Feferman, Solomon, et al. Kurt Gödel: Collected Works, Vol. II N.Y Oxford U.P 1995

GÖDEL

# What is Cantor's continuum problem? (*1964*)



⟦This article is a revised and expanded version of *Gödel 1947*. The introductory note to both *1947* and *1964* is found on page 154, immediately preceding *1947*.⟧

## 1. The concept of cardinal number

Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space? An equivalent question is: How many different sets of integers do there exist?

This question, of course, could arise only after the concept of "number" had been extended to infinite sets; hence it might be doubted if this extension can be effected in a uniquely determined manner and if, therefore, the statement of the problem in the simple terms used above is justified. Closer examination, however, shows that Cantor's definition of infinite numbers really has this character of uniqueness. For whatever "number" as applied to infinite sets may mean, we certainly want it to have the property that the number of objects belonging to some class does not change if, leaving the objects the same, one changes in any way whatsoever their properties or mutual relations (e.g., their colors or their distribution in space). From this, however, it follows at once that two sets (at least two sets of changeable objects of the space-time world) will have the same cardinal number if their elements can be brought into a one-to-one correspondence, which is Cantor's definition of equality between numbers. For if there exists such a correspondence for two sets $A$ and $B$ it is possible (at least theoretically) to change the properties and relations of each element of $A$ into those of the corresponding element of $B$, whereby $A$ is transformed into a set completely indistinguishable from $B$, hence of the same cardinal number. For example, assuming a square and a line segment both completely filled with mass points (so that at each point of them exactly one mass point
259 is situated), it follows, owing to the | demonstrable fact that there exists a one-to-one correspondence between the points of a square and of a line segment and, therefore, also between the corresponding mass points, that the mass points of the square can be so rearranged as exactly to fill out the line segment, and vice versa. Such considerations, it is true, apply directly only to physical objects, but a definition of the concept of "number" which would depend on the kind of objects that are numbered could hardly be considered to be satisfactory.



So there is hardly any choice left but to accept Cantor's definition of equality between numbers, which can easily be extended to a definition of "greater" and "less" for infinite numbers by stipulating that the cardinal number $M$ of a set $A$ is to be called less than the cardinal number $N$ of a set $B$ if $M$ is different from $N$ but equal to the cardinal number of some subset of $B$. That a cardinal number having a certain property exists is defined to mean that a set of such a cardinal number exists. On the basis of these definitions, it becomes possible to prove that there exist infinitely many different infinite cardinal numbers or "powers", and that, in particular, the number of subsets of a set is always greater than the number of its elements; furthermore, it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite numbers (including sums and products with any infinite number of terms or factors) and to prove practically all ordinary rules of computation.

But, even after that, the problem of identifying the cardinal number of an individual set, such as the linear continuum, would not be well-defined if there did not exist some systematic representation of the infinite cardinal numbers, comparable to the decimal notation of the integers. Such a systematic representation, however, does exist, owing to the theorem that for each cardinal number and each set of cardinal numbers[1] there exists exactly one cardinal number immediately succeeding in magnitude and that the cardinal number of every set occurs in the series thus obtained.[2] This theorem makes it possible to denote the cardinal number immediately succeeding the set of finite numbers by $\aleph_0$ (which is the power of the "denumerably infinite" sets), the next one by $\aleph_1$, etc.; the one immediately succeeding all $\aleph_i$ (where $i$ is an integer) by $\aleph_\omega$, the next one by $\aleph_{\omega+1}$, etc. The theory of ordinal numbers provides the means for extending this series further and further.

[1] As to the question of why there does not exist a set of all cardinal numbers, see footnote 15.

[2] The axiom of choice is needed for the proof of this theorem (see *Fraenkel and Bar-Hillel 1958*). But it may be said that this axiom, from almost every possible point of view, is as well-founded today as the other axioms of set theory. It has been proved consistent with the other axioms of set theory which are usually assumed, provided that these other axioms are consistent (see my *1940*). Moreover, it is possible to define in terms of any system of objects satisfying the other axioms a system of objects satisfying those axioms *and* the axiom of choice. Finally, the axiom of choice is just as evident as the other set-theoretical axioms for the "pure" concept of set explained in footnote 14.

## 260 | 2. The continuum problem, the continuum hypothesis, and the partial results concerning its truth obtained so far

So the analysis of the phrase "how many" unambiguously leads to a definite meaning for the question stated in the second line of this paper: The problem is to find out which one of the $\aleph$'s is the number of points of a straight line or (which is the same) of any other continuum (of any number of dimensions) in a Euclidean space. Cantor, after having proved that this number is greater than $\aleph_0$, conjectured that it is $\aleph_1$. An equivalent proposition is this: Any infinite subset of the continuum has the power either of the set of integers or of the whole continuum. This is Cantor's continuum hypothesis.

