Computing the Hilbert Transform of the Generalized Laguerre and Hermite Weight Functions

Walter Gautschi and Jörg Waldvogel

Abstract. We give explicit formulae for the Hilbert transform $⨍_{\mathbb{R}} w(t)dt/(t-x)$, where w is either the generalized Laguerre weight function $w(t) = 0$ if $t \leq 0$, $w(t) = t^\alpha e^{-t}$ if $0 < t < \infty$, and $\alpha > -1$, $x > 0$, or the Hermite weight function $w(t) = e^{-t^2}$, $-\infty < t < \infty$, and $-\infty < x < \infty$. Furthermore, several numerical evaluation schemes are discussed, based on various representations of the objects under consideration. In this connection we study the numerical stability of the three-term recurrence relation satisfied by the integrals $⨍_{\mathbb{R}} \pi_n(t; w)w(t)dt/(t-x)$, $n = 0, 1, 2, \ldots$, where $\pi_n(\,\cdot\,; w)$ is the generalized Laguerre, resp. the Hermite, polynomial of degree n.

AMS Categories: 65D30, 65D32, 65R10.

0. Introduction

In [5] we considered the Hilbert transform of the Jacobi weight function and set forth a combination of analytic and numerical methods for its evaluation. In this paper we consider the Hilbert transform of the remaining classical weight functions, namely the generalized Laguerre, and the Hermite, weight function. For the former, unlike the Jacobi weight, no analytic results seem to exist in the literature (except for the ordinary Laguerre weight). Here we express it in terms of Tricomi's incomplete gamma function and also propose numerical methods using Gaussian quadrature rules, contour integration, and a recurrence relation. For the Hermite weight, the Hilbert transform is expressed in terms of Dawson's integral. A similar expression holds for the generalized Laguerre weight with parameter $\alpha = -\frac{1}{2}$.

1. The Generalized Laguerre Weight

The problem we wish to consider is to evaluate

$$I_\alpha(x) = \int_0^\infty \frac{t^\alpha e^{-t}}{t-x}\,dt, \quad \alpha > -1, \quad x > 0, \tag{1.1}$$

where the integral is taken in the sense of the Cauchy principal value. For $\alpha = 0$, the result is expressible in terms of an exponential integral (cf. [1, Ch.5]),

$$I_0(x) = -e^{-x} \int_{-\infty}^{x} \frac{e^t}{t}\,dt = -e^{-x}\,\mathrm{Ei}(x), \tag{1.2}$$

and for general α, as will be shown in §1.5, in terms of Tricomi's incomplete gamma function.

1.1. *Recurrence relation.* In principle it suffices to know $I_\alpha(x)$ for $0 < \alpha \leq 1$, since there is a simple relationship between $I_\alpha(x)$ and $I_{\alpha-1}(x)$. Indeed, from (1.1),writing

$$\frac{t^\alpha}{t-x} = \frac{t}{t-x}\,t^{\alpha-1} = \left(1 + \frac{x}{t-x}\right) t^{\alpha-1},$$

one obtains the recurrence relation

$$I_\alpha(x) = \Gamma(\alpha) + x I_{\alpha-1}(x). \tag{1.3}$$

If $0 < \alpha \leq 1$ and the left side is known, we can solve for the last term on the right and obtain $I_{\alpha-1}(x)$ for $-1 < \alpha - 1 \leq 0$. For $\alpha = 1$, we simply have $I_1(x) = 1 + xI_0(x) = 1 - xe^{-x}\,\mathrm{Ei}(x)$. To compute $I_{\alpha+n}(x)$ for $0 < \alpha \leq 1$ and $n = 1, 2, \ldots$, let

$$y_n = \frac{I_{\alpha+n}(x)}{\Gamma(\alpha+n)}, \quad n = 0, 1, 2, \ldots, \quad 0 < \alpha \leq 1. \tag{1.4}$$

Then (1.3) yields the first-order inhomogeneous difference equation

$$\begin{aligned} y_n &= 1 + \frac{x}{\alpha+n-1}\,y_{n-1}, \quad n = 1, 2, 3, \ldots, \\ y_0 &= I_\alpha(x)/\Gamma(\alpha). \end{aligned} \tag{1.5}$$

