu}(x) := x^{-\mu+\frac{1}{2}} e^{-\frac{x}{2}} {}_1F_1(-\mu-\lambda+\frac{1}{2};-2\mu+1;x) . $$

Remark: From the general rule

$$ \frac{d}{dx}\left[x^{c-1} {}_1F_1(a;c,x)\right] = (c-1)x^{c-2} {}_1F_1(a;c-1,x) $$

it follows especially

$$ 2\frac{d}{dx}\left[x {}_1F_1(\frac{1}{2};\frac{3}{2},x^2)\right] = \frac{e^{x^2}}{x} \; , \; 2\frac{d}{dx}\left[x {}_1F_1(\frac{1}{2};\frac{3}{2},-x^2)\right] = -\frac{e^{-x^2}}{x} \; , $$

i.e.

$$ -\frac{e^{-x}}{x}dx = \frac{2}{\sqrt{x}} d\left[\sqrt{x} {}_1F_1(\frac{1}{2};\frac{3}{2},-x)\right] $$

Lemma: For the Gaussian, the Kummer and the Whittaker functions the following identities are valid:

i) $$ 2 {}_1F_1(\frac{1}{2};\frac{3}{2},\pm ix) = \int_0^1 \frac{e^{\pm ixt}}{\sqrt{t}} dt = \int_0^1 (\cos xt \pm i \sin(xt)) \frac{dt}{\sqrt{t}} $$ $x>0$, [GrI] 3.76

ii) $$ 2 {}_1F_1(\frac{1}{2};\frac{3}{2},x) = \sum_{n=0}^{\infty} \frac{2}{2n+1} \frac{x^n}{n!} = \int_0^1 (\cosh(xt) + \sinh(xt)) \frac{dt}{\sqrt{t}} $$ $x>0$, [GrI] 3.76

iii) $$ x {}_1F_1(\frac{1}{2};\frac{3}{2},-x^2) = \sqrt{\frac{\pi}{2}} - \frac{e^{-x^2/2}}{\sqrt{2x}} M_{-\frac{1}{4},\frac{1}{4}}(x^2) $$ [GrI] 9.236

iv) $$ x {}_1F_1(\frac{1}{2};\frac{3}{2},-x^2) = e^{-x^2} \sum_{n=0}^{\infty} \frac{2^n}{1\cdot 3 \cdots (2n+1)} x^{2n+1} $$ [AbM] 7.16

v) $$ M_{\frac{1}{4},\frac{1}{4}}(x^2) = x\sqrt{x} e^{-x^2/2} {}_1F_1(\frac{1}{2};\frac{3}{2},x^2) $$ [GrI] 9.22-9.23

vi) $$ \int_0^{\infty} e^{-\pi y^2} \sin(2\pi xy) dy = x e^{-\pi x^2} {}_1F_1(\frac{1}{2};\frac{3}{2},\pi x^2) $$ [GrI] 3.952.

For the Gauss error function and the Dawson integral

$$erf(z) = 2\int_0^z e^{-\pi t^2} dt \quad , \quad F(z) := e^{-z^2}\int_0^z e^{-u^2} du$$

we recall from [LeN]:

Lemma $F(z)$ is an entire function with the following properties

i) $F(z) = \sum_0^\infty (-1)^k \frac{2^k z^{2k+1}}{1\cdot 3 .. \cdot (2k+1)} = z \cdot {}_1F_1(1, \frac{3}{2}, -z^2) = \frac{1}{4\sqrt{\pi}} f_H(\frac{z}{\sqrt{\pi}})$

ii) $F(0) = 0$, $F'(x) + 2xF(x) = 1$, $\lim_{z\to\infty} 2xF(x) = 1$, i.e. $F(x) \approx \frac{1}{2x}$ for $z \to \infty$

iii) for the single maximum and inflection points it holds

$F(0.924...) = 0.541.. > 1/2$ and $F(1.50...) = 0.42...$ ([AbM] 7.1.17)

iv) the function $xF(x)$ is monoton increasing ([AbM] table 7.5)

v) $erf(x) = \frac{2x}{\sqrt{\pi}} {}_1F_1(\frac{1}{2}; \frac{3}{2}, -x^2) = \frac{2}{\sqrt{\pi}} \sum_0^\infty \frac{(-1)^k x^{2k+1}}{k!(2k+1)}$.

Remark: From [AbM] 13, ([GaW], [GrI] we recall the formulas

i) ${}_1F_1(\frac{1}{2}; \frac{3}{2}, -\pi x^2) = \frac{1}{x}\int_0^x e^{-\pi t^2} dt = \frac{1}{x}\int_0^x f(t) dt$

ii) ${}_1F_1(1; \frac{3}{2}, \pi x^2) = \frac{f(x)}{x}\int_0^x f(t) dt$

iii) $\sqrt{\frac{x}{\pi}} f_H(\sqrt{\frac{x}{\pi}}) = 4xe^{-x} {}_1F_1(\frac{1}{2}; \frac{3}{2}, x) = 4e^{\frac{x}{2}} M_{\frac{1}{4}, \frac{1}{4}}(x)$, $f_H(x)$ denotes the Hilbert transformation of $f(x)$

iv) $e^{-x^2} {}_1F_1(\frac{1}{2}; \frac{3}{2}, x^2) = \sum_{n=0}^\infty \frac{2^n}{1\cdot 3 \cdots (2n+1)} (ix)^{2n}$

v) $(xH - Hx)[f_H] = 0$

vi) $H\left[\frac{e^{-x}}{x}\right] = c\frac{F(\sqrt{x})}{\sqrt{x}}$ ([GaW] 7.1.17).

The Hilbert transformation $f_H(x)$ of $f(x)$ is given by ([AbM], [Grl] 3.952, 7.612, 9.212)

$$f_H(x) = 2\int_0^\infty e^{-\pi y^2}\sin(2\pi xy)dy = 2xe^{-\pi x^2}{}_1F_1(\frac{1}{2};\frac{3}{2},\pi x^2) = 2x{}_1F_1(1;\frac{3}{2},-\pi x^2)$$

.

Proposition: The replacement of the exponential function by the non-decreasing function $f_H(x)$

$$e^{-x} \to f_H(\sqrt{\frac{x}{\pi}})$$

defines a modified Gamma function

$$\Gamma^*(s) := \int_0^\infty x^s d\mu(x)$$

in the critical stripe with the measure

$$d\mu(x) := \sqrt{x}\sum_0^\infty \frac{(-1)^k 2^k}{(k+1/2)}\frac{x^k}{1\cdot 3..\cdot(2k+1)} =: d(Ei^*(-x)) \quad .$$

It enables the building of an alternative Theta function

$$\vartheta_H(x^2) := 2\sum_1^{-\infty} f_H(nx) = \frac{2}{x}\vartheta_H(\frac{1}{x^2}) \, ,$$

which supports the Polya criteria above. The corresponding trigonometric series representation of the Zeta function on the critical line is given by

$$\Xi^*(t) := \xi^*(\frac{1}{2}+it) := 2\int_0^\infty x^s \psi_H(x^2)\frac{dx}{x} = 4\int_1^\infty \sqrt{y}\cos(t\log y)\psi_H(y^2)\frac{dy}{y} = 4\int_0^\infty e^{z/2}\psi_H(e^{2z})\cos(tz)dz \quad .$$

with its related series representation in the form

$$\Xi^*(t) = \xi^*(s) = \sum_0^\infty \frac{a_{2n}}{(2n)!}(s-\frac{1}{2})^{2n} = \sum_0^\infty (-1)^n \frac{a_{2n}}{(2n)!}t^{2n}$$

with

$$a_{2n} := 4\int_1^\infty y^{1/2}\psi_H(y^2)(\log y)^{2n}\frac{dy}{y} \quad .$$

Remark: The first term of the $\psi_H(y^2)-$series predominates for y large.

Remark: We claim that the (polynomial) convergence behavior of

$$f_H(x) \approx \frac{1}{x} \quad \text{for } x \to \infty$$

also supports the RH criterion with respect to an appropriate Li function approximation property to the Riemann function $\pi(x)$, i.e.

The Riemann Hypothesis is true iff $\pi(x) = Li(x) + O(x^{1/2+\varepsilon})$.

Proof: Due to the corresponding property of the Hilbert transform the functions $f_H(x)$ and $f(x)$ are norm-equivalent with respect to the Hilbert space $L_2(-\infty,\infty)$ ([BrK]), i.e.

$$(f,\chi)_{L_2(-\infty,\infty)}=(f_H,\chi)_{L_2(-\infty,\infty)}\quad \forall\chi\in L_2(-\infty,\infty).$$

Therefore the Fourier series properties of the Gaussian functions (e.g. the Poisson summation formula) are also valid in a weak $L_2(-\infty,\infty)-$sense for its Hilbert transform. The analogue Poisson summation formula to

$$\vartheta(x^2):=\sum_{-\infty}^{-\infty}e^{-\pi n^2x^2}=\sum_{-\infty}^{-\infty}f(nx)=\frac{1}{x}\sum_{-\infty}^{-\infty}e^{-\frac{\pi n^2}{x^2}}$$

follows from the transformation rule

$$f_H(x)\leftarrow 4\pi\int_0^\infty f(y)\sin(2\pi xy)dy\,.$$

The constant Fourier term of f_H vanishes, as a consequence of the corresponding property of the Hilbert transform (PeB] examples 2.99, 2.9.11), i.e. it holds

$$f_H(0)=\hat{f}_H(0)=0$$

leading to

$$\vartheta_H(x^2):=\sum_{-\infty}^{-\infty}f_H(nx)=2\sum_{1}^{-\infty}f_H(nx)=\frac{1}{x}\sum_{-\infty}^{-\infty}f_H(\frac{n}{x})=\frac{2}{x}\sum_{1}^{-\infty}f_H(\frac{n}{x})\,.$$

This alternative Theta function ϑ_H enables an alternative Zeta function definition on the critical line analogue to [EdH] 1.8. The corresponding property of the Dawson integral function (see lemma ii) above) gives that $xf_H(x)$ is monoton increasing.

Therefore $f_H(x)$ fulfills the essential "Polya-function" properties ([EdH] 12.5 and the cited reference there). At the same time the invariant multiplicative measure ([EdH] 10.2, ([LeN])

$$d\sigma(x):=-\int_x^\infty e^{-t}\frac{dt}{t}=-2\int_{\sqrt{\frac{x}{\pi}}}^\infty f(t)\frac{dt}{t}=\frac{1}{2}Ei(-x)$$

on the group R^+ is replaced by the invariant (Kummer function weighted) measure (see also lemma vii) above)

$$d\mu(x):=\frac{1}{2}Ei^*(-x):=-2\int_{\sqrt{\frac{x}{\pi}}}^\infty f_H(t)\frac{dt}{t}=-8\pi\int_{\sqrt{\frac{x}{\pi}}}^\infty {}_1F_1(1;\frac{3}{2},-\pi t^2)dt$$

$$=-4\sqrt{\pi}\int_x^\infty {}_1F_1(1;\frac{3}{2},-s)\frac{ds}{\sqrt{s}}=-4\sqrt{\pi}\int_x^\infty F(\sqrt{s})\frac{ds}{s}=-4\sqrt{\pi}\int_x^\infty\sum_0^\infty(-1)^k\frac{(2s)^k}{1\cdot 3..\cdot(2k+1)}\frac{ds}{\sqrt{s}}\,.$$

The relationships to the Gamma function is given by the corresponding Mellin transforms (being valid in the critical stripe) of the Gaussian function, its related Kummer function $F(x)$ and the Whittaker function ([GrI], [AbM])

i) $$M\left[\sqrt{\frac{x}{\pi}}f(_H(\frac{x}{\sqrt{\pi}})\right](s-\frac{1}{2})=\pi^{\frac{s-1}{2}}M[f_H](s)=\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2})=\sin(\frac{\pi}{2}s)\frac{2^{1-s}}{\sqrt{\pi}}\Gamma(s)\Gamma(\frac{1-s}{2})=:\Gamma^*(\frac{s}{2})$$

ii) $$\frac{1}{2}\Gamma(\frac{s}{2})=\int_0^\infty x^se^{-x^2}\frac{dx}{x}=(1-s)\int_0^\infty x^s{}_1F_1(\frac{1}{2};\frac{3}{2},-x^2)\frac{dx}{x}=\int_0^\infty x^s\frac{d}{dx}\left[x_1F_1(\frac{1}{2};\frac{3}{2},-x^2)\right]\frac{dx}{x}$$

iii) $$\frac{\Gamma(\frac{s}{2})}{1-s}=\int_0^\infty x^{s/2}{}_1F_1(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x}$$ [GrI] 7.612.