But, although Cantor's set theory now has had a development of more than seventy years and the problem evidently is of great importance for it, nothing has been proved so far about the question what the power of the continuum is or whether its subsets satisfy the condition just stated, except (1) that the power of the continuum is not a cardinal number of a certain special kind, namely, not a limit of denumerably many smaller cardinal numbers,[3] and (2) that the proposition just mentioned about the subsets of the continuum is true for a certain infinitesimal fraction of these subsets, the analytic[4] sets.[5] Not even an upper bound, however large, can be assigned for the power of the continuum. Nor is the quality of the cardinal number of the continuum known any better than its quantity. It is undecided whether this number is regular or singular, accessible or inaccessible, and (except for König's negative result) what its character of cofinality (see footnote 4) is. The only thing that is known, in addition to the results just mentioned, is a great number of consequences of, and some propositions equivalent to, Cantor's conjecture.[6]

This pronounced failure becomes still more striking if the problem is considered in its connection with general questions of cardinal arithmetic. It is easily proved that the power of the continuum is equal to $2^{\aleph_0}$. So the continuum problem turns out to be a question from the "multiplication table" of cardinal numbers, namely, the problem of evaluating a certain

[3] See *Hausdorff 1914*, p. 68, or *Bachmann 1955*, p. 167. The discoverer of this theorem, J. König, asserted more than he had actually proved (see his *1905*.)

[4] See the list of definitions on pp. 268–9.

[5] See *Hausdorff 1935*, p. 32. Even for complements of analytic sets the question is undecided at present, and it can be proved only that they either have the power $\aleph_0$ or $\aleph_1$ or that of the continuum or are finite (see *Kuratowski 1933*, p. 246.)

[6] See *Sierpiński 1934* and *1956*.



infinite product (in fact the simplest non-trivial one that can be formed). There is, however, not one infinite product (of factors $> 1$) for which so much as an upper bound for its value can be assigned. All one knows about the evaluation of infinite products are two lower bounds due to Cantor and König (the latter of which implies the aforementioned negative theorem on the power of | the continuum), and some theorems concerning 261
the reduction of products with different factors to exponentiations and of exponentiations to exponentiations with smaller bases or exponents. These theorems reduce[7] the whole problem of computing infinite products to the evaluation of $\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)}$ and the performance of certain fundamental operations on ordinal numbers, such as determining the limit of a series of them. All products and powers can easily be computed[8] if the "generalized continuum hypothesis" is assumed, i.e., if it is assumed that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\alpha$, or, in other terms, that the number of subsets of a set of power $\aleph_\alpha$ is $\aleph_{\alpha+1}$. But, without making any undemonstrated assumption, it is not even known whether or not $m < n$ implies $2^m < 2^n$ (although it is trivial that it implies $2^m \leq 2^n$), nor even whether $2^{\aleph_0} < 2^{\aleph_1}$.

## 3. Restatement of the problem on the basis of an analysis of the foundations of set theory and results obtained along these lines

This scarcity of results, even as to the most fundamental questions in this field, to some extent may be due to purely mathematical difficulties; it seems, however (see Section 4), that there are also deeper reasons involved and that a complete solution of these problems can be obtained only by a more profound analysis (than mathematics is accustomed to giving) of the meanings of the terms occurring in them (such as "set", "one-to-one correspondence", etc.) and of the axioms underlying their use. Several such analyses have already been proposed. Let us see then what they give for our problem.

First of all there is Brouwer's intuitionism, which is utterly destructive in its results. The whole theory of the $\aleph$'s greater than $\aleph_1$ is rejected as meaningless.[9] Cantor's conjecture itself receives several different meanings, all of which, though very interesting in themselves, are quite different from

[7] This reduction can be effected, owing to the results and methods of *Tarski 1925*.

[8] For regular numbers $\aleph_\alpha$, one obtains immediately:
$$\aleph_\alpha^{\mathrm{cf}(\aleph_\alpha)} = \aleph_\alpha^{\aleph_\alpha} = 2^{\aleph_\alpha} = \aleph_{\alpha+1}.$$

[9] See *Brouwer 1909*.

the original problem. They lead partly to affirmative, partly to negative answers.[10] Not everything in this field, however, has been sufficiently clarified. The "semi-intuitionistic" standpoint along the lines of H. Poincaré and H. Weyl[11] would hardly preserve substantially more of set theory.