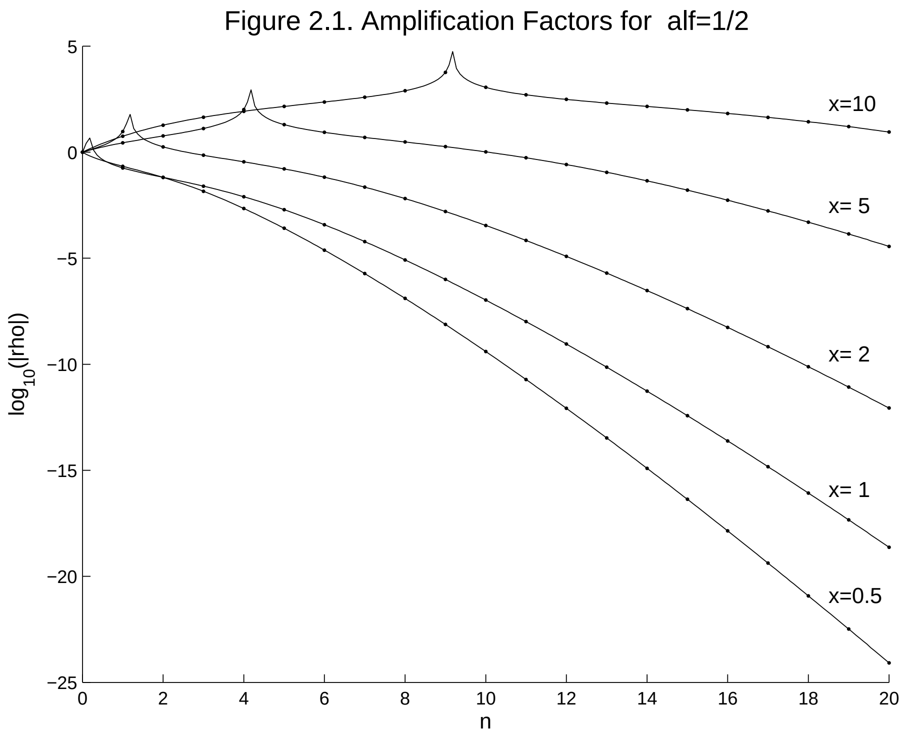

Figure 1: Amplification factors for the recursion (1.9)

This recursion is quite stable, as can be seen from the behavior of the "amplification factors"

$$\rho_n = \frac{y_0 h_n}{y_n}, \quad h_n = \frac{x^n \Gamma(\alpha)}{\Gamma(\alpha+n)}. \tag{1.6}$$

See Fig. 1, where they are plotted on a logarithmic scale for $\alpha = \frac{1}{2}$ and $x = .5, 1, 2, 5, 10$. The value $\alpha = \frac{1}{2}$ is representative for all other values of α in the interval $(0, 1]$, even very small ones. The quantity h_n in (1.6) is the solution, with $h_0 = 1$, of the homogeneous recurrence relation associated with (1.5).

The significance of the ρ's (cf., e.g., [7]) is that ρ_t/ρ_s measures the amplification (or damping, as it were) at $n = t$ of a small relative error committed at $n = s$.

1.2. *Gaussian quadrature.* We write

$$I_\alpha(x) = \int_0^\infty \frac{t^\alpha - x^\alpha}{t-x}\, e^{-t}dt + x^\alpha \fint_0^\infty \frac{e^{-t}}{t-x}\, dt,$$

that is, in view of (1.2),

$$I_\alpha(x) = \int_0^\infty \frac{t^\alpha - x^\alpha}{t-x}\, e^{-t}dt - x^\alpha e^{-x}\,\mathrm{Ei}(x). \tag{1.7}$$

Although the integral on the right is an ordinary integral, it requires some care to evaluate.

We first make the change of variables $t = (1+\tau)x$ and write

$$\int_0^\infty \frac{t^\alpha - x^\alpha}{t-x}\, e^{-t}dt = x^\alpha e^{-x} \int_{-1}^\infty \frac{(1+\tau)^\alpha - 1}{\tau}\, e^{-\tau x} d\tau.$$

Splitting the integral on the right in two parts and changing variables in an obvious manner yields

$$\begin{aligned}\int_0^\infty \frac{t^\alpha - x^\alpha}{t-x}\, e^{-t}dt = x^\alpha e^{-x} \Bigg\{ \int_0^1 &\frac{1-(1-t)^\alpha}{t}\, e^{tx}dt \\ &+ \frac{1}{x}\int_0^\infty \frac{(1+t/x)^\alpha - 1}{t/x}\, e^{-t}dt \Bigg\}.\end{aligned}$$