Remark: In the neighborhood of $x=1$ it holds

$$\frac{\pi}{2}\tan(\frac{\pi}{2}x)=\frac{1}{1-x}+O(|1-x|)\ .$$

Remark: The function $\phi(x\sqrt{2\pi})$ with

$$\phi(x):=\frac{1}{e^x-1}-\frac{1}{x}$$

is self-reciprocal with respect to the sine transform. This property is used ([TiE]) to deduce the functional equation for the zeta function. For

$$\mathrm{H}_\nu(x):=x^{2\nu}e^{-\frac{x^2}{2}}{}_1F_1(\frac{1}{2}-\nu;1+\nu,\frac{x^2}{2})=x^{2\nu}{}_1F_1(2\nu+\frac{1}{2};\nu+1,-\frac{x^2}{2})\ ,\ \mathrm{Re}(\nu)>-\frac{1}{2}$$

an analogue self-reciprocal property is valid ([GrI] 7.643) in the form ($x>0$)

$$\int_0^\infty \mathrm{H}_{2\nu}(y)\sin(xy)dy=\sqrt{\frac{\pi}{2}}\mathrm{H}_{2\nu}(x)\ .$$

The special case $\nu=-1/4$ gives

$$\mathrm{H}_{-1/4}(x)=\frac{1}{\sqrt{x}}e^{-x^2/2}{}_1F_1(\frac{3}{4},\frac{3}{4},\frac{x^2}{2})=\frac{1}{\sqrt{x}}\ .$$

We note that in the critical stripe for ([GrI] 9.215)

$$\mathrm{H}_0(\sqrt{2}x)=e^{-x^2}{}_1F_1(\frac{1}{2};1,x^2)=e^{-x^2/2}J_0(\frac{x^2}{2}e^{\frac{\pi}{2}i})={}_1F_1(\frac{1}{2};1,-x^2)$$

it holds ([GrI] 7.612)

$$2\int_0^\infty x^s\mathrm{H}_0(\sqrt{2}x)\frac{dx}{x}=\frac{1}{\sqrt{2\pi}}\frac{\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(\frac{3}{4}-\frac{1}{2}(s-\frac{1}{2}))}\ .$$

Remark: The alternative measure $d\mu(x)$ to the exponential integral function resp. the corresponding alternative Gamma function definition provide opportunities to open issues from several mathematical areas (Math), but also from mathematical physics modelling (Physics) areas. We mention the following:

Math:

i) Transcendence of the Euler constant γ, E-function concept (Note 28)
ii) Goldbach conjectures
iii) Ramanujan's master theorem, divergent series (note 20, Note 33)
iv) Wavelet theory, Bessel spaces & Lommel polynomials, (Note 28

Physics:

i) Ground state zero model for harmonic quantum oscillator (Hermite resp. alternatively Lommel polynomials, wave package, infinite average energy, renormalization, bounded Hermitian operator, not compatible energy state of n photons with relativistic co-variant description of photons, infinite ground state energy without photons)
ii) Yukawa potential (leveraged on Plemelj-Stieltjes potential and Dawson integral function) to synchronize with Coulomb (electromagnetic) and Newton (gravitation) force, Note 34
iii) G. Veneziano string theory to describe interactions of elementary particles using Euler Beta function (elastic scattering process with 2 incoming spin-less particles of transverse momenta, s-channel, t-channels input diagrams A(s,t), Regge trajectories, resonance formula)
iv) Black body radiation, Helmholtz free energy (mass as manifestation of vacuum energy).

Further Analysis

From ([BrK], [GrI] 1.512, 7.621, 9.121, [WaG] 13-41, 13-72, note 18) we recall several Mellin transform equalities:

Lemma: It holds

i) $$M[f_H](s) == \frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}$$

ii) $$4(1-s)\int_0^\infty x^{s/2}{}_1F_1(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x} = 4\Gamma(\frac{s}{2}) = \Gamma^{-1}(\frac{s}{2})\int_0^\infty x^s K_0(2x)\frac{dx}{x}$$ for $0 < \mathrm{Re}(s) < 1$

iii) $$\int_0^\infty x^{s/2+1}{}_1F_1'(\frac{1}{2};\frac{3}{2},-x)\frac{dx}{x} = \frac{1}{s-1}\Gamma(1+\frac{s}{2})$$ for $0 < \mathrm{Re}(s) < 1$

iv) $$\int_0^\infty x^s\left[x^{-3/4}e^{-x/2}M_{\frac{1}{4},\frac{1}{4}}(x)\right]\frac{dx}{x} = \frac{\Gamma(s)\Gamma(1-s)}{(1-2s)\Gamma(\frac{1}{2}-s)}$$ for $0 < \mathrm{Re}(s) < 1$

v) $$\int_0^\infty x^{1-2s}J_{2\nu}^2(2x)\frac{dx}{x} = \frac{\Gamma(2s)\Gamma(\nu - s + \frac{1}{2})}{2\Gamma(s+\nu+\frac{1}{2})\Gamma^2(s+\frac{1}{2})}$$ for $0 < \mathrm{Re}(2s) < \mathrm{Re}(2\nu + 1)$.

Remark: Replacing

$$e^{-x} \to \mathrm{B}(x) := 2x{}_1F_1'(\frac{1}{2};\frac{3}{2},-x)$$

the corresponding Mellin transforms in the critical stripe moves

$$\frac{s}{2}\Gamma(\frac{s}{2}) \to \frac{s}{1-s}\Gamma(\frac{s}{2})$$

whereby it holds ([GrI] 8.321)

$$\Gamma(\frac{s}{2}) = \frac{2}{s} - \gamma + ... \quad .$$

We recall that

$$\zeta(s) = \frac{1}{1-s} + O(1) \quad , \quad -\zeta'(s)/\zeta(s) = \frac{1}{1-s} + O(1)$$ for $s \to 1^+$

and (in the critical stripe)

$$\zeta(s) = -s\int_0^\infty [x]x^{-s-1}dx = M\left[x\frac{d}{dx}[x]\right](s) \quad .$$

The corresponding effect for an alternative $Li-$function definition gives (lemma iv), [LeN])

$$Li(e^x) = Ei(x) = -\int_{-x}^\infty \frac{e^{-t}}{t}dt \to {}^*Li(e^x) := {}^*Ei(x) = -2\int_{-x}^\infty {}_1F_1'(\frac{1}{2};\frac{3}{2},-t)dt = \sum_1^\infty \frac{1}{n+1/2}\frac{x^n}{n!} \quad .$$

Remark: In the context of property lemma vii) we refer to [GaG] with its relationship to [PoG], using sums of squares to prove that certain entire functions have only real zeros. The paper gives a relationship to the ${}_1F_1(a;c,z)$ function and the Laguerre polynomials.

Remark: The function

$$\alpha(x) := 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x})$$

plays a key role in the Voronoi formula [NaC]. Because of

$$2\pi\int_0^\infty x^{s/2}Y_0(4\pi\sqrt{x})\frac{dx}{x} = \frac{2}{(2\pi)^s}\Gamma^2(\frac{s}{2})\cos(\frac{\pi}{2}s) = \cos(\frac{\pi}{2}s)\left[4\int_0^\infty x^{s/2}K_0(4\pi\sqrt{x})dx\right]$$

its Mellin transform is given by (see also notes 13, 20, 22, 26)

$$\int_0^\infty x^s\alpha(x)\frac{dx}{x} = \Gamma(s)\left[\frac{2(\cos(\pi s)-1)}{(2\pi)^{2s}}\Gamma(s)\right] =: \Gamma(s)\varphi(-s) \quad .$$

Remark: The split

$$x^{s/2} + x^{(1-s)/2} = x^{1/4}\left[x^{-1/4+s/2} + x^{1/4-s/2}\right] = 2x^{1/4}\cosh(\frac{it}{2}\log x) = 2x^{1/4}\cos(\frac{t}{2}\log x)$$

enables the integral representation on the critical line in the form

$$\Xi^*(t) := \xi^*(\frac{1}{2}+it) = 4\int_1^\infty y^{1/2}\psi_H(y^2)\cos(t\log y)\frac{dy}{y}$$

$$\Xi^*(t) := \xi^*(\frac{1}{2}+it) = 2\int_0^\infty 2e^{z/2}\psi_H(e^{2z})\cos(tz)dz \quad .$$

This leads to the representation ([TiE] 2.16, 10.2)

$$\Xi^*(t) = \xi^*(\frac{1}{2}+it) = 2\int_{-\infty}^\infty \Theta(y)\cos(ty)dy$$

resp.

$$\Theta(y) = \frac{1}{\pi}\int_{-\infty}^\infty \Xi^*(t)\cos(yt)dt$$

with

$$\Theta(y) := 2e^{y/2}\psi_H(e^{2y}) = 2e^{y/2}\sum_1^\infty f_H(ne^y) = 8\pi e^{y/2}\sum_1^\infty ne^y f(ne^y)\,{}_1F_1(\frac{1}{2};\frac{3}{2},\pi n^2e^{2y}) \, .$$

The power series ([EdH] 1.8)

$$\cosh y = \sum_1^\infty \frac{y^{2n}}{(2n)!}$$

enables the series representation of $\xi^*(s)$ as an even function of $(s-1/2)$ in the form

$$\Xi^*(t) := \xi^*(\frac{1}{2}+it) = 4\int_1^\infty y^{1/2}\psi_H(y^2)\sum\frac{t^{2n}(\log y)^{2n}}{(2n)!}\frac{dy}{y} \quad .$$

Applying the series representation

$$\cos x = \sum_0^\infty (-1)^n\frac{x^{2n}}{(2n)!}$$

gives the appreciated integral form representation ([EdH] 12.5, [PoG2]):

Theorem: The series representation

$$\Xi^*(t)=\xi^*(s)=\sum_0^\infty \frac{a_{2n}}{(2n)!}(s-\frac{1}{2})^{2n}=\sum_0^\infty (-1)^n \frac{a_{2n}}{(2n)!}t^{2n}$$

as an even function of $s-1/2$ has all its zeros on the critical line, whereby it holds

$$a_{2n}:=4\int_1^\infty y^{1/2}\psi_H(y^2)(\log y)^{2n}\frac{dy}{y} .$$

Remark: The Hilbert-Polya conjecture states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self adjoint operator. A proof of this conjecture is basically about finding the right function space and operator, enabling appropriate symmetry properties. The Hilbert transform is a one-dimensional (convolution) integral transformation, which is an isomorphism on the Hilbert space $L_2^\#(0,1)$ and which transfers the constant Fourier term of a Fourier series to zero.

The “trick” to build a trigonometric integral representation of the Zeta function on the critical line is the split

$$2x^{s/2}\frac{dx}{x}=\left[x^{s/2}+x^{(1-s)/2}\right]\frac{dx}{x}=2x^{1/4}\cos(\frac{t}{2}\log x)\frac{dx}{x} .$$

This is enabled by the Theta function property. Unfortunately, the corresponding Mellin transform of the Theta function $\vartheta(x)$ is not defined, due to not convergent integrals for any $s\in C$, [EdH] 10.3. The root cause of this is the not vanishing constant Fourier term of both, the Gauss-Weierstrass based and the fractional part function based Theta function. In [BrK] two modified Theta functions are defined, which have vanishing constant Fourier terms. This is achieved by applying the Hilbert transformation to both Theta functions.

Remark: With respect to the proposed trigonometric integral and series representation (which can be interpreted also as a norm in an appropriately defined Hilbert space) we recall with the notations from [BrK] the following properties:

It holds

$$f_H(x) := 2\int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi \in L_2^\#(0,1) \quad \text{and} \quad f_H(0) = \hat{f}_H(0) = 0$$

$$\rho_H(x) := \log(2\sin\frac{\pi x}{2}) \in L_2^\#(0,1),\ f_H'(x), \rho_H' \in H_{-1}^\#(0,1) \quad \text{and} \quad \hat{\rho}_H(0) = 0 .$$

For the corresponding Hilbert space framework we note, that

i) if $\{\psi_n\}$ is an orthogonal system of $L_2^\#(0,1)$, then $\{\overline{\psi}_n\} := \{H\psi_n\}$ is also an orthogonal system of $L_2^\#(0,1)$

ii) $(\overline{\psi}_n, \psi_n)_0 = 0$

iii) $\overline{\psi}_n' \approx n\overline{\psi}_n$ is an orthogonal system of $H_{-1}^\#(0,1)$

iv) for $g \in H_0^\#$ it holds

$$\|g'\|_{-1} = \|Ag'\|_0 = \|Hg\|_0 = \|g\|_0 \ ,$$

which provides the appreciated isomorph mappings between the concerned Hilbert spaces. Putting

$$^x g(x) := x * g(x) \quad , \quad \varsigma(x) := x\rho_H'(x) = \frac{\pi x}{2}\cot(\frac{\pi x}{2})$$

leads to ([BrK]):

Corollary: It holds

i) $\|f\| = \|f_H\| \cong \|f_H'\|_{-1} \quad , \quad \|\overline{\vartheta}_H\| = \|\vartheta_H\| \quad , \|\rho\| = \|\rho_H\| \cong \|\rho_H'\|_{-1}$

ii) $\|\varsigma - \rho_H\|_0 \cong \|^x\rho_H'\|_0 \cong \|^x\rho'\|_0 \cong \|^x\rho_H\|_{-1} \cong \|^x\rho\|_{-1}$.