2 | However, this negative attitude toward Cantor's set theory, and toward classical mathematics, of which it is a natural generalization, is by no means a necessary outcome of a closer examination of their foundations, but only the result of a certain philosophical conception of the nature of mathematics, which admits mathematical objects only to the extent to which they are interpretable as our own constructions or, at least, can be completely given in mathematical intuition. For someone who considers mathematical objects to exist independently of our constructions and of our having an intuition of them individually, and who requires only that the general mathematical concepts must be sufficiently clear for us to be able to recognize their soundness and the truth of the axioms concerning them, there exists, I believe, a satisfactory foundation of Cantor's set theory in its whole original extent and meaning, namely, axiomatics of set theory interpreted in the way sketched below.

It might seem at first that the set-theoretical paradoxes would doom to failure such an undertaking, but closer examination shows that they cause no trouble at all. They are a very serious problem, not for mathematics, however, but rather for logic and epistemology. As far as sets occur in mathematics (at least in the mathematics of today, including all of Cantor's set theory), they are sets of integers, or of rational numbers (i.e., of pairs of integers), or of real numbers (i.e., of sets of rational numbers), or of functions of real numbers (i.e., of sets of pairs of real numbers), etc. When theorems about all sets (or the existence of sets in general) are asserted, they can always be interpreted without any difficulty to mean that they hold for sets of integers as well as for sets of sets of integers, etc. (respectively, that there either exist sets of integers, or sets of sets of integers, or ... etc., which have the asserted property). This concept of set,[12] how-

[10]See *Brouwer 1907*, I, 9; III, 2.

[11]See *Weyl 1932*. If the procedure of construction of sets described there (p. 20) is iterated a sufficiently large (transfinite) number of times, one gets exactly the real numbers of the model for set theory mentioned in Section 4, in which the continuum hypothesis is true. But this iteration is not possible within the limits of the semi-intuitionistic standpoint.

[12]It must be admitted that the spirit of the modern abstract disciplines of mathematics, in particular of the theory of categories, transcends this concept of set, as becomes apparent, e.g., by the self-applicability of categories (see *Mac Lane 1961*). It does not seem, however, that anything is lost from the mathematical content of the theory if categories of different levels are distinguished. If there existed mathematically interesting proofs that would not go through under this interpretation, then the paradoxes of set theory would become a serious problem for mathematics.



ever, according to which a set is something obtainable from the integers (or some other well-defined objects) by iterated application[13] of the operation "set of",[14] not something obtained by | dividing the totality of all existing 263
things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.[15]

But, furthermore, the axioms underlying the unrestricted use of this concept of set or, at least, a subset of them which suffices for all mathematical proofs devised up to now (except for theorems depending on the existence of extremely large cardinal numbers, see footnote 20), have been formulated so precisely in axiomatic set theory[16] that the question of whether some given proposition follows from them can be transformed, by means of mathematical logic, into a purely combinatorial problem concerning the manipulation of symbols which even the most radical intuitionist must acknowledge as meaningful. So Cantor's continuum problem, no matter what philosophical standpoint is taken, undeniably retains at least this meaning: to find out whether an answer, and if so which answer, can be derived from the axioms of set theory as formulated in the systems cited.

Of course, if it is interpreted in this way, there are (assuming the consistency of the axioms) a priori three possibilities for Cantor's conjecture: It may be demonstrable, disprovable, or undecidable.[17] The third alternative (which is only a precise formulation of the foregoing conjecture, that the difficulties of the problem are probably not purely mathematical) is the most likely. To seek a proof for it is, at present, perhaps the most promising way of attacking the problem. One result along these lines has been

[13]This phrase is meant to include transfinite iteration, i.e., the totality of sets obtained by finite iteration is considered to be itself a set and a basis for further applications of the operation "set of".

[14]The operation "set of $x$'s" (where the variable "$x$" ranges over some given kind of objects) cannot be defined satisfactorily (at least not in the present state of knowledge), but can only be paraphrased by other expressions involving again the concept of set, such as: "multitude of $x$'s", "combination of any number of $x$'s", "part of the totality of $x$'s", where a "multitude" ("combination", "part") is conceived of as something which exists in itself no matter whether we can define it in a finite number of words (so that random sets are not excluded).

[15]It follows at once from this explanation of the term "set" that a set of all sets or other sets of a similar extension cannot exist, since every set obtained in this way immediately gives rise to further applications of the operation "set of" and, therefore, to the existence of larger sets.