It is convenient to introduce the function

$$h_\alpha(u) = \frac{(1+u)^\alpha - 1}{u}, \quad u > 0, \quad \alpha > -1, \tag{1.8}$$

in terms of which

$$\int_0^\infty \frac{t^\alpha - x^\alpha}{t-x}\, e^{-t}dt = x^\alpha e^{-x} \left\{ \int_0^1 h_\alpha(-t) e^{tx}dt + \frac{1}{x}\int_0^\infty h_\alpha(t/x) e^{-t}dt \right\}.$$

For computational purposes, the first integral on the right is split into one from 0 to $\frac{1}{2}$ and another from $\frac{1}{2}$ to 1. This yields the final formula to be used as a basis of our computation,

$$\begin{aligned} I_\alpha(x) = x^\alpha e^{-x} &\Bigg\{ \int_0^{1/2} h_\alpha(-t) e^{tx}dt + \int_{1/2}^1 \frac{e^{tx}}{t} dt \\ &- \int_{1/2}^1 \frac{e^{tx}}{t}\,(1-t)^\alpha dt + \frac{1}{x}\int_0^\infty h_\alpha(t/x) e^{-t}dt - \mathrm{Ei}(x) \Bigg\}. \end{aligned} \tag{1.9}$$

The function $h_\alpha(u)$ occurring in this formula is easily and accurately evaluated by Taylor expansion, when $|u| \le \frac{1}{2}$,

$$\frac{(1+u)^\alpha - 1}{u} = \sum_{k=0}^{\infty} \binom{\alpha}{k+1} u^k, \tag{1.10}$$

and directly otherwise. The first two integrals in (1.9) are now computed by Gauss-Legendre quadrature respectively on the interval $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$, the third integral by Gauss-Jacobi quadrature on the interval $[\frac{1}{2}, 1]$ with Jacobi parameters α and $\beta = 0$, and the fourth by Gauss-Laguerre quadrature. For the exponential integral $\mathrm{Ei}(x)$ rational approximations [2] and software [9] are available, but straightforward Taylor expansion of $\mathrm{Ei}(x) - \gamma - \ln x$ is also quite feasible and accurate, at least for x not excessively large, say less than 100.

If $\alpha = -\frac{1}{2}$, the integral in (1.1), by a change of variables $t \mapsto t^2$, becomes

$$I_{-\frac{1}{2}}(x) = ⨍_{-\infty}^{\infty} \frac{e^{-t^2}}{t^2 - \xi^2}\, dt, \quad \xi = \sqrt{x}.$$

Using here

$$\frac{1}{t^2 - \xi^2} = \frac{1}{2\xi}\left(\frac{1}{t-\xi} - \frac{1}{t+\xi}\right),$$

and changing the variable of integration from t to $-t$ in the integral corresponding to the second term, yields

$$I_{-\frac{1}{2}}(x) = \frac{1}{\xi} ⨍_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\xi}\, dt, \tag{1.11}$$

where the integral on the right is the Hilbert transform $I(\xi)$ of the Hermite weight (cf. (2.1) below). By virtue of (2.6), we therefore get

$$I_{-\frac{1}{2}}(x) = -2\sqrt{\frac{\pi}{x}}\, F(\sqrt{x}), \tag{1.12}$$

with F denoting Dawson's integral.

1.3. *Numerical examples.* The procedure based on (1.9) was programmed and run in IEEE double precision on a Sun SPARCstation 2. The various

Gaussian quadrature rules required were generated by the routines `drecur` and `dgauss` of [6]. In order to obtain an idea of how fast these quadrature rules converge, we printed for each quadrature the smallest value n^* of n for which the n-point and $(n-1)$-point rule agreed within a relative accuracy ε. For the first three integrals in (1.9) we chose $\varepsilon = 10^2 \times$eps, where eps is the machine (double) precision, and for the last integral (which has a tendency to converge more slowly) we chose $\varepsilon = 10^3 \times$eps. (In single precision, the