The Hilbert-Polya conjecture has been solved in [BrK] for both entire Zeta functions, ξ and ξ^*, based on the invariant (primes density related) measure

$$d\mu(x) = d\log x = \frac{dx}{x}$$

on the multiplicative group R^+ ([EdH] 10.2). The self-adjoint operators with transform ξ and ξ^* (in the sense of [EdH] chapter 10) are built on the Hilbert transforms of the Gauss-Weierstrass function $f(y)$ and the fractional part function ([TiE] 2.1). The “appropriate” function spaces are the (distributional) Hilbert spaces $L_2^\#(0,1)$, $H_{-1}^\#(0,1)$.

From [WaG] 13-41, 13-42 we recall

$$\int_0^\infty J_{2(n+1)}(2\sqrt{x})J_{2(m+1)}(2\sqrt{x})\frac{dx}{x} = \frac{\delta_{n,m}}{2(n+1)} \ .$$

Remark: From [BeB1] 9.6 we recall

$$\sum_{1}^{\infty}\frac{\log n}{n^s}=\frac{1}{(s-1)^2}+K+o(1) \quad \text{for} \quad s\to 1$$

Remark: It holds in a distributional sense (notes 23 below)

$$\xi^*(s):=2\int_0^{\infty}x^{s/2}\psi_H(x)\frac{dx}{x} \quad \text{for} \quad 0<\operatorname{Re}(s)<1 \, .$$

and therefore

$$\xi^*(s)=\int_0^{\infty}\left[x^{s/2}+x^{(1-s)/2}\right]\psi_H(x)\frac{dx}{x}=\tilde{F}_H(s)\zeta(s)+\tilde{F}_H(1-s)\zeta(1-s)=\xi^*(1-s) \, .$$

The Mellin transform of the Hilbert transform of the Gauss-Weierstrass function indicates an alternative $\tilde{L}i-$function in the form

$$2Li(x)\to\tilde{L}i(x)=\frac{\pi}{2}\int_2^x(1-t)\tan(\frac{\pi}{2}t)+t\cot(\frac{\pi}{2}t)\frac{dt}{\log t} \, ,$$

which also fulfills the Chebyshey convergence criterion ([EdH] 1.1). From [GrI] 1.421, we recall

$$\frac{\pi}{2}x\tan(\frac{\pi}{2}x)=\sum_{1}^{\infty}\frac{2x^2}{(2k-1)^2-x^2} \, , \quad \frac{\pi}{2}x\cot(\frac{\pi}{2}x)=1-\sum_{1}^{\infty}\frac{2x^2}{k^2-x^2} \, .$$

In the neighborhood of $x=1$ it holds

$$\frac{\pi}{2}\tan(\frac{\pi}{2}x)=\frac{1}{1-x}+O(|1-x|) \, .$$

Remark: As ξ^* is analytical and real for real $s\in R$ with $\xi^*(s)=\xi^*(1-s)$ the Schwartz reflection formula gives

$$\bar{\xi}^*(\frac{1}{2}+it)=\xi^*(\frac{1}{2}+it)$$

i.e. the ξ^*-function is real on the critical line (same argument as for the entire $\xi-$function). Both functions $\xi(s)$ and $\xi^*(s)$ have the same set of zeros.

Remark: The Riemann duality equation representation in the form

$$(*) \qquad \xi(s):=\zeta(s)(s-1)\left[\int_0^{\infty}(s-1)x^sB'(\pi x^2)dx\right]=\xi(1-s) \quad \text{for} \quad 0<\operatorname{Re}(s)<1 \, .$$

enables an alternative analysis of the following RH criterion:

The Riemann Hypothesis is true iff $\pi(x)=Li(x)+O(x^{1/2+\varepsilon})$.

The “primes number density” dJ is defined by ([EdH] 1.13 ff.)

$$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^{s}\frac{ds}{s}\quad .$$

The representation (*) above leads to

$$\log\zeta(s)=-2\log(s-1)-\log 2+\sum_{\rho}\log(1-\frac{s}{\rho})-\log\left[\frac{1}{2}\pi^{-(1+s)/2}\int_{0}^{\infty}y^{(s-1)/2}B'(y)dy\right]$$

with

$$\int_{0}^{\infty}y^{(s+1)/2}{}_1F_1'(\frac{1}{2};\frac{3}{2},-y)\frac{dy}{y}=\int_{0}^{\infty}y^{(s+1)/2}\left[\sum_{1}^{\infty}\frac{2n}{2n+1}\frac{(-y)^{n}}{n!}\right]\frac{dy}{y}=\frac{s+1}{2s}\Gamma(\frac{s+1}{2})\,.$$

An alternative approximation error term for the Riemann primes number formula is then given by

$$J(x)=2Li(x)-\sum_{\mathrm{Im}(\rho)>0}Li(x^{\rho})+Li(x^{1-\rho})+R(x)$$

with ([EdH] 1.16)

$$R(x):=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\log\pi^{-(1+s)/2}+\log\frac{1}{s}\Gamma(1+\frac{s+1}{2})\right]x^{s}\frac{ds}{s}$$

$$R(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left[\log\pi^{-(1+s)/2}+\log\int_{0}^{\infty}y^{(s-1)/2}d(\sum_{1}^{\infty}\frac{2n}{2n+1}\frac{(-y)^{n}}{n!})\right]x^{s}\frac{ds}{s}\quad .$$

Some notes

Note 1: Concerning invariant operators, adjoints and their transforms we quote from [EdH], chapter 10):

Let V *be the vector space of all complex-valued functions on* R^+ *with the inner product*

$$(u,v):=\int_0^\infty u(x)v(x)dx\,.$$

By

$$I:v(x)\mapsto\int_0^\infty v(ux)F(u)du$$

an integral operator $I:V\to V$ *is defined. An operator is said to be invariant if it commutes with all translation operators* $T_u:v(x)\mapsto v(ux)$*. The transform of an invariant operator is the function whose domain is the set of complex numbers* s *such that the function* $v(x):=x^{-s}$ *lies in the domain of the operator and whose value for such an s is the factor by which the operator multiplies* $v(x):=x^{-s}$ *. Thus e.g. the Zeta function* $\zeta(s)$ *for* $\mathrm{Re}(s)>1$ *is the transform of the summation operator*

$$v(x)\mapsto\sum_1^\infty v(nx)\,.$$

When defining the adjoint of an invariant operator on V the inner product is defined on a rather small subset of V, whenever both side of $(Lu,v)=(u,L^*v)$ *are defined. There is an only formally valid representation of Riemann's duality equation as transform of an integral operator*

$$I:v(x)\mapsto\int_0^\infty v(ux)G(u)du$$

in the form

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\frac{2\xi(s)}{s(s-1)}\,.$$

But the operator I *has no transform at all, as the integral does not converge for any s. The integral would converge at* ∞ *if the constant term* $f(0)=\hat{f}(0)=1$ *is absent. If one would find an integral operator in the form* I *satisfying the same functional equation than G does and if*

$$\int_0^\infty x^{1-s}G(x)\frac{dx}{x}\quad \text{converges} \quad \text{and} \quad \int_0^\infty x^{1-s}G(x)\frac{dx}{x}=\int_0^\infty x^{s}G(x)\frac{dx}{x}$$

then this operator would be self-adjoint in the sense of above.

Note 2 ([CaD]): If the function

$$\Xi(t):=\xi(\frac{1}{2}+it)$$

can be realized as a convolution $\Xi(t)=(K*dF)(t)$ where $K(t)\in LP^*$, i.e. is an entire function from the Laguerre-Polya class of order <2, i.e.

$$\Phi(z)=c^z me^{\alpha z-\beta z^2}\prod_k(1-z/\alpha_k)e^{z/\alpha_k}\ ,$$

where c,α,α_k are real, $\beta\geq 0$ and m is a nonnegative integer, this would prove the RH.

Note 3: Ramanujan's Theory of Divergent Series and the Hilbert space $H_{-1}^{\#}(0,1)$ **, (**[BeB], [BrK]):

Ramanujan's theory focuses upon the "constant" c of a series

$$\sum_{k=1}^{\infty} f(k)$$

which may converge or diverge. It is emanated from the Euler-Maclaurin summation formula. Ramanujan claims that the constant of a series "is like the centre of gravity of a body". From ([BeB] 8, Entry 17 (iv)) we quote:

"*Ramanujan informs us to note that*

$$\sum_{1}^{\infty} \sin(2\pi\nu x) = \frac{1}{2}\cot(\pi x) ,$$

which also is devoid of meaning, ... may be formally established by differentiating the equality

$$\sum_{1}^{\infty} \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x)$$ ".

The equivalence of a cot(x) series representation with the standard Theta function and the related Riemann duality equation is given in [HaH]. In the context of elliptical theta functions in [DoG] corresponding properties for the Bessel functions are provided.
Putting

$$(A\sigma)(x) := \sum_{1}^{\infty} \frac{\sin(2\pi\nu x)}{\nu} \in H_0^{\#}$$

it holds [BrK]):

$$\sigma = \sum_{1}^{\infty} \sin(2\pi\nu\circ) \in H_{-1}^{\#} .$$

Therefore, "differentiating" is defined in a weak sense with respect to the inner product of the Hilbert space $H_{-1}^{\#}$ i.e.

$$(u,v)_{-1} = (Au,Av)_0 \;,\quad (u,v)_{-1/2} = (Au,v)_0 \;,\quad (u',v')_{-1} = (Au',Av')_0 = (Hu,Hv)_0 = (u,v)_0 .$$

From ([GrI] 1.441 we recall the formula

$$\sum_{1}^{\infty} (-1)^{\nu-1} \frac{\cos 2\pi\nu x}{\nu} = \log 2\cos(\pi x) \;,\quad -\frac{1}{2} < x < \frac{1}{2}$$

Let $\zeta(s,a)$ be the Hurwitz Zeta function (see also note 10 below) and

$$\psi(x) := \sum_{k=o}^{\infty} \binom{2k}{k} \frac{\sin^{2k+1}(\pi x)}{2^{2k}(2k+1)^2} .$$

Then it holds ([BeB] 9. Entry 31(a) (b))

$$\frac{1}{\pi}\psi(x) = \zeta'(-1,x) - \zeta'(-1,1-x) + x\log(2\sin(\pi x)) .$$

Ramanujan asserts (which is also stated as wrong ([BeB] 9. Entry 31 (b)) that

$$\psi(\frac{1}{2}-x) + \psi(\frac{1}{2}+x) = \pi\log(2\sin(\pi x))$$

applying the following formula twice with x replaced by ([BeB] 9. Entry 16)

$$\psi(x) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{\sin(2kx)}{k^2} + x\log|2\sin x| .$$

Note 4: The Riemann formula for $J(x)$ ([EdH] 1.13-1.16) is built on the representation

$$J(x)=\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\log\zeta(s)x^{s}\frac{ds}{s} \quad , \quad a>0 .$$

It is evaluated based on the entire Zeta function by the following formulas for $\xi(s)$:

$$\xi(s)=\Gamma(1+\frac{s}{2})\pi^{-s/2}(s-1)\zeta(s) \quad ,$$

$$\xi(0)=\frac{1}{2}=-\zeta(0) \quad , \ \log\xi(0)=-\log 2$$

$$\xi(s)=\xi(0)\prod(1-\frac{s}{\rho})$$

i.e.

$$\log\zeta(s)=\log\xi(0)+\sum(1-\frac{s}{\rho})-\log\left[\Gamma(1+\frac{s}{2})\pi^{-s/2}(s-1)\right].$$

The Gamma function

$$\Gamma(s):=\int_{0}^{\infty}x^{s}e^{-x}\frac{dx}{x}=\int_{0}^{\infty}x^{s}\int_{x}^{\infty}e^{-t}\frac{dt}{t}dx$$

plays a key role or the definition of the Zeta function, for Riemann's main formula and the Riemann Hypothesis analysis ([EdH] 1.4, 1.17). The famous Riemann "error" integral for his "primes" formula ([EdH] 1.16) is given by

$$\frac{1}{2\pi i}\frac{1}{\log x}\int_{a-i\infty}^{a+i\infty}\frac{d}{ds}\left[\frac{\log\Gamma(1+\frac{s}{2})}{s}\right]x^{s}ds=\sum_{1}^{\infty}\int_{x}^{\infty}\frac{t^{-2n}}{\log t}\frac{dt}{t}=\int_{x}^{\infty}\frac{dt}{t(t^{2}-1)\log t} \quad .$$

It holds ([EdH] 3.2)

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}y^{s}\frac{ds}{s}=\begin{cases}0 \ if \ 0<y<1\\ \frac{1}{2} if \quad y=1\\ 1 \ if \quad y>1\end{cases}$$

Our alternative Zeta function and related alternative Gamma function is given by

$$\xi^{*}(s)=\frac{1}{2}\pi^{-s/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-s)}\zeta(s) \quad .$$

Note 5: For real $t\in R$ it holds

$$\zeta(1+it)\neq 0 \quad .$$

Note 6: The n^th Hermite polynomial is given by ([SzG])

$$H_n(z)=\frac{2^n}{\sqrt{\pi}}\int_{-\infty}^{\infty}(z-it)^n e^{-t^2}dt$$

and Hermite polynomials have only real zeros all of which are simple. The weighted Hermite polynomials

$$\varphi_n(x):=\frac{e^{-\frac{x^2}{2}}H_n(x)}{\sqrt{2^n n!\sqrt{\pi}}} \quad \text{with} \quad H_n(x):=(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}\ ,\quad H_0(x)=1\ ,\quad H_1(x)=x$$

form a set of orthonormal functions in $L_2(-\infty,\infty)$, i.e. the Hermite polynomials have only real zeros. The relation to the Gauss-Weierstrass function is given by

$$f(x)=\pi^{1/4}\varphi_0(\sqrt{2\pi}x)\ .$$

It holds $\varphi_n\in L_2$, $H\varphi_n\in L_2$, $(\varphi_n,H\varphi_n)=0$ and

$$L_2:=H:=span[\varphi_n(x)]=span[H(\varphi_n(x))]\ .$$

All zeros of the Mellin transforms of the weighted Hermite polynomials lie on the critical line [BuD]. The proof is based on the recursion formula of the Hermite polynomials in combination with an argument from Polya ([PoG2]). Integral representations and examples of expansions in series of the Hermite polynomials like

$$e^{ax}=e^{a^2/4}\sum_0^\infty\frac{a^n}{2^n n!}H_n(x) \quad \text{for any real or complex } a$$

are given in ([LeN] 4.11, 4.16).