[16]See, e.g., *Bernays 1937*, *1941*, *1942*, *1943*, *von Neumann 1925*; cf. also *von Neumann 1928a* and *1929*, *Gödel 1940*, *Bernays and Fraenkel 1958*. By including very strong axioms of infinity, much more elegant axiomatizations have recently become possible. (See *Bernays 1961*.)

[17]In case the axioms were inconsistent the last one of the four a priori possible alternatives for Cantor's conjecture would occur, namely, it would then be both demonstrable and disprovable by the axioms of set theory.

obtained already, namely, that Cantor's conjecture is not disprovable from the axioms of set theory, provided that these axioms are consistent (see Section 4).

It is to be noted, however, that on the basis of the point of view here adopted, a proof of the undecidability of Cantor's conjecture from the accepted axioms of set theory (in contradistinction, e.g., to the proof of the transcendency of $\pi$) would by no means solve the problem. For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Can-
264 tor's conjecture | must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality. Such a belief is by no means chimerical, since it is possible to point out ways in which the decision of a question, which is undecidable from the usual axioms, might nevertheless be obtained.

First of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[18] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of". These axioms can be formulated also as propositions asserting the existence of very great cardinal numbers (i.e., of sets having these cardinal numbers). The simplest of these strong "axioms of infinity" asserts the existence of inaccessible numbers (in the weaker or stronger sense) $> \aleph_0$. The latter axiom, roughly speaking, means nothing else but that the totality of sets obtainable by use of the procedures of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for further applications of these procedures).[19] Other axioms of infinity have first been formulated by P. Mahlo.[20] These axioms show clearly, not only that the axiomatic

[18]Similarly the concept "property of set" (the second of the primitive terms of set theory) suggests continued extensions of the axioms referring to it. Furthermore, concepts of "property of property of set" etc. can be introduced. The new axioms thus obtained, however, as to their consequences for propositions referring to limited domains of sets (such as the continuum hypothesis) are contained (as far as they are known today) in the axioms about sets.

[19]See *Zermelo 1930*.

[20][*Revised note of September 1966*: See *Mahlo 1911*, pp. 190–200, and *1913*, pp. 269–276. From Mahlo's presentation of the subject, however, it does not appear that the numbers he defines actually exist. In recent years great progress has been made in the area of axioms of infinity. In particular, some propositions have been formulated which, if consistent, are extremely strong axioms of infinity of an entirely new kind (see *Keisler and Tarski 1964* and the material cited there). Dana Scott (*1961*) has proved that one of them implies the existence of non-constructible sets. That these axioms are implied by the general concept of set in the same sense as Mahlo's has not been made clear



system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set explained above.

It can be proved that these axioms also have consequences far outside the domain of very great transfinite numbers, which is their immediate subject matter: each of them, under the assumption of its consistency, can be shown to increase the number of decidable propositions even in the field of Diophantine equations. As for the continuum problem, there is little
hope of | solving it by means of those axioms of infinity which can be set 265
up on the basis of Mahlo's principles (the aforementioned proof for the undisprovability of the continuum hypothesis goes through for all of them without any change). But there exist others based on different principles (see footnote 20); also there may exist, besides the usual axioms, the axioms of infinity, and the axioms mentioned in footnote 18, other (hitherto unknown) axioms of set theory which a more profound understanding of the concepts underlying logic and mathematics would enable us to recognize as implied by these concepts (see, e.g., footnote 23).

Secondly, however, even disregarding the intrinsic necessity of some new axiom, and even in case it has no intrinsic necessity at all, a probable decision about its truth is possible also in another way, namely, inductively by studying its "success". Success here means fruitfulness in consequences, in particular in "verifiable" consequences, i.e., consequences demonstrable without the new axiom, whose proofs with the help of the new axiom, however, are considerably simpler and easier to discover, and make it possible to contract into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent, owing to the fact that analytical number theory frequently allows one to prove number-theoretical theorems which, in a more cumbersome way, can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is conceivable. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole field, and yielding such powerful methods for solving problems (and even solving them constructively, as far as that is possible) that, no matter whether or not they are intrinsically necessary, they would have to be accepted at least in the same sense as any well-established physical theory.

yet (see *Tarski 1962*, p. 134). However, they are supported by strong arguments from analogy, e.g., by the fact that they follow from the existence of generalizations of Stone's representation theorem to Boolean algebras with operations on infinitely many elements. Mahlo's axioms of infinity have been derived from a general principle about the totality of sets which was first introduced by A. Levy (*1960*). It gives rise to a hierarchy of different precise formulations. One, given by P. Bernays (*1961*), implies all of Mahlo's axioms.]