			single				double			
x	alpha	hilbert	n1	n2	n3	n4	n1	n2	n3	n4
0.50	0.10	-0.645766100214D-01	5	6	6	21	10	11	11	92
0.50	0.30	0.256532993500D+00	5	6	5	20	10	11	11	87
0.50	0.50	0.487817480185D+00	5	6	6	18	9	11	11	82
0.50	0.70	0.662529121674D+00	4	6	6	17	9	11	11	76
0.50	0.90	0.801524138808D+00	4	6	5	13	9	11	11	67
1.50	0.10	-0.332759050435D+00	6	6	5	11	10	10	10	38
1.50	0.30	-0.171460868311D+00	5	6	5	10	10	10	10	36
1.50	0.50	-0.216602175465D-01	5	6	5	10	10	10	10	34
1.50	0.70	0.127377364345D+00	5	6	5	9	9	10	10	32
1.50	0.90	0.283237714160D+00	4	6	5	7	9	10	10	28
4.50	0.10	-0.805375913333D-02	6	6	5	7	11	10	10	17
4.50	0.30	0.955567141732D-01	6	6	5	7	10	10	10	16
4.50	0.50	0.223371446826D+00	6	6	5	5	10	10	10	16
4.50	0.70	0.389899605377D+00	6	6	5	5	10	10	10	15
4.50	0.90	0.612817575593D+00	6	6	5	5	10	10	10	14
13.50	0.10	0.271303905850D-01	9	9	8	5	12	11	11	9
13.50	0.30	0.101560021816D+00	9	9	9	5	12	11	11	9
13.50	0.50	0.221256809530D+00	9	9	7	5	12	11	11	9
13.50	0.70	0.420091222360D+00	9	9	7	4	12	11	11	8
13.50	0.90	0.754188711212D+00	9	9	9	3	11	11	11	8

TABLE 1.1. Numerical results for $I_\alpha(x)$

value taken for ε was $10 \times$eps for all integrals, with eps the machine single precision.) Table 1.1 shows the results for selected values of x and α. The first four integers after the double-precision result for $I_\alpha(x)$ are the values of n^* in single precision for the four integrals of (1.9), the next four integers the analogous values of n^* for double precision.

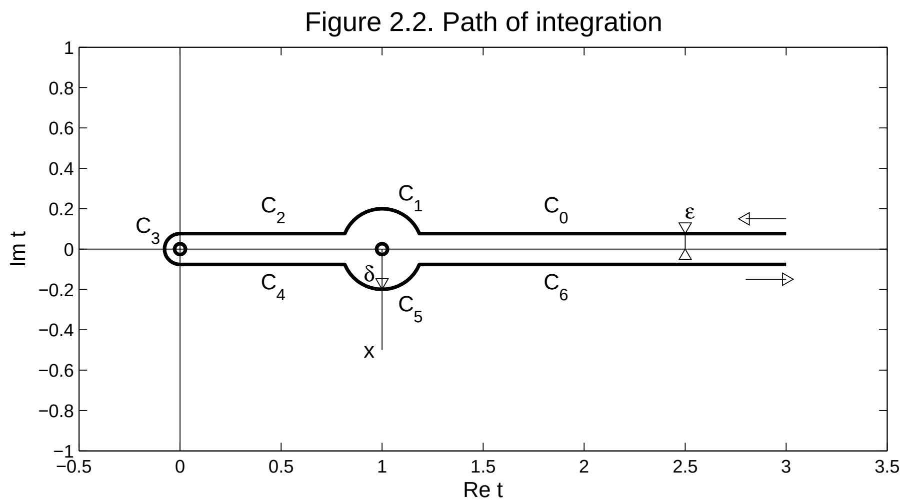

Figure 2: Path C ofintegration (for $x = 1$).

Generally, n^* is quite small except for the last integral, particularly when x and α are both small. In this case the function $h_\alpha(t/x)$ is quite steep near the origin, the derivative at $t = 0$ being $\alpha(\alpha - 1)/(2x)$.

1.4. *Contour integration.* Closely related to the integral $I_\alpha(x)$ is the contour integral

(1.13) $$G_\alpha(x) = \frac{1}{2\pi i}\int_C \frac{(-t)^\alpha e^{-t}}{t - x}dt,$$

where the contour C starts at $+\infty + i\epsilon$, $\epsilon > 0$, encircles the origin once counterclockwise, and returns to $+\infty - i\epsilon$. In order to compute this integral, we decompose the path of integration into $C = C_0 \cup C_1 \cup \cdots \cup C_6$ as shown in Fig. 2. For brevity we write

(1.14) $$f_\alpha(t) = \frac{(-t)^\alpha e^{-t}}{t - x}, \quad t \in \mathbb{C} \setminus \mathbb{R}_+$$