Note 7: For convergence analysis of the integral

$$2\int_0^\infty y^{1/2}f_H(ky)(\log y)^{2n}\frac{dy}{y}$$

we note the relationship to the confluent hypergeometric function ${}_1F_1(a;c,z)$ ([GrI] 3.952)

$$f_H(y):=[H(f)](y)=4\pi\int_0^\infty f(\xi)\sin(2\pi\xi y)d\xi=2yf(y){}_1F_1(\frac{1}{2};\frac{3}{2},\pi y^2)\ .$$

It holds ([GrI] 7.612)

$$\int_0^\infty t^{s-1}{}_1F_1(\frac{1}{2};\frac{3}{2},-t)dt=\frac{1}{2}\frac{\Gamma(s)}{(\frac{1}{2}-s)} \quad \text{for} \quad 0<\mathrm{Re}(s)<\frac{1}{2}\ .$$

As $e^{-x}\approx 2xB'(x)$ the corresponding Mellin transformations give

$$\frac{s}{2}\Gamma(\frac{s}{2})\approx\frac{s}{1-s}\Gamma(\frac{s}{2}) \quad \text{for} \quad 0<\mathrm{Re}(s)<1\ .$$

Note 8: The following series converge

$$\sum_p\frac{\sin\frac{\pi}{2}p}{p}$$

Note 9: The integral

$$\int_{-\infty}^{\infty} e^{-t^2} \cos(zt)dt = \frac{\sqrt{\pi}}{2} e^{\frac{z^2}{4}}$$

does not have any zeros at all.

Note 10: With respect to the density of the primes we recall (e.g. [EdH] 1.1, 1.13):

$$\int_2^{\infty} \frac{t}{\log t} \frac{dt}{t} < \pi(x) < 1.11 \int_2^{\infty} \frac{t}{\log t} \frac{dt}{t} \quad , \quad \frac{1}{2} \log 2 \frac{x}{\log x} < \pi(x) < 8 \log 2 \frac{x}{\log}$$

and

$$Li(x) = \frac{1}{2\pi i} \frac{1}{\log x} \int_{a-i\infty}^{a+i\infty} \frac{d}{ds} \left[\frac{\log(s-1)}{s} \right] x^s ds = \lim_{\varepsilon \to 0} \left[\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^{x} \frac{dt}{\log t} \right] .$$

An analogue density to $1/\log x$ (with analogue $Li^*(x)$ definition) with same convergence properties could be

$$\frac{dt}{\log t} \approx \frac{\pi}{2} \left[(1-t)\tan(\frac{\pi}{2}t) + t\tan(\frac{\pi}{2}(1-t)) \right] \frac{dt}{\log t} .$$

whereby it holds ([TiE] II, 2.4) in the neighborhood of $s = 1$

$$\frac{\pi}{2} \tan(\frac{\pi}{2}s) = \frac{1}{1-s} + O(|1-s|) .$$

Note 11: [SzG], 6.4, "*Theorem of Polya-Szegö on trigonometric polynomials with monotonic coefficients*":
Let $a_0 > a_1 > ... > a_m > 0$, *then the functions*

$$f(t) = a_0 \cos(mt) + a_1 \cos((m-1)t) + ... + a_{m-1} \cos t + a_m$$

$$g(t) = a_0 \cos((m+\frac{1}{2})t) + a_1 \cos((m-\frac{1}{2})t) + ... + a_{m-1} \cos((\frac{3}{2})t) + a_m \cos((\frac{1}{2})t)$$

have only real and simple zeros; there is, respectively, exactly one zero in each of the intervals

$$\frac{(\mu-\frac{1}{2})}{(m+\frac{1}{2})}\pi < t < \frac{(\mu+\frac{1}{2})}{(m+\frac{1}{2})}\pi \quad \textit{and} \quad \frac{(\mu-\frac{1}{2})}{(m+1)}\pi < t < \frac{(\mu+\frac{1}{2})}{(m+1)}\pi$$

where $\mu = 1,2,....,2m$, *and* $\mu = 1,2,....,2m+1$, *respectively.*

Note 12 (Goldbach conjecture): A function, which is in a sense a generalization of ζ is the Hurwitz Zeta function, defined by ([TiE] 2.17:

$$\zeta(s,a) := \sum_{n=1}^{\infty} \frac{1}{(n+a)^s} \quad \text{for} \quad \operatorname{Re}(s) > 1 \;,\; 0 < a \le 1 \,.$$

Related to the Hurwitz Zeta function is the Dirichlet series ([IvA] 1.8), defined by

$$L(s,\chi) := \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1-\chi(n)p^{-s})^{-1} \quad \text{for} \quad \operatorname{Re}(s) > 1 \quad .$$

Assuming that the zeros of all Dirichlet L-functions have real part less than ¾ Hardy and Littlewood applied the circle method to prove that every sufficiently large odd number can be written as the sum of three primes ([HaG]). The Goldbach conjecture is that every even number greater than 2 can be expressed as the sum of two primes. The analysis from Hardy and Littlewood gives heuristically an asymptotic formula for the number of representations for sufficiently large even numbers.

Note 13 ([IvA] 3.1, 3.4, 13.1, 13.8, [BeB1] 2, 9.7) The Voronoi summation formula concerning the error in the divisor problem and the circle problem**:**

Let $r(n)$ be the number of ways n may be written as a sum of two integer squares and let

$$\sum_{n\le x}{}^{*} r(n) := \pi x - 1 + \sqrt{x}\sum_{1}^{\infty} r(n) n^{-1/2} J_1(2\pi\sqrt{nx})$$

and

$$P(x) := \sum_{n\le x}{}^{*} r(n) - \pi x = -1 + x\sum_{1}^{\infty} r(n)\frac{J_1(2\pi\sqrt{nx})}{\sqrt{nx}} = .-1 + \frac{1}{\pi}\sum_{1}^{\infty} r(n) J_o'(2\pi\sqrt{nx}) \,.$$

In this context we refer to Weber's first exponential integral (see above)

$$\mathrm{B}(-z) := {}_1F_1(\frac{1}{2};\frac{3}{2},-z) = \int_0^{\infty} \frac{J_{1/2}(2\sqrt{xt})}{(2\sqrt{xt})^{1/2}} e^{-t^2} dt$$

and note 18 below. Let $0 < a < b < \infty$, $f \in C^2(a,b)$ and $K_0(x)$ $Y_0(x)$ be the modified Bessel functions of order 0. Let further denote

$$d(n) := \sum_{d|n} 1 \quad \text{and} \quad \sum_{a\le n\le b}{}^{*}$$

be the sum, where the last term is to be halved in case x is an integer. Then it holds for $f \in C^2(a,b)$

$$\sum_{a\le n\le b}{}^{*} d(n) f(n) = \int_a^b (\log(x) + 2\gamma) f(x) dx + \sum_{1}^{\infty} d(n) \int_a^b f(x)\alpha(\pi x) dx$$

with

$$\alpha(x) := 4K_0(4\pi\sqrt{x}) - 2\pi Y_0(4\pi\sqrt{x}) \quad .$$

With respect to [NaC] we note the formula ([WaG] 6-23):

$$-\frac{\pi}{2} Y_0(4\pi\sqrt{x}) + K_0(4\pi\sqrt{x}) = \int_0^{\infty} \cos t \cos\frac{4\pi x}{t}\frac{dt}{t}$$

From [GrI] 6.534 and [WaG] 6-22, 6-23, 13-21, 13-24, 13-41, 13-72, 17-73, 17-74 we recall

$$\alpha(x)=4\int_0^\infty \frac{tJ_0(2\sqrt{t})}{t^2-(4\pi)^4x^2}dt=4\int_0^\infty(\cos t)(\cos\frac{4\pi x}{t})\frac{dt}{t}$$

$$K_0(z)=\int_0^\infty e^{-z\cosh t}dt=\frac{1}{2}\int_0^\infty e^{-t-\frac{z^2}{4t}}\frac{dt}{t}=\int_0^\infty\cos(z\sinh t)dt$$

$$J_0^2(z)+Y_0^2(z)=\frac{8}{\pi^2}\int_0^\infty K_0(2z\sinh t)dt \quad , \quad \frac{8}{\pi^2}K_0^2(z)=(\frac{4}{\pi})^2\int_0^\infty K_0(2z\cosh t)dt$$

$$\int_0^\infty x^s J_0^2(4\pi x)\frac{dx}{x}=\frac{\Gamma(1-s)}{(2\pi)^s}\frac{\Gamma(\frac{s}{2})}{\Gamma^3(1-\frac{s}{2})}=\frac{\Gamma(1-s)}{(2\pi)^s}\left[\frac{\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\right]^2\frac{\sin(\frac{\pi}{2}s)}{\pi}=2^{-2s}\pi^{-s-1/2}\frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{s}{2})}{\Gamma^2(1-\frac{s}{2})}$$

$$4\int_0^\infty x^{s/2}K_0(4\pi\sqrt{x})\frac{dx}{x}=\frac{2}{(2\pi)^s}\Gamma^2(\frac{s}{2})$$

$$2\pi\int_0^\infty x^{s/2}Y_0(4\pi\sqrt{x}\frac{dx}{x}=\frac{2}{(2\pi)^s}\Gamma^2(\frac{s}{2})\cos(\frac{\pi}{2}s)=\cos(\frac{\pi}{2}s)\left[4\int_0^\infty x^{s/2}K_0(4\pi\sqrt{x})dx\right]$$

and therefore

$$\int_0^\infty x^s\alpha(x)\frac{dx}{x}=\Gamma(s)\left[\frac{2(\cos(\pi s)-1)}{(2\pi)^{2s}}\Gamma(s)\right]=:\Gamma(s)\varphi(-s) \quad .$$

We note that ([WaG] 7-34, 13-74, 13-75)

$$(\frac{2}{\pi})^{1/2}e^{x/2}K_0(\frac{x}{2})=\sum_0^\infty\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-\frac{1}{2})}\frac{(\frac{1}{x})^n}{n!}$$

and that

$$\frac{\pi^2}{4}x\left[J_\nu^2(x)+Y_\nu^2(x)\right]\approx e^{2x}K_0^2(x)\approx\frac{\pi}{2}\frac{1}{x}$$

is an increasing function for $\nu<1/2$. The Mellin transform of the first summand is given by

$$2\int_0^\infty x^s\left[xJ_0^2(2x)\right]\frac{dx}{x}=\frac{\Gamma(-s)}{\Gamma(\frac{1-s}{2})}\frac{\Gamma(\frac{3+2s}{4})}{\Gamma(\frac{3-2s}{4})\Gamma(\frac{1-2s}{4})} \qquad \text{for } \operatorname{Re}(s)<1$$

From [WaG] 13-21, 13-73, 13-72 we recall the formulas

$$\frac{\pi^2}{4}\left[J_\nu^2(x)+Y_\nu^2(x)\right]=2\int_0^\infty K_0(2z\sinh t)\cosh 2\nu t dt \quad ,$$

$$K_\nu^2(x)=2\int_0^\infty K_0(2x\cosh t)\cosh 2\nu t dt=2\int_0^\infty K_{2\nu}(2x\cosh t)dt$$

leading to ([GrI] 3.512)

$$\int_0^\infty x^{2s}K_\nu^2(\frac{x}{2})\frac{dx}{x}=2\int_0^\infty\int_0^\infty x^{2s}K_0(x\cosh t)\cosh(2\nu t)dt\frac{dx}{x}=2\int_0^\infty\frac{\cosh(2\nu t)}{\cosh^{2s}t}\int_0^\infty y^{2s}K_0(y)\frac{dy}{y}dt$$

$$=2^{2s-1}\Gamma^2(s)\int_0^\infty\frac{\cosh(2\nu t)}{\cosh^{2s}t}=2^{2s-1}\Gamma^2(s)2^{2(s-1)}B(s+\nu,s-\nu) \, .$$