## 4. Some observations about the question: In what sense and in which direction may a solution of the continuum problem be expected?

But are such considerations appropriate for the continuum problem? Are there really any clear indications for its unsolvability by the accepted axioms? I think there are at least two:

The first results from the fact that there are two quite differently defined classes of objects both of which satisfy all axioms of set theory that have been set up so far. One class consists of the sets definable in a certain manner by properties of their elements;[21] the other of the sets in the sense of arbitrary multitudes, regardless of if, or how, they can be defined. Now, before it has | been settled what objects are to be numbered, and on the basis of what one-to-one correspondences, one can hardly expect to be able to determine their number, except perhaps in the case of some fortunate coincidence. If, however, one believes that it is meaningless to speak of sets except in the sense of extensions of definable properties, then, too, he can hardly expect more than a small fraction of the problems of set theory to be solvable without making use of this, in his opinion essential, characteristic of sets, namely, that they are extensions of definable properties. This characteristic of sets, however, is neither formulated explicitly nor contained implicitly in the accepted axioms of set theory. So from either point of view, if in addition one takes into account what was said in Section 2, it may be conjectured that the continuum problem cannot be solved on the basis of the axioms set up so far, but, on the other hand, may be solvable with the help of some new axiom which would state or imply something about the definability of sets.[22]

The latter half of this conjecture has already been verified; namely, the concept of definability mentioned in footnote 21 (which itself is definable in axiomatic set theory) makes it possible to derive, in axiomatic set theory, the generalized continuum hypothesis from the axiom that every set is definable in this sense.[23] Since this axiom (let us call it "A") turns

[21]Namely, definable by certain procedures, "in terms of ordinal numbers" (i.e., roughly speaking, under the assumption that for each ordinal number a symbol denoting it is given). See my papers *1939a* and *1940*. The paradox of Richard, of course, does not apply to this kind of definability, since the totality of ordinals is certainly not denumerable.

[22]D. Hilbert's program for a solution of the continuum problem (see his *1926*), which, however, has never been carried through, also was based on a consideration of all possible definitions of real numbers.

[23]On the other hand, from an axiom in some sense opposite to this one, the negation of Cantor's conjecture could perhaps be derived. I am thinking of an axiom which (similar to Hilbert's completeness axiom in geometry) would state some maximum



out to be demonstrably consistent with the other axioms, under the assumption of the consistency of these other axioms, this result (regardless of the philosophical position taken toward definability) shows the consistency of the continuum hypothesis with the axioms of set theory, provided that these axioms themselves are consistent.[24] This proof in its structure is similar to the consistency proof of non-Euclidean geometry by means of a model within Euclidean geometry. Namely, it follows from the axioms of set theory that the sets definable in the aforementioned sense form a model of set theory in which the proposition A and, therefore, the generalized continuum hypothesis is true.

A second argument in favor of the unsolvability of the continuum problem on the basis of the usual axioms can be based on certain facts (not known at Cantor's time) which seem to indicate that Cantor's conjecture will turn out | to be wrong,[25] while, on the other hand, a disproof of it is 267
demonstrably impossible on the basis of the axioms being assumed today.

One such fact is the existence of certain properties of point sets (asserting an extreme rareness of the sets concerned) for which one has succeeded in proving the existence of non-denumerable sets having these properties, but no way is apparent in which one could expect to prove the existence of examples of the power of the continuum. Properties of this type (of subsets of a straight line) are: (1) being of the first category on every perfect set,[26] (2) being carried into a zero set by every continuous one-to-one mapping of the line onto itself.[27] Another property of a similar nature is that of being coverable by infinitely many intervals of any given lengths. But in this case one has so far not even succeeded in proving the existence of non-denumerable examples. From the continuum hypothesis, however, it follows in all three cases that there exist, not only examples of the power of the continuum,[28] but even such as are carried into themselves (up to denumerably many points) by *every* translation of the straight line.[29]

Other highly implausible consequences of the continuum hypothesis are that there exist: (1) subsets of a straight line of the power of the continuum which are covered (up to denumerably many points) by *every* dense set

property of the system of all sets, whereas axiom A states a minimum property. Note that only a maximum property would seem to harmonize with the concept of set explained in footnote 14.

[24]See my monograph *1940* and my paper *1939a*. For a carrying through of the proof in all details, my *1940* is to be consulted.