(suppressing in the notation the dependence on x) and restrict t to the cut plane as shown. The αth power is understood in the sense of the principal value. We then have the following limit relations, as ϵ and δ tend to zero in an appropriate manner, assuming $\alpha > -1$:

$$\begin{aligned}
&\lim \frac{1}{2\pi i} \int_{C_0 \cup C_2} f_\alpha(t)dt = -\frac{1}{2\pi i} e^{-i\alpha\pi} I_\alpha(x), \\
&\lim \frac{1}{2\pi i} \int_{C_4 \cup C_6} f_\alpha(t)dt = \frac{1}{2\pi i} e^{i\alpha\pi} I_\alpha(x), \\
&\lim \frac{1}{2\pi i} \int_{C_3} f_\alpha(t)dt = 0, \\
&\lim \frac{1}{2\pi i} \int_{C_1} f_\alpha(t)dt = \tfrac{1}{2} e^{-i\alpha\pi} x^\alpha e^{-x},
\end{aligned}$$

and

$$\lim \frac{1}{2\pi i} \int_{C_5} f_\alpha(t)dt = \tfrac{1}{2} e^{i\alpha\pi} x^\alpha e^{-x}.$$

Adding up all contributions yields

$$G_\alpha(x) = \frac{\sin \alpha\pi}{\pi} I_\alpha(x) + \cos \alpha\pi \cdot x^\alpha e^{-x}. \tag{1.15}$$

We remark that the definition (1.13) of $G_\alpha(x)$ makes sense for arbitrary $\alpha \in \mathbb{C}$. Moreover, the path C may be deformed to any path $\tilde{C}$ in $\mathbb{C} \setminus \mathbb{R}_+$ without changing the value of $G_\alpha(x)$ as long as $\tilde{C}$ enters through the first quadrant and leaves the fourth (with $\mathrm{Re}\, t \to \infty$). Therefore, the variable x need not be confined to $\mathbb{R}_+$ but can be arbitrary complex, provided the path $\tilde{C}$ is chosen to leave x on its left as it is run through. Consequently, $G_\alpha(x)$ is an entire function in both variables α and x. It will be identified in the next subsection.

Eq. (1.15) may serve to define the analytic continuation of $I_\alpha(x)$ to every $x \in \mathbb{C}$ and $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, by solving (1.15) for $I_\alpha(x)$:

$$I_\alpha(x) = \frac{\pi}{\sin \alpha\pi} \left[G_\alpha(x) - \cos \alpha\pi \cdot x^\alpha e^{-x} \right]. \tag{1.16}$$

When $\alpha \to n$, $n \in \mathbb{Z}$, since $I_\alpha(x)$ remains finite, one needs to use the rule of Bernoulli-L'Hospital, with

$$\left. \frac{\partial}{\partial \alpha} I_\alpha(x) \right|_{\alpha=n} = \frac{1}{2\pi i} \int_{\tilde{C}} f_n(t) \log(-t) dt. \tag{1.17}$$

The formula (1.16) lends itself also to numerical evaluation. Assuming $x > 0$, $\alpha > -1$, we take a path $C = \tilde{C}$ in (1.13) defined by $t = s^2$, $s = \sigma + i s_0$,

$-\infty < \sigma < \infty$, $s_0 > 0$, which represents a parabola with focus at $t = 0$ opening in the direction of the positive real axis. The integral

$$G_\alpha(x) = \frac{1}{2\pi i}\int_{\tilde{C}} f_\alpha(t)dt = \frac{1}{\pi i}\int_{\infty}^{-\infty} f_\alpha((\sigma + is_0)^2)(\sigma + is_0)d\sigma$$

is conveniently approximated by the composite trapezoidal rule, with step $h > 0$,

$$G_\alpha(x) = -\frac{h}{\pi i}\sum_{k=-\infty}^{\infty} f_\alpha((kh + is_0)^2)(kh + is_0) + R(h), \tag{1.18}$$

which, because of the exponential decay of the integrand at both ends of the path, has an error term going to zero exponentially fast in h^{-1}, i.e.,

$$R(h) = O(e^{-\gamma/h}), \quad h \to 0, \tag{1.19}$$

for some $\gamma > 0$ (cf. [4, Eq.]). Moreover, it suffices to approximate the infinite sum in (1.18) by a finite sum which, by symmetry, can be taken twice the sum from $k = 1$ to n plus the term for $k = 0$. Here, n is typically a relatively small positive integer. For example, if $\alpha = \frac{1}{32}$, $x = 1$, taking $s_0 = 3$, we achieve a relative accuracy of 28 decimal digits with $h = \frac{17}{64}$ and $n = 38$. (The calculations were done with the symbolic package Pari.)