Analogue it holds

$$\frac{\pi^2}{4}\int_0^\infty x^{2s}x(J_\nu^2(\frac{x}{2})+Y_\nu^2(\frac{x}{2}))\frac{dx}{x}=2\int_0^\infty\int_0^\infty x^{2s}K_0(x\sinh t)\cosh 2\nu t dt\frac{dx}{x}=2\int_0^\infty\frac{\cosh(2\nu t)}{\sinh^{2s}t}\int_0^\infty y^{2s}K_0(y)\frac{dy}{y}dt$$

$$=2^{2s-1}\Gamma^2(s)\int_0^\infty\frac{\cosh(2\nu t)}{\sinh^{2s}t}\cdot$$

For $\nu=0$ one gets

$$\frac{\pi^2}{4}\int_0^\infty x^{2s}x(J_0^2(\frac{x}{2})+Y_0^2(\frac{x}{2}))\frac{dx}{x}=2^{2s-1}\Gamma^2(s)\int_0^\infty\sinh^{-2s}tdt=2^{2s-1}\Gamma^2(s)\frac{1}{2\sqrt{\pi}}\Gamma(s)\Gamma(1-s) \quad \text{for } 0<\operatorname{Re}(s)<1\cdot$$

From [WaG] 13-41 we recall the Struve integral

$$\int_0^\infty x^{1-2s}J_{2\nu}^2(2x)\frac{dx}{x}=\frac{\Gamma(2s)\Gamma(\nu-s+\frac{1}{2})}{2\Gamma(s+\nu+\frac{1}{2})\Gamma^2(s+\frac{1}{2})} \quad \text{for } 0<\operatorname{Re}(2s)<\operatorname{Re}(2\nu+1)\cdot$$

We therefore have

$$\int_0^\infty x^{2s}(J_0^2(\frac{x}{2})+Y_0^2(\frac{x}{2}))\frac{dx}{x}=\frac{2^{2s-1}}{\pi^2\sqrt{\pi}}\Gamma^3(s-\frac{1}{2})\Gamma(\frac{3}{2}-s) \quad \text{for } \frac{1}{2}<s<\frac{3}{2}$$

$$\int_0^\infty x^{2s}K_0^2(\frac{x}{2})\frac{dx}{x}=\frac{\Gamma^3(s)}{2^3}$$

$$\int_0^\infty x^{2s}J_0^2(\frac{x}{2})\frac{dx}{x}=2^{2s}\frac{\Gamma(1-2s)\Gamma(s)}{2\Gamma^3(1-s)}$$

leading to

$$\int_0^\infty x^{2s}Y_0^2(\frac{x}{2})\frac{dx}{x}=\frac{2^{2s-1}}{\pi^2\sqrt{\pi}}\Gamma^3(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)-\int_0^\infty x^{2s}J_0^2(\frac{x}{2})\frac{dx}{x}=\frac{2^{2s-1}}{\pi^2\sqrt{\pi}}\Gamma^3(s-\frac{1}{2})\Gamma(\frac{3}{2}-s)-2^{2s}\frac{\Gamma(1-2s)\Gamma(s)}{2\Gamma^3(1-s)}\cdot$$

It is proposed to analyze alternatively the cases $\nu=0,1/4$ e.g. for the function

$$\alpha^*(x):=4K_0^2(2\sqrt{\pi\sqrt{x}})-2\pi Y_0^2(2\sqrt{\pi\sqrt{x}})\cdot$$

We note ([GrI] 1.411)

$$\sec h(x)=1+\sum_1^\infty\frac{B_{2k}}{(2k)!}x^{2k}\ , \quad \cos ech(x)=\frac{1}{x}+\sum_1^\infty\frac{2(2^{2k-1}-1)B_{2k}}{(2k)!}x^{2k-1}\cdot$$

The connection between $d_k(n)$ and $\zeta(s)$ is given by

$$\zeta^k(s)=\sum_1^\infty d_k(n)n^{-s} \quad \text{for } R(s)>0$$

respectively

$$\sum_{n\le x}{}^* d_k(n)=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta^k x^s\frac{ds}{s}\cdot$$

In the context of the circle method we refer to the (distributional) Hilbert spaces $L_2^{\#}(0,1)$, $H_{-\beta}^{\#}(0,1)$ ([BrK], [BrK1], section “Hardy space and Hilbert scale” with e.g. the Szegö singular integral operator) and to the Mellin transform formulas for $J_0(2\sqrt{x})$, $J_1(2\sqrt{x})$, $K_0(2\sqrt{x})$, $Y_0(2\sqrt{x})$ (note below).

Note 14: The zero function can be expressed as Schlömilch series ([WaG] 19-41), provided that $0<x<\pi$

$$\frac{1}{2}+\sum_{k=1}^{\infty}(-1)^k J_0(kx)=0 \quad .$$

The series oscillates when $x=0$ and diverges to $+\infty$ when $x=\pi$.

Note 15 Lemma (“counting sums” and Stieltjes integrals):

Let $(a_n)_{n\in N}$ a sequence of complex numbers and $(t_n)_{n\in N}$ a strictly increasing, unbounded sequence of real numbers. Let further $A(x)$ be the sum over all those a_n with index n with $t_n \le x$.

If $g:[t_1,\infty[\to C$ is continuous differentiable, then it holds for all $x\ge t_1$

$$\sum_{\substack{n \\ t_n\le x}} a_n g(t_n)=A(x)g(x)-\int_{t_1}^{x}A(t)g'(t)dt \quad .$$

The inversion formula for the Dirichlet series

$$A(s):=\sum_{1}^{\infty}a_n n^{-s}$$

is given by ($c>0$ that ($A(s)$ is absolute convergent for ($\mathrm{Re}(s)=c$)

$$\sum_{n\le x}{}^{*}a_n=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}A(s)x^s\frac{ds}{s} \, .$$

<u>Example</u>: Putting

$$a_n:=\frac{\log p_n}{p_n} \quad , \quad t_n:=p_n \quad , \quad g(t):=\frac{1}{\log t}$$

leads to

$$\sum_{p\le x}\frac{1}{p}=\log\log x+B+O\left(\frac{1}{\log x}\right) \quad .$$

Note 16: Consecutive zeros on the critical line ([IvA] 10.3): For any $\varepsilon>0$ and $n>n_0(\varepsilon)$ it holds

$$\gamma_{n+1}-\gamma_n<\gamma_n^{\vartheta+\varepsilon} \quad \text{with} \quad \vartheta=0.1559458...<0.15625=\frac{5}{32} \, .$$

Note 17: If the RH is true, then ([IvA] theorem 12.10):

$$p_{n+1}-p_n \ll p_n^{1/2}\log p_n \quad .$$

Note 18: The Bessel functions and the Gamma function are linked e.g. by Weber's infinite integral ([WaG] 13-24, 13-21, [BeB1] 3.2): For $0<\mathrm{Re}(s)<3/2$ and

$$\mu'(x)=\frac{d\mu}{dx}=\frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})}=\frac{d(\log J_0(2\sqrt{x}))}{dx}$$ for $x\neq\alpha_k$,

it holds

i) $$\frac{\Gamma(1+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}=\frac{s}{2}\int_0^\infty x^{s/2}J_0(2\sqrt{x})\frac{dx}{x}=\frac{s}{2}(1-\frac{s}{2})\int_0^\infty x^{\frac{s-1}{2}}J_1(2\sqrt{x})\frac{dx}{x}$$

ii) $$\frac{\Gamma(1+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}=\frac{s}{2}(1-\frac{s}{2})\int_0^\infty x^{\frac{s}{2}}J_0(2\sqrt{x})\frac{d\mu}{x}$$ i.e. $$\frac{\Gamma(1+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}=)\int_0^\infty x^{\frac{s}{2}}J_0(2\sqrt{x})d\mu$$

iii) $$4\Gamma^2(\frac{s}{2})=\int_0^\infty x^sK_0(2x)\frac{dx}{x}$$

iv) $$\int_0^\infty x^{s/2}\frac{J_1(2\pi\sqrt{x})}{2\pi\sqrt{x}}\frac{dx}{x}=\frac{1}{4\pi}\frac{\pi^{1-s}\Gamma(\frac{s}{2})}{(1-\frac{s}{2})\Gamma(1-\frac{s}{2})}$$

v) $$\frac{1}{2\pi}\Gamma(1+\frac{s}{2})\cos(\frac{\pi}{2}s)=\int_0^\infty x^sY_0(2x)\frac{dx}{x}=\frac{1}{2}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}=\frac{\pi^{\frac{1-s}{2}}}{2}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\Gamma(\frac{s}{2})\tan(\frac{\pi}{2}s)$$

vii) $$\int_0^\infty x^sY_0(2x)\frac{dx}{x}=\frac{1}{2}\frac{\Gamma(1+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\int_0^\infty x^sf_H(x)\frac{dx}{x}$$

Note 19: The Hermite polynomials $H_n(x)$ fulfill the recursion formula

$$H_n(\sqrt{2\pi}x)=2xH_{n-1}(\sqrt{2\pi}x)-(n-1)b_n\varphi_{n-2}(x)-2(n-1)H_{n-2}(\sqrt{2\pi}x)\ .$$

Using the abbreviation

$$a_n:=\sqrt{\frac{2(n-1)!}{n!}}\qquad b_n:=\sqrt{\frac{(n-2)!}{n!}}$$

this gives the recursion formula

$$\varphi_n(x):=a_nx\varphi_{n-1}(x)-(n-1)b_n\varphi_{n-2}(x),\quad \varphi_0(x):=\pi^{-1/4}e^{-\frac{x^2}{2}},\quad \varphi_1(x):=2^{-1/2}\pi^{-1/4}xe^{-\frac{x^2}{2}}$$

,
from which the recursion formula for the corresponding Hilbert transforms can be calculated

$$\hat{\varphi}_n(x):=a_n\left[x\hat{\varphi}_{n-1}(x)-\frac{1}{\pi}\int_{-\infty}^{\infty}\varphi_{n-1}(y)dy\right]-(n-1)b_n\hat{\varphi}_{n-2}(x)$$

$$\hat{\varphi}_0(x)=\pi^{1/4}\int_{-\infty}^{\infty}e^{-\frac{\omega^2}{2}}\sin(\omega x)d\omega\ .$$

Note 20: Ramanujan's Master Theorem ([BeB] The first quarterly report, 1.2 Theorem I):

$$\int_0^\infty F(x)x^{s-1}dx=\Gamma(s)\varphi(-s) \quad \text{for} \quad F(x)=\sum_0^\infty \frac{\varphi(k)}{k!}(-x)^k \quad \text{in the neighborhood of } x=0 .$$

Note 21: The linkage of the distributional Hilbert space $H_{-1}^{\#}(0,2\pi)$ to the Mellin transform challenge of the standard trinometric series $C^{\#}(0,2\pi)$ ([RiB]). and to the Bagchi formulation of the Nyman criterion ([BaB]). Let

$$f(x)=\frac{a_0}{2}+\sum_1^\infty a_n\sin(nx)+b_n\cos(nx)$$

with vanishing constant Fourier term, i.e. $a_0=0$. On the one hand side it holds

$$\int_0^\infty \sin(x)x^s\frac{dx}{x}=\Gamma(s)\sin(\frac{\pi}{2}s) \qquad \int_0^\infty \cos(x)x^s\frac{dx}{x}=\Gamma(s)\cos(\frac{\pi}{2}s) \quad \text{for } 0<\operatorname{Re}(s)<1$$

and on the other side it holds

$$\int_0^\infty\left[\sum_1^\infty a_n\sin(nx)+b_n\cos(nx)\right]x^s\frac{dx}{x}=\Gamma(s)\left[\sin(\frac{\pi}{2}s)+\cos(\frac{\pi}{2}s)\right]\sum_1^\infty\frac{a_n+b_n}{n^s} \quad \text{for } 1<\operatorname{Re}(s).$$

A "function" $g\in H_{-1}^{\#}$ with $f=Ag\in H_0^{\#}$ gives the (convergence) answer to the trigonometric series characterization question from [RiB]. The corresponding definition of the Zeta function gives a shift from $s\to z+1$ with $\operatorname{Re}(z)>0$.