[25]Views tending in this direction have been expressed also by N. Luzin in his *1935*, pp. 129 ff. See also *Sierpiński 1935*.

[26]See *Sierpiński 1934a* and *Kuratowski 1933*, pp. 269 ff.

[27]See *Luzin and Sierpiński 1918* and *Sierpiński 1934a*.

[28]For the third case see *Sierpiński 1934*, p. 39, Theorem 1.

[29]See *Sierpiński 1935a*.

of intervals;[30] (2) infinite-dimensional subsets of Hilbert space which contain no non-denumerable finite-dimensional subset (in the sense of Menger-Urysohn);[31] (3) an infinite sequence $A^i$ of decompositions of any set $M$ of the power of the continuum into continuum-many mutually exclusive sets $A^i_x$ such that, in whichever way a set $A^i_{x_i}$ is chosen for each $i$, $\prod_{i=0}^{\infty}(M - A^i_{x_i})$ is denumerable.[32] (1) and (3) are very implausible even if "power of the continuum" is replaced by "$\aleph_1$".

One may say that many results of point-set theory obtained without using the continuum hypothesis also are highly unexpected and implausible.[33] But, true as that may be, still the situation is different there, in that, in most of those instances (such as, e.g., Peano's curves) the appearance to the contrary can be explained by a lack of agreement between our intuitive geometrical concepts and the set-theoretical ones occurring in the theo-
268 rems. Also, it is very | suspicious that, as against the numerous plausible propositions which imply the negation of the continuum hypothesis, not one plausible proposition is known which would imply the continuum hypothesis. I believe that adding up all that has been said one has good reason for suspecting that the role of the continuum problem in set theory will be to lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

## Definitions of some of the technical terms

Definitions 4–15 refer to subsets of a straight line, but can be literally transferred to subsets of Euclidean spaces of any number of dimensions if "interval" is identified with "interior of a parallelepipedon".

1. I call *the character of cofinality* of a cardinal number $m$ (abbreviated by "cf($m$)") the smallest number $n$ such that $m$ is the sum of $n$ numbers $< m$.
2. A cardinal number $m$ is *regular* if $\mathrm{cf}(m) = m$, otherwise singular.
3. An infinite cardinal number $m$ is *inaccessible* if it is regular and has no immediate predecessor (i.e., if, although it is a limit of numbers $< m$, it is not a limit of fewer than $m$ such numbers); it is *strongly inaccessible* if each product (and, therefore, also each sum) of fewer than $m$ numbers $< m$ is $< m$. (See *Sierpiński and Tarski 1930*, *Tarski 1938*.)

[30] See *Luzin 1914*, p. 1259.

[31] See *Hurewicz 1932*.

[32] See *Braun and Sierpiński 1932*, p. 1, proposition ($Q$). This proposition is equivalent with the continuum hypothesis.

[33] See, e.g., *Blumenthal 1940*.



It follows from the generalized continuum hypothesis that these two concepts are equivalent. $\aleph_0$ is evidently inaccessible, and also strongly inaccessible. As for finite numbers, 0 and 2 and no others are strongly inaccessible. A definition of inaccessibility, applicable to finite numbers, is this: $m$ is inaccessible if (1) any sum of fewer than $m$ numbers $< m$ is $< m$, and (2) the number of numbers $< m$ is $m$. This definition, for transfinite numbers, agrees with that given above and, for finite numbers, yields 0, 1, 2 as inaccessible. So inaccessibility and strong inaccessibility turn out not to be equivalent for finite numbers. This casts some doubt on their equivalence for transfinite numbers, which follows from the generalized continuum hypothesis.

4. A set of intervals is *dense* if every interval has points in common with some interval of the set. (The endpoints of an interval are not considered as points of the interval.)
5. A *zero set* is a set which can be covered by infinite sets of intervals with arbitrarily small lengths-sum.
6. A *neighborhood* of a point $P$ is an interval containing $P$.
7. A subset $A$ of $B$ is *dense in B* if every neighborhood of any point of $B$ contains points of $A$.
8. A point $P$ is in the *exterior* of $A$ if it has a neighborhood containing no point of $A$.
9. A subset $A$ of $B$ is *nowhere dense in B* if those points of $B$ which are in the exterior of $A$ are dense in $B$, or (which is equivalent) if for no interval $I$ the intersection $IA$ is dense in $IB$.
10. A subset $A$ of $B$ is *of the first category in B* if it is the sum of denumerably many sets nowhere dense in $B$.
11. A set $A$ is *of the first category on B* if the intersection $AB$ is of the first category in $B$.
12. A point $P$ is called *a limit point* of a set $A$ if any neighborhood of $P$ contains infinitely many points of $A$.
13. A set $A$ is called *closed* if it contains all its limit points.
14. A set is *perfect* if it is closed and has no isolated point (i.e., no point with a neighborhood containing no other point of the set).
15. *Borel sets* are defined as the smallest system of sets satisfying the postulates:
    (1) The closed sets are Borel sets.
    (2) The complement of a Borel set is a Borel set.
    (3) The sum of denumerably many Borel sets is a Borel set.
16. A set is *analytic* if it is the orthogonal projection of some Borel set of a space of next higher dimension. (Every Borel set therefore is, of course, analytic.)