1.5. *Closed-form expression for $G_\alpha(x)$.* We now show that

$$G_\alpha(x) = e^{-x}\gamma^*(-\alpha, -x), \tag{1.20}$$

where γ^* is the incomplete gamma function as defined by Tricomi (cf. [8]), an entire function in both of its variables. It has the power series expansion

$$e^z\gamma^*(a, z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(a + k + 1)}, \quad a \in \mathbb{C},\ z \in \mathbb{C}. \tag{1.21}$$

We choose a contour $C = \tilde{C}$ in (1.13) such that the geometric series

$$\frac{1}{t - x} = \sum_{k=0}^{\infty} x^k t^{-(1+k)}$$

converges for every $t \in \tilde{C}$. Substitution in (1.13) yields

$$G_\alpha(x) = \sum_{k=0}^{\infty}(-x)^k \cdot \frac{-1}{2\pi i}\int_{\tilde{C}}(-t)^{\alpha-k-1}e^{-t}dt.$$

The integral on the right can be evaluated by the same integration technique employed in §1.4. Using, in addition, the reflection formula for the gamma function,

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

with $z = k - \alpha + 1$, we find

$$-\frac{1}{2\pi i}\int_{\tilde{C}} (-t)^{\alpha-k-1} e^{-t} dt = \Gamma(\alpha - k)\frac{1}{\pi}\sin(k-\alpha+1)\pi = \frac{1}{\Gamma(k-\alpha+1)},$$

and thus

$$G_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-x)^k}{\Gamma(k-\alpha+1)}\,.$$

Comparison with (1.21) shows the validity of (1.20).

For $\alpha = \pm n$, $n \in \mathbb{N}_0$, the function $\gamma^*(-\alpha, -x)$ is elementary (cf. [8, Eq]), giving

$$\begin{aligned} G_n(x) &= (-x)^n e^{-x}, \\ G_{-n} &= \frac{(-1)^n}{x^n}[e^{-x} - e_{n-1}(-x)], \end{aligned}$$

where e_{n-1} is the $(n-1)$st partial sum of the exponential series. We also note that there are efficient expansions of $\gamma^*(-\alpha, -x)$ in terms of Bessel functions ([8, Eqs]). The function γ^* is included among the special function routines in the package Pari.

2. The Hermite Weight

The problem now is to compute

$$I(x) = ⨍_{-\infty}^{\infty} \frac{e^{-t^2}}{t-x}\, dt, \quad x \in \mathbb{R}. \tag{2.1}$$

Since $I(-x) = -I(x)$, it suffices to consider positive values of x.

By definition of the Cauchy principal value integral, we have

$$\begin{aligned} I(x) &= \lim_{\varepsilon \downarrow 0}\left(\int_{-\infty}^{x-\varepsilon} + \int_{x+\varepsilon}^{\infty}\right)\frac{e^{-t^2}}{t-x}\, dt \\ &= \lim_{\varepsilon \downarrow 0}\int_{\varepsilon}^{\infty} \frac{-e^{-(x-t)^2} + e^{-(x+t)^2}}{t}\, dt \\ &= -e^{-x^2}\int_0^{\infty} \frac{e^{2xt} - e^{-2xt}}{t}\, e^{-t^2} dt, \end{aligned}$$

that is,

$$I(x) = -4xe^{-x^2} \int_0^\infty \frac{\sinh(2xt)}{2xt}\, e^{-t^2} dt. \tag{2.2}$$

Using Taylor expansion of the hyperbolic sine, term-by-term integration, and $\int_0^\infty t^{2k} e^{-t^2} dt = \frac{1}{2}\Gamma(k+\frac{1}{2})$, yields

$$\int_0^\infty \frac{\sinh(2xt)}{2xt}\, e^{-t^2} dt = \frac{1}{2} \sum_{k=0}^\infty \frac{\Gamma(k+\frac{1}{2})}{\Gamma(2k+2)} (2x)^{2k}. \tag{2.3}$$

The duplication formula for the gamma function,

$$\Gamma(2k+2) = \frac{2^{2k+3/2}}{\sqrt{2\pi}} \Gamma(k+1)\Gamma(k+\tfrac{3}{2}),$$

allows us to write (2.3) in the form

$$\int_0^\infty \frac{\sinh(2xt)}{2xt}\, e^{-t^2} dt = \frac{\sqrt{\pi}}{4} \sum_{k=0}^\infty \frac{1}{k+\frac{1}{2}} \frac{x^{2k}}{k!}. \tag{2.4}$$