For a complex valued function $2\pi-$periodic function $f(\varphi)=u(\varphi)+iv(\varphi)$ its conjugated function can be represented by

$$\bar{f}(\varphi)=-\lim_{\varepsilon\to0}\frac{1}{2\pi i}\oint_{\varepsilon,\pi}f(\varphi+\vartheta)-f(\varphi-\vartheta)\cot\frac{\vartheta}{2}d\vartheta=\frac{1}{2\pi i}\oint_{0,2\pi}f(\vartheta)\cot\frac{\varphi-\vartheta}{2}d\vartheta .$$

Let $a_0;a_n,b_n$ be the Fourier coefficients of f . Then $0;-b_n,a_n$ are the Fourier coefficients of its conjugate and it holds

$$\frac{1}{\pi}\int_0^{2\pi}f^2(\varphi)d\varphi=\frac{a_0^2}{2}+\frac{1}{\pi}\int_0^{2\pi}\bar{f}^2(\varphi)d\varphi \quad \text{resp.} \quad \frac{1}{\pi}\int_0^{2\pi}\bar{f}^2(\varphi)d\varphi=\sum_1^n a_n^2+b_n^2 .$$

The Nyman criterion is based on the Zeta function representation in the critical stripe

$$-\frac{\zeta(s)}{s}=\int_0^\infty x^{-s}\rho(x)\frac{dx}{x}$$

with $\rho(x):=x-[x]$ being the fractional part function. With respect to the distributional Hilbert space $H_{-\beta}$, $\beta>0$ we refer to [BaB] the Nyman criterion is reformulated within a purely functional analysis weighted l^2- Hilbert space framework, which is about of all sequences $a=\{a_n|n\in N\}$ of complex numbers such that

$$\sum_{n=1}^\infty\omega_n|a_n|^2<\infty \quad \text{with} \quad \frac{c_1}{n^2}\le\omega_n\le\frac{c_2}{n^2} \text{, i.e. isomorph to the Hilbert space } l_{-1} .$$

According to [BaB] all bounded sequences of complex numbers are vectors in the separable (!) Hilbert space l_{-1} (separavle means, that the metric space is a closed hull of countable sets). This fulfills the of the Bagchi's criterion.

We note that in harmonic analysis by

$$[\varphi]^2 := \frac{\pi}{2}\sum_1^\infty \nu(a_\nu^2 + b_\nu^2) = \frac{1}{2}\iint |dh(z)|^2 dxdy = \frac{1}{4\pi}\int_{\partial B}\int_{\partial B}\frac{|\varphi(w)-\varphi(\zeta)|^2}{|w-\zeta|^2}ds(w)d\zeta < \infty$$

the energy of the harmonic continuation $h = \mathrm{E}(\varphi)$ to the boundary is given.

Note 22: The Voronoi Summation Formula ([BeB1]) Theorem 19.5.1, 2.4, 2.5): Let, $f \in C^1(0,\infty)$ and

$$\phi(s) := \sum_1^\infty a(n)\lambda_n^{-s} \quad , \quad \psi(s) := \sum_1^\infty b(n)\mu_n^{-s} \qquad 0 < \lambda_n, \mu_n \to \infty$$

be two Dirichlet series with abscissas of absolute convergence σ_a and σ_a^*, respectively.
Let $r > 0$, and suppose that $\phi(s)$ and $\psi(s)$ satisfy a functional equation of the type

$$\Gamma(s)\phi(s) = \Gamma(r-s)\psi(r-s) .$$

Putting

$$Q(x) := \frac{1}{2\pi i}\int_C \phi(s)x^s \frac{ds}{s}$$

where C is a simple closed curve(s) containing the integrand's poles in its interior, then if $0 < a < \lambda_1 < x < \infty$,

$$\sum_{\lambda_n \le x}{}^* a(n) f(\lambda_n) = \int_a^x Q'(t)f(t)dt + \sum_1^\infty b(n)\int_a^x \left(\frac{t}{\mu_n}\right)^{(r-1)/2} J_{r-1}(2\sqrt{\mu_n t})f(t)dt \ .$$

In [NaC] L_2 – theory of Mellin and Fourier transformation is applied to derive a more symmetrical Voronoi formula (with similar structure as in [DuR], where the fractional part function is applied):
For $f, g \in G_1^2(0,\infty)$ properly defined it holds

$$\lim_{N\to\infty}\left\{\sum_{n=1}^N d(n)f(n) - \int_0^N (\log x + 2\gamma)f(x)dx\right\} = \lim_{N\to\infty}\left\{\sum_{n=1}^N d(n)g(n) - \int_0^N (\log x + 2\gamma)g(x)dx\right\} \ .$$

Dixon & Ferrar ("*On the summation formula of Voronoi and Poisson*") provided a formula with reduced regularity assumptions (bounded variation) to the function g :

$$\sum_1^\infty g(nt) = -\frac{1}{2}g(0^+) + \frac{1}{t}\int_0^\infty g(x)dx + \sum_1^\infty 0\int^\infty g(x)\cos(\frac{2\pi n}{t}x)dx \ .$$

Note 23: Holomorphic function in the distributional sense **(**[PeB], chapter 1, §15):
Let $z \to g_z$ be a function defined on a open subset $U \subset C$ with values in the distribution space. Then g_z is called a holomorphic in $U \subset C$ (or $g(z) := g_z$ is called holomorphic in $U \subset C$ in the distribution sense), if for each $\varphi \in C_c^\infty$ the function $z \to (g_s, \varphi)$ is holomorphic in $U \subset C$ in the usual sense.

Note 24: Müntz formula and Bessel function ([TiE], **(**[WaG] 13-2, 13-3).

For $\omega(x),\omega'(x)$ continuous and bounded in any finite interval with $\omega(x)=o(x^{-\alpha})$ and $\omega(x)=o(x^{-\beta})$ for $x\to\infty$ and $\alpha,\beta>1$ it holds

$$\zeta(s)\int_0^\infty x^s\omega(x)\frac{dx}{x}=\int_0^\infty x^s\left[\sum_1^\infty\omega(nx)-\frac{1}{x}\int_0^\infty\omega(t)dt\right]\frac{dx}{x}\qquad \text{for } 0<\mathrm{Re}(s)<1.$$

$$2\pi\int_0^\infty x^sY_0(2x)\frac{dx}{x}=\Gamma^2(\frac{s}{2})\cos(\frac{\pi}{2}s)=\frac{1}{4}\cos(\frac{\pi}{2}s)\int_0^\infty x^sK_0(2x)\frac{dx}{x}\qquad \text{for } 0<\mathrm{Re}(s)<3/2$$

$$\int_0^\infty Y_0(x)dx=0\quad,\quad\int_0^\infty K_0(x)dx=\frac{\pi}{2}\ .$$

Note 25: The Gauss-Weierstrass related "density" function

$$f(\sqrt{x})\frac{dx}{x}$$

plays a key role in the Riemann primes number theory. It also defines the $\Gamma(s)-$function by

$$\Gamma(s)=\int_0^\infty x^s\int_x^\infty e^{-t}\frac{dt}{t}dx\ .$$

The corresponding alternative "density" function would be given by replacing

$$e^{-t}\frac{dt}{t}\to 2dB(t)\ ,$$

leads to a modified "Gamma function" in the form

$$\Gamma^*(s)=\int_0^\infty x^s\int_x^\infty 2B'(t)dt\frac{dx}{x}=2\int_0^\infty\int_0^t x^sB'(t)\frac{dx}{x}dt=\frac{2}{s}\int_0^\infty t^sB'(t)dt\ .$$

With respect to

$$f_H(y)=4\pi\int_0^\infty f(\xi)\sin(2\pi\xi y)d\xi$$

we note ([GrI] 3.757)

$$2\int_0^\infty\sqrt{\frac{\xi}{x}}\sin(2\pi\xi x)dx=2\int_0^\infty\sqrt{\frac{\xi}{x}}\cos(2\pi\xi x)dx=1\ .$$

Note 26: The Mellin transform fulfill the following properties

i) $M[h'](s)=(1-s)M[h](s-1)$

ii) $M[xh'](s)=-sM[h](s)$

iii) $M[(xh)'](s)=(1-s)M[h](s)$

iv) $M[h''](s)=(s-1)(s-2)M[h](s-2)$

v) $(h*M[g])(s)=\int_0^\infty h(\frac{x}{u})g(u)\frac{du}{u}=\int_0^\infty u^{-s}h(\frac{x}{u})g(u)u^s\frac{du}{u}\ .$

Note 27: The Mellin transform of the special Lommel functions $s_{-1,-1}$ is given by ([GrI] 6.861)

$$(s-1)\int_0^\infty (2x)^s s_{-1,-1}(x)\frac{dx}{x} = \frac{\pi}{2}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{1-s}{2})} \quad \text{for} \quad 0<\mathrm{Re}(s)<3/2 \,.$$

Note 28: With respect to a still missing proof of the transcendence of the Euler constant γ we mention the following: Both, the exponential integral function

$$v(x) := -\int_x^\infty \frac{e^{-t}}{t}dt$$

and the Bessel functions (e.g. $J_0(2\sqrt{x})$) are E-functions ([SiC1], [WaG] 15-28). The derivative, the sum, the product and the integral of E-functions are again E-functions, as well as the multiplication with any real number ([SiC1]). Therefore the degenerated hypergeometric functions above are E-functions, as well.

The inverse of the logarithmic derivative of $v(x)$ is used ([LaE] [NiN], §84 [SiC]) to build a CF representation for

$$\frac{w(x)}{w'(x)} = xe^x\int_x^\infty e^{-t}\frac{dt}{t} \,.$$

The required polynomial system is given by the Lommel polynomials ([DiD], [WaG] 10-7). CF representations for

$$-e^x\int_0^x e^{-t}\frac{dt}{t} \quad , \quad 2xe^{x^2}\int_x^\infty e^{-t^2}dt$$

are given in [PeO] §20, (15),(16). "*Iterates of the exponential function*" are treated in [BeB] chapter IV, example 2. Bessel integral functions are analyzed in [HuP]. A CF analysis of degenerated hypergeometric functions are given in [PeO] §81.

Applying Gauss's CF leads to the specific CF representation

$$\frac{1}{2}\sqrt{\pi}e^z erf(\sqrt{z}) = e^{z^2}\,{}_1F_1(\frac{1}{2},\frac{3}{2},-z) = \cfrac{z}{1-\cfrac{z}{\frac{3}{2}+\cfrac{z}{\frac{5}{2}-\cfrac{\frac{3}{2}z}{\frac{7}{2}+\cfrac{2z}{\frac{9}{2}-\cfrac{\frac{5}{2}z}{\frac{11}{2}+\cfrac{3z}{\frac{13}{2}-\cfrac{\frac{7}{2}z}{\frac{15}{2}\pm\ldots\ldots}}}}}}}}$$

A proof of the irrationality of CF function enables a proof of the irrationality of certain numbers from the domain of those functions. For example the irrationality of π is a consequence of the fact, that $\tan(x)$ is irrational for rational $x \neq 0$ based on the identity $\tan(\pi/4)=1$.

The fractional part function resp. its Hilbert transform provides the link to Hilbert spaces of periodic function $H_\alpha^\#$ (e.g. $\alpha = 0,\pm 1/2,\pm 1$) on the unit circle. Orthogonal polynomials on the unit circle are analyzed in [SzG] XI. An analysis of related Lommel polynomials is given in [DiD].

Lommel polynomials are also related to Bessel functions and its asymptotic expansion remainders functions [WaG] 9-6, 7-3.

The beautiful proof from I. Niven ([NiI] theorem 6.18) that π is irrational is based on his brilliant idea to define an appropriate auxiliary function for assumed positive integers a,b with $\pi = a/b$ in the form

$$f_n(x) := g(x)/n! := x^n (a-bx)^n / n! = b^n x^n (\pi - x)^n / n! \ .$$

The function $g(x)$ and all its derivative, evaluated at $x=0$, are integers divisible by $n!$ and therefore $f_n(x)$ and all its derivatives are integers. At the same time, as $f_n(x)\sin x > 0$ it holds

$$0 < I_n := \int_0^\infty f_n(x)\sin(x)dx < \frac{\pi^n a^n}{n!}\pi \quad ,$$

i.e. $I_n \in (0,1)$ for $n > n_0$, while on the other hand side I_n are integers, which finally leads to a contradiction to the assumption, that π is rational.
The following properties of the fractional part function from number theory might be helpful.

Lemma: i) For $(m,n)=1$ it holds

$$\sum_{k=1}^{n-1}\left[\frac{mk}{n}\right] = \frac{(m-1)(n-1)}{2}$$

ii) Let θ be real, and $0<\theta<1$. Then for

$$g_n := \begin{cases} 0 & if\ [n\theta] = [(n-1)\theta] \\ 1 & otherwise \end{cases}$$

it holds

$$\lim_{n\to\infty}\frac{g_1 + g_2 ..... + g_n}{n} = \theta$$

iii) For any real number ξ and $n \in N$ it holds

$$[n\theta] = [(\xi] + \left[(\xi + \frac{1}{n}\right] + ... + \left[(\xi + \frac{n-1}{n}\right] \ .$$

In order to complete the tool set to enable a proof of the irrationality of the Euler Gamma constant γ it requires an appropriate representation of γ building on hypergeometric functions. Corresponding logarithmic derivatives is expected to enable a CF representation.