## Supplement to the second edition

Since the publication of the preceding paper, a number of new results have been obtained; I would like to mention those that are of special interest in connection with the foregoing discussions.

1. A. Hajnal has proved[34] that, if $2^{\aleph_0} \neq \aleph_2$ could be derived from the axioms of set theory, so could $2^{\aleph_0} = \aleph_1$. This surprising result could greatly facilitate the solution of the continuum problem, should Cantor's continuum hypothesis be demonstrable from the axioms of set theory, which, however, probably is not the case.

2. Some new consequences of, and propositions equivalent with, Cantor's hypothesis can be found in the new edition of W. Sierpiński's book.[35] In the first edition, it had been proved that the continuum hypothesis is equivalent with the proposition that the Euclidean plane is the sum of denumerably many "generalized curves" (where a generalized curve is a point
270 set definable | by an equation $y = f(x)$ in some Cartesian coordinate system). In the second edition, it is pointed out[36] that the Euclidean plane can be proved to be the sum of fewer than continuum-many generalized curves under the much weaker assumption that the power of the continuum is not an inaccessible number. A proof of the converse of this theorem would give some plausibility to the hypothesis $2^{\aleph_0}$ = the smallest inaccessible number $> \aleph_0$. However, great caution is called for with regard to this inference,[36a] because the paradoxical appearance in this case (like in Peano's "curves") is due (at least in part) to a transference of our geometrical intuition of curves to something which has only some of the characteristics of curves. Note that nothing of this kind is involved in the counterintuitive consequences of the continuum hypothesis mentioned on page 267.

3. C. Kuratowski has formulated a strengthening of the continuum hypothesis,[37] whose consistency follows from the consistency proof mentioned in Section 4. He then drew various consequences from this new hypothesis.

4. Very interesting new results about the axioms of infinity have been obtained in recent years (see footnotes 20 and 16).

In opposition to the viewpoint advocated in Section 4 it has been suggested[38] that, in case Cantor's continuum problem should turn out to be

[34] See his *1956*.

[35] See *Sierpiński 1956*.

[36] See his *1956*, p. 207 or his *1951*, p. 9. Related results are given by C. Kuratowski (*1951*, p. 15) and R. Sikorski (*1951*).

[36a] [*Note added September 1966*: It seems that this warning has since been vindicated by Roy O. Davies (*1963*).]

[37] See his *1948*.

[38] See *Errera 1952*.



undecidable from the accepted axioms of set theory, the question of its truth would lose its meaning, exactly as the question of the truth of Euclid's fifth postulate by the proof of the consistency of non-Euclidean geometry became meaningless for the mathematician. I therefore would like to point out that the situation in set theory is very different from that in geometry, both from the mathematical and from the epistemological point of view.

In the case of the axiom of the existence of inaccessible numbers, e.g., (which can be proved to be undecidable from the von Neumann-Bernays axioms of set theory provided that it is consistent with them) there is a striking asymmetry, mathematically, between the system in which it is asserted and the one in which it is negated.[39]

Namely, the latter (but not the former) has a model which can be defined and proved to be a model in the original (unextended) system. This means that the former is an extension in a much stronger sense. A closely related fact is that the assertion (but not the negation) of the axiom implies new theorems about integers (the individual instances of which can be verified by computation). So the criterion of truth explained on page 264 is satisfied, to some extent, for the assertion, but not for the negation. Briefly speaking, only the assertion | yields a "fruitful" extension, while the negation is sterile outside its own very limited domain. The generalized continuum hypothesis, too, can be shown to be sterile for number theory and to be true in a model constructible in the original system, whereas for some other assumption about the power of $2^{\aleph_\alpha}$ this perhaps is not so. On the other hand, neither one of those asymmetries applies to Euclid's fifth postulate. To be more precise, both it and its negation are extensions in the weak sense.