The series on the right can be recognized as the error function of a purely imaginary argument, namely [1, Eq. 7.1.5]

$$\sum_{k=0}^\infty \frac{1}{k+\frac{1}{2}} \frac{x^{2k}}{k!} = \sqrt{\pi}\, \frac{\operatorname{erf} ix}{ix} = \frac{2}{x} \int_0^x e^{t^2} dt. \tag{2.5}$$

Thus, substitution of (2.4) and (2.5) in (2.2) expresses the Hilbert transform in terms of Dawson's integral,

$$I(x) = -2\sqrt{\pi} F(x), \quad F(x) = e^{-x^2} \int_0^x e^{t^2} dt. \tag{2.6}$$

¿From the asymptotic expansion [1, Eq. 7.1.23] of the error function one finds

$$I(x) \sim -\frac{\sqrt{\pi}}{x} \left(1 + \sum_{k=1}^\infty \frac{1 \cdot 3 \cdots (2k-1)}{(2x^2)^k} \right) \quad \text{as } x \to \infty. \tag{2.7}$$

For computation, one can use the series expansion (2.4) in (2.2) when x is small or moderately large, and the asymptotic expansion (2.7) when x is large. For single-precision accuracy, $x = 5$ may be taken as the separation point, for double precision $x = 7.5$. Results thus obtained are displayed in Table 2.1. The last two columns show the number of terms required in single resp. double precision for the series in (2.4), or the series in (2.7), to yield full machine precision.

x	hilbert	n	nd
0.50	-0.15045878048051D+01	6	12
1.00	-0.19074421882418D+01	10	17
2.00	-0.10682238655627D+01	18	29
5.00	-0.36205586704396D+00	8	76
7.50	-0.23848654284464D+00	5	17
10.00	-0.17814524994095D+00	5	12
50.00	-0.35456171091663D-01	3	6
100.00	-0.17725424868948D-01	2	5
200.00	-0.88623800370477D-02	2	4

TABLE 2.1. Numerical results for $I(x)$

There are also rational approximations for $F(x)$ that could be used, e.g., those in [3].

Following the Eulerian habit of providing alternative proofs, here is a complex-variable derivation of (2.6) using the Cauchy transform

$$C(z) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-z}\, dt, \quad z \in \mathbb{C} \setminus \mathbb{R}, \tag{2.8}$$

and the well-known fact that

$$I(x) = \lim_{\epsilon \downarrow 0} \tfrac{1}{2}[C(x+i\epsilon) + C(x-i\epsilon)]. \tag{2.9}$$

By [1, Eq. 7.1.4] we have, for $\epsilon > 0$,

$$C(x+i\epsilon) = \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-x-i\epsilon}\, dt = i\pi w(x+i\epsilon), \tag{2.10}$$

where

$$w(z) = e^{-z^2}\left(1 + \frac{2i}{\sqrt{\pi}} \int_0^z e^{t^2}\, dt\right) = e^{-z^2} \operatorname{erfc}(-iz)$$

is the complex error function [1, Eq. 7.1.3]. Multiplying (2.10) by -1 and making the change of variables $t \mapsto -t$ in the resulting integral yields

$$C(x - i\epsilon) = -i\pi w(-x + i\epsilon). \tag{2.11}$$

Therefore, the average of (2.10) and (2.11) becomes

$$\tfrac{1}{2}[C(x + i\epsilon) + C(x - i\epsilon)] = \tfrac{1}{2} i\pi[w(x + i\epsilon) - w(-x + i\epsilon)].$$

Now apply $w(-z) = 2e^{-z^2} - w(z)$ (cf. [1, Eq. 7.1.11]) with $z = x - i\epsilon$ to obtain

$$\tfrac{1}{2}[C(x + i\epsilon) + C(x - i\epsilon)] = \tfrac{1}{2} i\pi[w(x + i\epsilon) - 2e^{-(x-i\epsilon)^2} + w(x - i\epsilon)].$$

Going to the limit $\epsilon \downarrow 0$, and using (2.9), we find

$$\begin{aligned} I(x) &= i\pi[w(x) - e^{-x^2}] \\ &= i\pi\left[e^{-x^2}\left(1 + \frac{2i}{\sqrt{\pi}}\int_0^x e^{t^2}dt\right) - e^{-x^2}\right] \\ &= -2\sqrt{\pi}F(x), \end{aligned}$$

in agreement with (2.6).