Follow the idea from [BrR] there is a representation in the form

$$\gamma = \int_0^\infty x^s d\sigma = \int_0^\infty x^s\left[e^{-x} - J_0(2\sqrt{x})\right]\frac{dx}{x} (= -\int_0^\infty e^{-t}\log t dt = -\Gamma'(1)) \ .$$

It is derived from corresponding Mellin transforms in combination with $\Gamma(1)=1$, $\Gamma'(1) = -\gamma$, $s\Gamma(s) = \Gamma(1+s)$ to show the convergence of

$$\int_0^\infty x^s d\sigma = \Gamma(s) - \frac{\Gamma(s)}{\Gamma(1-s)} = \frac{1}{s}\left[(1-\gamma s) - \frac{(1-\gamma s)}{(1+\gamma s)} + O(s^2)\right] = \gamma\frac{(1-\gamma s)}{(1+\gamma s)} + O(s) \underset{s\to 0^+}{\to} \gamma \ .$$

The analogue relationship comparing the Gaussian function with its Hilbert transform is given by

$$\int_0^\infty x^s d\mu = \Gamma(s)\left[1 - c\cdot\tan(\pi s)\right]$$

with

$$\mu(x) := \int_x^\infty \left[e^{-t} - f_H(\sqrt{\frac{t}{\pi}})\right]\frac{dt}{t} \ .$$

Note 29: The asymptotic of the zeros of the degenerate hypergeometric functions are analyzed in [SeA]. There is only one independent solution of the Kummer differential equation with the solution

$$ {}_1F_1(\frac{1}{2};1,2z) = e^z J_0(ze^{\pi i/2}) \cdot $$

Note 30: We claim that there is a relationship of the function

$$ f_H'(\frac{x}{\sqrt{\pi}}) \approx {}_1F_1(\frac{1}{2};\frac{3}{2},-x) = \frac{1}{2}\int_0^1 e^{-xt} t^{-1/2} dt $$

with

$$ 2\log(2 \cdot {}_1F_1(\frac{1}{2};\frac{3}{2},-x)) = 1 - x $$

with the distributional Hilbert space $H_{-1/2}$ and the spectrum of harmonic music signals ([VoR] [VoR1]), which lie exactly between the red (Brownian motion) and the white (the derivative of the Brownian motion) noise signals.

Note 31: The Gaussian function and a related Kummer function are also linked by the Gauss transform ([KaA] 7.621, 9.215, [KaA]), i.e. for $\mathrm{Re}(\xi^2) > 1$

$$ G\left[{}_1F_1(\frac{1}{2};1,x) \right](\xi) := 2\xi \int_0^\infty {}_1F_1(\frac{1}{2};1,t) e^{-\xi^2 t} dt = 2\xi \int_0^\infty e^{t/2} J_0(\frac{t}{2} e^{\pi i/2}) e^{-\xi^2 t} dt = \frac{2}{\sqrt{\xi^2 - 1}} \cdot $$

Note 32: Let

$$ E(x) := e^{-x} \quad , \quad \varphi(n) := \frac{2n}{2n+1} \quad , \quad \Phi(x) := xF'(x) := 2x\, {}_1F_1'(\frac{1}{2};\frac{3}{2},-x) \cdot $$

It holds

i) $E(x) \approx \Phi(x) = \sum_0^\infty \varphi(n) \frac{(-x)^n}{n!}$, $\mathrm{Res}_{s=-n}[\Gamma(s)] = \frac{(-1)^n}{n!}$

ii) $E(x)$ decreases faster than $\Phi(x)$ for all $x > 0$

iii) $Ei(-x) := -\int_x^\infty e^{-t} \frac{dt}{t} \approx F(x) = -\int_x^\infty dF$

iv) $Li(x) = Ei(\log x) \approx Li^*(x) := F(\log x) = \sum_0^\infty \frac{\varphi(n)}{n} \frac{\log^n x}{n!}$

v) $M[E](s) = \Gamma(s)$

vi) $M[\Phi](s) = \Gamma(s) \frac{2s}{2s-1} = \Gamma(s)\varphi(-s)$ for $0 < \mathrm{Re}(s) < \frac{1}{2}$,

which corresponds to the Ramanujan Master Theorem representation ([BeB] IV, Entry 11) resp.

$$ \int_0^\infty x^{s/2} d\, {}_1F_1(\frac{1}{2};\frac{3}{2},-x) = \frac{\Gamma(1+\frac{s}{2})}{s-1} = \frac{\Pi(\frac{s}{2})}{s-1} \quad \text{for} \quad 0 < \mathrm{Re}(s) < 1 \quad , $$

which corresponds to the Gauss notation of the Gamma function ([EdH] 1.3).

Note 33: (ball symmetric potential of linear oscillator): for the energy $E=\frac{1}{2}\hbar\omega$ the radial function $f(r)$ of the corresponding eigenfunction of the Schrödinger equation is given by ([FlS] p 99)

$$f(r)=e^{\lambda\frac{r^2}{2}}\left[c_1\cdot {}_1F_1(\frac{1}{2};\frac{3}{2},\lambda r^2)+c_2\frac{1}{r}{}_1F_1(\frac{1}{2};\frac{3}{2},\lambda r^2)\right] .$$

Remark (Yukawa potential): The $Ei(-x)-$function is also related to the Yukawa potential of a point charge in the form

$$e^{-\mu r}/r \quad , \quad \mu>0$$

in order to define a nuclear force potential which decays rapidly at infinity. Yukawa assumed that μ^{-1} was of the order of magnitude of a nuclear radius. It results that the potential u of a charge distribution satisfies the Yukawa equation

$$\Delta u=\mu^2 u$$

at points of free space. Thus the Yukawa potentials are invariant under the group of rotations and translations of space, like those of Newton potentials. As μ approaches zero the Yukawa potentials approach those of Newton. The function $e^{-\mu r}/r$ is a member of the Bessel family of functions. Bessel functions have certain advantages over Newtonian potentials in functional analysis.

Analog to the above we propose a modified Yukawa potential dY^* by the substitution

$$e^{-\mu r}/r\to\mu F'(\mu r) .$$

The Dawson function $F(x)$ shows similar behavior as the Yukawa potential with the additional properties that $F(0)=0$ and the long distance behavior

$$F(x)\approx x^{-1} .$$

We recall the alternative option for the exponential integral function density (and the eigenfunction of the ground state energy of the harmonic quantum oscillator)

$$\frac{e^{-x}}{x}dx\to -2{}_1F_1'(\frac{1}{2};\frac{3}{2},-x)dx$$

leading to an alternative "Yukawa" potential with same long range behavior as both, the gravitation (Newton) and the electromagnetic (Coulomb) potential (whereby the Coulomb potential is aligned with weak interaction potential by the Higgs mechanism) in the form

$$-g^2\frac{e^{-\frac{mc}{\hbar}x}}{x}\to -2ag^2{}_1F_1'(\frac{1}{2};\frac{3}{2},-ax)=-2g^2\frac{mc}{\hbar}{}_1F_1'(\frac{1}{2};\frac{3}{2},-\frac{mc}{\hbar}x) .$$

With the standard notation and corresponding abbreviation

$$n\cdot\sigma_n:=\frac{Zme^2}{\hbar^2} \quad , \sigma:=\sigma_1 \quad V^*(r):=\frac{2\sigma}{r} \quad , \quad E^*:=\frac{2mE}{\hbar^2}$$

the Coulomb eigenvalue problem, based on the corresponding Schrödinger equation,

$$u''(r)+\left[E^*+V^*(r)\right]u(r)=0$$

has the eigen-pair solutions

$$\psi^*(r) := \frac{1}{\sqrt{\pi}}\sigma_n^{3/2} e^{-\sigma_n r} {}_1F_1'(1-n;2,\sigma_n r)$$

$$E_n := -\frac{1}{2}\sigma_n^2 \hbar^2 \qquad \text{Rydberg formula.}$$

The Yang-Mills theory seeks to describe the behavior of elementary particles and is the core of the unification of electromagnetic and weak forces, as well as quantum chromodynamics, the theory of strong forces. It is the basis of the Standard Model. It is a special example of gauge theory with non-abelian symmetry group.

According to the above we propose a modified potential $V^{**}(r)$ of the Schrödinger equation, which delivers (not symmetric) eigen-functions in the form

$$\psi^{**}(r) \approx \sqrt{\frac{\sigma_n}{\pi}} \frac{\sigma_n r \cdot e^{-\sigma_n r}}{r} {}_1F_1(\frac{1}{2}-n;\frac{3}{2},\sigma_n r) \quad .$$

Applying the Dawson function to define the total energy in a one-dimensional Schrödinger equation by

$$E(x) := E_{kin}(x) + E_{pot}(x) := \alpha x F(\gamma x) + \frac{\beta}{x} F(\gamma x) = \frac{1}{2\gamma}(\alpha x + \frac{\beta}{x}) \cdot \frac{d}{dx}\left[erf^2(\gamma x)\right]$$

with some constants α, β, γ provides a model with finite zero and infinity energy according to

$$E(x) \to \begin{cases} \beta & for \quad x \to 0 \\ \frac{\alpha}{2\gamma} & for \quad x \to \infty \end{cases}$$

References

[AbM] Abramowitz M., Stegen I. A., "*Handbook of Mathematical Functions*", Dover Publications, Inc., New York, 1965

[BaB] Bagchi B., "*On Nyman, Beurling and Baez-Duarte's Hilbert space reformulation of the Riemann Hypothesis*", Indian Statistical Institute, Bangalore Centre, (2005), www.isibang.ac.in

[BeB] Berndt B. C., Andrews G. E., "*Ramanujan´s Notebooks Part I*", Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1985

[BeB1] Berndt B. C., "*Ramanujan´s Notebooks Part IV*", Springer Verlag, New York, Berlin, Heidelberg, Tokyo, 1994

[BrK] Braun K., "*Spectral Analysis to prove the Riemann Hypothesis*", www.riemann-hypothesis.de

[BrK1] Braun K., "*A Note to the Bagchi Formulation of the Nyman-RH-Criterion*", www.riemann-hypothesis.de

[BrR] R. P. Brent R. P., "*An asymptotic expansion inspired by Ramanujan*", Austr. Math. Soc. Gaz. 20, 149-155 (1993)

[BuD] D. Bump D., Choi K.-K., Kurlberg P., Vaaler J., "*A Local Riemann Hypothesis I*", Math. Zeit. 233 (2000) pp. 1-19

[CaD] Cardon D. A., "*Convolution operators and zeros of entire functions*", Proc. Amer. Math. Soc., 130, 6 (2002) 1725-1734

[CsG] Csordas G., "*Fourier transforms of positive kernels and the Riemann* $\xi-$*function*", arXiv: 1309.0055 [math.CV] (2013)

[DeJ] Deshouileurs J.-M., Effinger G., Te Riele, Zinoview D., "*A complete Vinogradov 3-Primes Theorem under the Riemann Hypothesis*", Amer. Math. Soc., 3, (1997) 99-104

[DiD] Dickinson D., "*On Lommel and Bessel Polynomials*", Proc. Amer. Math., Soc. 5, 946-956 (1954)

[DoG] Doetsch G., "*Summatorische Eigenschaften der Besselschen Funktionen und andere Funktionalrelationen, die mit der linearen Transformationsformel der Thetafunktion äquivalent sind*", Comp. Math., tome 1 (1935) 85-97

[DuR] Duffin R. J., Weinberger H. F., "*Dualizing the Poisson summation formula*", Proc. Natl. Acad. Sci. 88 (1991) pp. 7348-7350

[EdH] Edwards H. M., "*Riemann´s Zeta Function*", Dover Publications, Inc., Mineola, New York, 1974

[GaW] Gautschi W., Waldvogel J., Computing the Hilbert Transform of the Generalized Laguerre and Hermit Weight Functions

[GaG] Gasper G., "*Using integrals of squares of certain real-valued special functions to prove that the Polya function* $\Xi^{*}(t)$*, the functions* $K_{iz}(a), a>0$ *and some other entire functions have only real zeros",* 2008, http://www.math.northwestern.edu/~george/preprints/ggUsingIntegrals11508.pdf

[GrI] Gradshteyn I. S., Ryzhik I. M., *Table of Integrals Series and Products*, Fourth Edition, Academic Press, New York, San Francisco, London, 1965

[HaH] Hamburger H., „*Über einige Beziehungen, die mit der Funktionalgleichung der Riemannschen Zeta-Funktion äquivalent sind*", Math. Ann. 85 (1922) pp. 129-140

[HaG] Hardy G. H., Littlewood J. E., "*Some problems of "Partitio Numerorum"; III, On the expression of a number as a sum of primes"*, Acta. Math., 44 (1923) 1-70

[HaJ] Havil J., "*Gamma: Exploring Euler's Constant*", Princeton University Press, 2003
[HuP] Humbert P., "*Bessel-integral functions*", Proceedings of the Edinburgh Mathematical Society (Series 2), 1933, 3:276-285

[IvA] Ivic A., "*The Riemann Zeta-Function, Theory and Applications*", Dover Publications, Inc., Mineola, NewYork, 1985

[KaA] Karlsson A., "*Applications of heat kernels on abelian groups:* $\zeta(2n)$, quadratic reciprocity, Bessel integrals", Number Theory, Analysis and Geometry, in memory of Serge Lang, Springer Verlag, Berlin, 396-418 (2012)

[LaE] de Laguerre E., "*Sur l'intégrale* $\int_x^\infty x^{-1}e^{-x}dx$", Bull. Soc.Math. France, 7 72-81(1879)

[LeN] Lebedev N. N., "*Special Functions and their Applications*", translated by R. A. Silverman, Prentice-Hall, Inc., Englewood Cliffs, N. Y., 1965