3. The Hilbert Transform of the Generalized Laguerre and Hermite Polynomials; Pseudostability of the Three-Term Recurrence Relation

Let $\{\pi_n(t; w)\}$ denote the (monic) orthogonal polynomials relative to the weight function w, and x be a point in the interior of the support of w. The Cauchy principal value integrals

$$\rho_n(x) = ⨍_{\mathbb{R}} \frac{\pi_n(t; w)}{t - x} w(t)dt, \quad n = 0, 1, 2, \ldots, \tag{3.1}$$

are of interest in connection with singular integral equations. They satisfy the same three-term recurrence relation as the orthogonal polynomials themselves, namely

$$y_{k+1} = (x - \alpha_k)y_k - \beta_k y_{k-1}, \quad k = 0, 1, 2, \ldots, \tag{3.2}$$

but with initial values

$$y_{-1} = -1, \quad y_0 = \fint_{\mathbb{R}} \frac{w(t)}{t-x}\, dt. \tag{3.3}$$

(It is assumed in (3.2) that $\beta_0 = \int_{\mathbb{R}} w(t)dt$.) Let $\{z_k\}$ be a second solution of (3.2) defined by

$$z_{-1} = y_0, \quad z_0 = 1. \tag{3.4}$$

It is shown in [5, §4] that the amplification of relative error in y_k due to small relative errors in y_0 and y_1 can be measured by the quantity

$$\omega_k = \frac{|z_1/y_1 - z_k/y_k| + |z_0/y_0 - z_k/y_k|}{|z_1/y_1 - z_0/y_0|}. \tag{3.5}$$

In this section, we wish to observe the behavior of $\{\omega_k\}$ in the two cases of the generalized Laguerre weight and the Hermite weight.

For the weight $w(t) = t^\alpha e^{-t}$, $0 < t < \infty$, it appears that serious growth of ω_k is most likely to occur when $x > 0$ in (3.1) is small and α large. We illustrate this by computing ω_k for $1 \le k \le 50$ in the case $\alpha = 10$ for selected values of x. The results, for $x = .5, 1, 2$, and 5, are plotted in the left frame of Fig. 3 with a logarithmic scale on the vertical axis. It would appear, but

Figure 3: Amplification factors ω_k for generalized Laguerre and Hermite weights

is not known, that ω_k remains finite as $k \to \infty$. Yet, Fig. 3 shows that it can assume rather large values — many decimal orders of magnitude — a phenomenon called pseudostability in [5]. A similar phenomenon takes place for the Hermite weight function $w(t) = e^{-t^2}$, $-\infty < t < \infty$, as is shown in the right frame of Fig. 3 for $x = 1, 3$, and 5. Here, it is large absolute values of x that give rise to pseudostability.

References

[1] Abramowitz, M. and Stegun, I.A. 1964. *Handbook of mathematical functions*, NBS Appl. Math. Ser., v. 55, U.S. Government Printing Office, Washington, DC.

[2] Cody, W.J. and Thacher, H.C., Jr. 1969. Chebyshev approximations for the exponential integral Ei(x), *Math. Comp. 23*, 289–303.

[3] Cody, W.J., Paciorek, K.A., and Thacher, H.C., Jr. 1970. Chebyshev approximations for Dawson's integral, *Math. Comp. 24*, 171–178.

[4] Davis, P.J and Rabinowitz, P 1984. *Methods of numerical integration*, 2d ed., Academic Press, Orlando, FL.

[5] Gautschi, W. and Wimp, J. 1987. Computing the Hilbert transform of a Jacobi weight function, *BIT 27*, 203–215.

[6] Gautschi, W. 1994. Algorithm 726: ORTHPOL — A package of routines for generating orthogonal polynomials and Gauss-type quadrature rules, *ACM Trans. Math. Software 20*, 21–62.

[7] Gautschi, W. 1997. The computation of special functions by linear difference equations. In *Advances in difference equations*, S. Elaydi, I. Győri, and G. Ladas, Eds., Gordon and Breach, Amsterdam, 213–243.

[8] Gautschi, W. 1998. The incomplete gamma function since Tricomi, in *Tricomi's ideas and contemporary applied mathematics*, Atti dei Convegni Lincei, n. 147, Accademia Nazionale dei Lincei, Roma, 203–237.

[9] Paciorek, K.A. 1970. Algorithm 385 — Exponential integral Ei(x), *Comm. ACM 13*, 446–447. Certification and Remarks, *ibid. 13*, 448–449, *15* (1972), 1074.