[MaS] Malgonde S. P., "*Some Abelian theorems for the distributional Kontorovich-Lebedev transformation*", Indian J. Pure Appl. Math., 25 (8) (1994) 861-867

[NaC] Nasim C., "*On the summation formula of Voronoi*", Trans. Amer. Math. Soc., 163 (1972) 35-45

[NiI] Niven I., Zuckerman H. S., Montgomery H. L., "*An Introduction to the Theory of Numbers* ", John Wiley & Sons, Inc., 1960

[NiN] Nielsen N., "*Handbuch der Theorie der Gammafunktion*", Chelsea Publishing Company, Bronx, New York, 1965

[PeO] Perron O., „*Die Lehre von den Kettenbrüchen, Analytisch-funktionentheoretische Kettenbrüche*“, B. G. Teubner Verlag, Stuttgart, 1957

[PeB] Petersen B. E., "*Introduction to the Fourier Transform & Pseudo-differential Operators*", Pitman Publishing Limited, Boston, London, Melbourne

[PoG] Polya G., "*Über die algebraisch-funktionentheoretischen Untersuchungen von J. L. W. V. Jensen",* Mathematisk-fysiske Meddelelsev VII 17 (1927), 1-33

[PoG1] Polya G., "*Bemerkungen über die Integraldarstellung der Riemannschen Zeta-Funktion*“, Acta Mathematica 48, 305-317 (1926)

[PoG2] G. Polya G., "*Über trigonometrische Integrale mit nur reellen Nullstellen*“, J. rein. und angew. Math. 158 (1927) pp. 6-18

[PoG3] G. Polya G., "*Über die Nullstellen gewisser ganzer Funktionen*“, Math. Zeit. 2 (1918), 352-383, also, Collected Papers, Vol II, 166-197

[RiB] Riemann B., „*Über die Darstellbarkeit einer Function durch eine trigonometrische Reihe*“

[SeA] Sedletskii A. M., „*Asymptotics of the Zeros of Degenerate Hypergeometric Functions*“, Mathematical Notes, Vol. 82, No. 2 (2007) 229-237

[SeA1] Sedletskii A. M., „*On the Zeros of Laplace Transforms*“, Mathematical Notes, Vol. 76, No. 6 (2004) 883-892

[SiC] Siegel C. L., „*Ueber einige Anwendungen diophantischer Gleichungen*“, Abhandlungen Akad. Berlin, 1-70, 1929

[SiC1] Siegel C. L., "*Transcendental Numbers*", Anuals of Mathematic Studies, No 16, Princeton University Press (BI Taschenbücher, Mannheim, 1967)

[SzG] Szegö G., „*Orthogonal Polynomials*”, Amer. Math. Soc. Collq. Publ., Vol. XXIII, 1939
[TiE] Titchmarsh E. C., „*The Theory of the Riemann Zeta-function*”, Clarendon Press, Oxford, 1951

[VoR] Voss R.F., Clarke, J., „*1/f noise in Music and Speech*”, Nature 258 (1975) 317-318

[VoR1] Voss R. F., “*1/f (flicker) noise: A brief Review*”, Proc. 32rd Annual Symposium on Frequency Control, Atlantic City, (1979) 40-46

[WaG] Watson G. N., “*A Treatise on the Theory of Bessel Functions*”, Cambridge University Press, Cambridge, 2nd edition first published 1944, reprinted 1996, 2003, 2004, 2006

[WhE] Whittaker E. T., Watson G. N., “*A Course of Modern Analysis*”, Cambridge University Press, Cambridge, 1902

Some formulas

$$\frac{\int_0^\infty e^{-\pi y^2}\sinh(xy)dy}{\int_0^\infty e^{-\pi y^2}\cosh(xy)dy} = \frac{1}{2}\int_0^x e^{-y^2/(4\pi)}dy$$ [Grl] 3.546

$$\int_0^\infty \frac{\pi dy}{\cosh(y)} = {}_1F_1(-1,1,\frac{1}{2})$$ [Grl] 3.523

$$\int_0^\infty x\, {}_1F_1(1,\frac{3}{2},\frac{x^2}{2})J_0(2xy)dx = \frac{1}{2\sqrt{\pi}}e^{-y^2}K_0(y^2)$$ [Grl] 7.663

$$\int_0^\infty \sqrt{x}\, {}_1F_1(\frac{1}{2},\frac{3}{2},\frac{x^2}{2})J_{3/2}(xy)dx = \frac{1}{\sqrt{y}}\, {}_1F_1(\frac{1}{2},\frac{3}{2},-\frac{y^2}{2})$$ [Grl] 7.663

$$\int_0^\infty \frac{e^{-xt}}{(t\pm\nu)^2}dt = xe^{\pm\nu x}Ei(\mp\nu x)\pm\frac{1}{\nu}$$ [Gri] 3.352

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}$$ i.e. $$\log\Gamma(s)+\log\Gamma(1-s) = -\log(\frac{1}{\pi}\sin\frac{\pi}{2}s)$$

$$\Gamma(\frac{s}{2})\Gamma(\frac{1-s}{2}) = \Gamma^*(\frac{s}{2})\Gamma^*(\frac{1-s}{2})$$

$$\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \frac{2^{1-s}}{\sqrt{\pi}}\cos(\frac{\pi}{2}s)\Gamma(s)$$, $$\Gamma(\frac{s}{2})\Gamma(\frac{s+1}{2}) = 2^{1-s}\sqrt{\pi}\Gamma(s)$$

$$\log\Gamma(1+\frac{s+1}{2}) = \sum_1^\infty\left[-\log(1+\frac{s+1}{2n})+\frac{s+1}{2}\log(1+\frac{1}{n})\right]$$

$$\int_{-\infty}^\infty e^{it}(it)^{s-1}dt = 2\Gamma(s)\sin(\pi s)$$, $$\int_{-\infty}^\infty e^{-it}(it)^{s-1}dt = 0$$ for $0<\mathrm{Re}(s)<1$.

$$\tan(\frac{\pi}{2}s) = \frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(\frac{s}{2})\Gamma(1-\frac{s}{2})}$$, $$\pi^{(1-s)/2}\tan(\pi\frac{s}{2})\Gamma(\frac{s}{2}) = \frac{\Gamma(\frac{1-s}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1-\frac{s}{2})} = \underset{s\to 0}{\to}\pi^{3/2}$$

$$\frac{1}{2}\frac{\tan(\frac{\pi}{2}x)}{\frac{\pi}{2}x} = \frac{4}{\pi^2}\sum_1^\infty\frac{1}{(2k-1)^2-x^2}$$ $$\frac{1}{2}\frac{\cot(\frac{\pi}{2}x)}{\frac{\pi}{2}(x-1)} = \frac{4}{\pi^2}\sum_1^\infty\frac{1}{(2k-1)^2-(x-1)^2}$$

$$\Gamma(1+\frac{s}{2}) = \frac{s}{2}\Gamma(\frac{s}{2}) = \prod_1^\infty(1+\frac{s}{2n})^{-1}(1+\frac{1}{n})^{s/2}$$, $$\frac{1}{s\Gamma(s)} = e^{\gamma s}\prod_1^\infty(1+\frac{s}{n})e^{-s/n}$$, $$\Gamma(s) = \frac{1}{s}-\gamma+\ldots$$

$$\frac{d}{ds}\left[\frac{\Gamma(1+\frac{s}{2})}{s}\right] = -\sum_1^\infty\frac{d}{ds}\frac{\log(1+(s/2n))}{s}$$, $$\int_0^\infty e^{-x}\log x dx = -\frac{1}{4}(\gamma+2\log 2)\sqrt{\pi}$$

$$\gamma = -\Gamma'(1),\ \log\xi(0) = -\log(2)$$

$$\xi(0) = -\zeta(0) = \frac{1}{2}$$, $$\zeta'(0) = -\frac{1}{2}\log(2\pi)$$, $$\zeta(-1) = -\frac{1}{12}$$, $$\zeta(-3) = \frac{1}{120}$$

$$\zeta(2n) = \frac{(2\pi)^{2n}(-1)^{n+1}B_{2n}}{2(2n)!}$$, $$\zeta(-(2n-1)) = (-1)^{2n-1}\frac{B_{2n}}{2n}$$

$$\tan x = \sum_{1}^{\infty} \frac{(-1)^{n-1} 2^{2n} (2^{2n}-1) B_{2n}}{(2n)!} x^{2n-1} \quad , \quad |x| < \frac{\pi}{2} \quad , \quad \cot x = \sum_{0}^{\infty} \frac{(-1)^{n-1} 2^{2n} B_{2n}}{(2n)!} x^{2n-1}$$

$${}_1F_1(\alpha;\beta;z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{\binom{-\alpha}{n}}{\binom{-\beta}{n}} \frac{z^n}{n!} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)} \int_0^1 e^{zt} t^{\alpha-1} (1-t)^{\beta-\alpha-1} dt$$

$$J_0(z) e^{-iz} = {}_1F_1(\frac{1}{2};1;-2iz)$$

$$-\log(1-e^{-x}) = \sum_{1}^{\infty} \frac{\cos(nx) - i\sin(nx)}{n} \text{ for } 0 < x < 2\pi$$

$$\sum \frac{1}{n^s} \approx \frac{1}{s-1} \text{ for } s \to 1 \quad , \quad \sum \frac{\log n}{n^s} \approx \frac{1}{(s-1)^2} \text{ for } s \to 1$$

$$\sum \frac{2\gamma + \log n}{n^s} \approx \frac{1}{(s-1)^2} + \frac{2\gamma}{(s-1)} \text{ for } s \to 1$$

$$\int_0^{\infty} li(e^{-t}) Y_0(2\sqrt{tx}) dt = \frac{\gamma + \log x + e^x li(e^{-x})}{\pi x} \quad , \quad \int_0^{\infty} li(e^{-t}) J_0(2\sqrt{tx}) dt = \frac{e^{-t} - 1}{x}$$

$$\frac{\sin(\frac{\pi}{2}s)}{\pi} \int_0^{\infty} x^s \left[K_0(2x) - Y_0(2x)\right] \frac{dx}{x} = \frac{1}{\Gamma(1-\frac{s}{2})} \left[4\Gamma(\frac{s}{2}) - \frac{1}{2\pi} \frac{s}{2} \cos(\frac{\pi}{2}s)\right] = \frac{4\Gamma(\frac{s}{2})}{\Gamma(1-\frac{s}{2})} \left[1 - \frac{s}{(s-1)\Gamma(\frac{1-s}{2})\Gamma(s-1)} \frac{\sqrt{\pi}}{2^{4-s}}\right]$$

$$\log\tan(\frac{\pi}{2}s) = \int_0^1 \frac{t^s - t^{1-s}}{(1+t)\log t} \frac{dt}{t} \text{ for } 0 < \mathrm{Re}(s) < 1$$

$${}_1F_1(\frac{1}{2};\frac{3}{2},x) = \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{x^n}{n!} = x^{-3/4} e^{x/2} M_{\frac{1}{4},\frac{1}{4}}(x) = 2\int_0^x \sqrt{tx} e^t \frac{dt}{t} = \frac{e^z}{2} \int_0^{\infty} (tx)^{-1/4} J_{1/2}(2\sqrt{xt}) e^{-t} dt$$

$${}_1F_1(\frac{1}{2};\frac{3}{2},-x) = e^{-x}\,{}_1F_1(1;\frac{3}{2},x) = \int_0^{\infty} \frac{J_{1/2}(2\sqrt{xt})}{(2\sqrt{xt})^{1/2}} e^{-t^2} dt$$

$${}_1F_1(\frac{1}{2},\frac{1}{2};-\pi y^2) = \int_0^{\infty} \sqrt{x} J_{1/2}(2\pi yx) f(x) \frac{dx}{x}$$

$${}^*B(z) = {}_1F_1(\frac{1}{2},\frac{3}{2};z) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)} \frac{z^n}{n!} = \int_0^{\infty} x^{1/2} e^{-zx} \frac{dx}{(1+x)^{1/2}}$$

$$z^* B'(z) = z\,{}_1F_1'(\frac{1}{2},\frac{3}{2};z) = \sum_{n=0}^{\infty} \frac{n}{2n+1} \frac{z^n}{n!} = -\int_0^{\infty} x^{1/2} (xz) e^{-zx} \frac{dx}{(1+x)^{1/2}}$$

$$\int_0^{\infty} e^{-\pi x^2} dx = \frac{1}{\pi} \int_0^{\infty} \frac{\tan x}{x} dx = \frac{1}{2}$$

$$\int_0^{\pi/2} x \cot x dx = \int_0^{\pi/2} (\frac{\pi}{2} - x) \tan x dx = \frac{\pi}{2} \log 2 = \frac{\pi}{2} \left[1 - 2\sum_{1}^{\infty} \frac{\zeta(2n)}{4^n (2n+1)}\right]$$

$$\int_0^{\pi/2} (\frac{1}{x} - \cot x) dx = \log \frac{\pi}{2}$$

$$\int_0^{\pi/2} \frac{1 - x\cot x}{\sin^2 x} dx = \frac{\pi}{4} .$$