As far as the epistemological situation is concerned, it is to be said that by a proof of undecidability a question loses its meaning only if the system of axioms under consideration is interpreted as a hypothetico-deductive system, i.e., if the meanings of the primitive terms are left undetermined. In geometry, e.g., the question as to whether Euclid's fifth postulate is true retains its meaning if the primitive terms are taken in a definite sense, i.e., as referring to the behavior of rigid bodies, rays of light, etc. The situation in set theory is similar; the difference is only that, in geometry, the meaning usually adopted today refers to physics rather than to mathematical intuition and that, therefore, a decision falls outside the range of mathematics. On the other hand, the objects of transfinite set theory, conceived in the manner explained on page 262 and in footnote 14, clearly do not belong to the physical world, and even their indirect connection with physical experience is very loose (owing primarily to the fact that set-theoretical concepts play only a minor role in the physical theories of today).

[39] The same asymmetry also occurs on the lowest levels of set theory, where the consistency of the axioms in question is less subject to being doubted by skeptics.

But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception, which induces us to build up physical theories and to expect that future sense perceptions will agree with them, and, moreover, to believe that a question not decidable now has meaning and may be decided in the future. The set-theoretical paradoxes are hardly any more troublesome for mathematics than deceptions of the senses are for physics. That new mathematical intuitions leading to a decision of such problems as Cantor's continuum hypothesis are perfectly possible was pointed out earlier (pages 264–265).

It should be noted that mathematical intuition need not be conceived of as a faculty giving an *immediate* knowledge of the objects concerned. Rather it seems that, as in the case of physical experience, we *form* our ideas also of those objects on the basis of something else which *is* immediately given. Only this something else here is *not*, or not primarily, the sensations. That something besides the sensations actually is immediately given follows (independently of mathematics) from the fact that even our ideas referring to physical objects contain constituents qualitatively different from sensations or mere combinations of sensations, e.g., the idea of object itself, whereas, on the other hand, by our thinking we cannot create any qualitatively new elements, but only | reproduce and combine those that are given. Evidently the "given" underlying mathematics is closely related to the abstract elements contained in our empirical ideas.[40] It by no means follows, however, that the data of this second kind, because they cannot be associated with actions of certain things upon our sense organs, are something purely subjective, as Kant asserted. Rather they, too, may represent an aspect of objective reality, but, as opposed to the sensations, their presence in us may be due to another kind of relationship between ourselves and reality.

However, the question of the objective existence of the objects of mathematical intuition (which, incidentally, is an exact replica of the question of the objective existence of the outer world) is not decisive for the problem under discussion here. The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis. What, however, perhaps more than anything else, justifies the acceptance of this criterion of truth in set theory is the fact that continued appeals to mathematical intuition are necessary not only for obtaining unambiguous answers to the questions of transfinite set theory, but also for the solution of the problems of finitary number theory[41] (of the type of Goldbach's conjecture),[42] where the meaningfulness and unambiguity of the concepts entering into them can hardly be doubted. This follows from the fact that for every axiomatic system there are infinitely many undecidable propositions of this type.

It was pointed out earlier (page 265) that, besides mathematical intuition, there exists another (though only probable) criterion of the truth of mathematical axioms, namely their fruitfulness in mathematics and, one may add, possibly also in physics. This criterion, however, though it may become decisive in the future, cannot yet be applied to the specifically set-theoretical axioms (such as those referring to great cardinal numbers), because very little is known about their consequences in other fields. The simplest case of an application of the criterion under discussion arises when some set-theoretical axiom has number-theoretical consequences verifiable by computation up to any given integer. On the basis of what is known today, however, it is not possible to make the truth of any set-theoretical axiom reasonably probable in this manner.

[40]Note that there is a close relationship between the concept of set explained in footnote 14 and the categories of pure understanding in Kant's sense. Namely, the function of both is "synthesis", i.e., the generating of unities out of manifolds (e.g., in Kant, of the idea of *one* object out of its various aspects).

## | Postscript 273

[*Revised postscript of September 1966*: Shortly after the completion of the manuscript of the second edition [[*1964*]] of this paper the question of whether Cantor's continuum hypothesis is decidable from the von Neumann–Bernays axioms of set theory (the axiom of choice included) was settled in the negative by Paul J. Cohen. A sketch of the proof has appeared in his *1963* and *1964*. It turns out that for all $\aleph_\tau$ defined by the usual devices and not excluded by König's theorem (see page 260 above) the equality $2^{\aleph_0} = \aleph_\tau$ is consistent and an extension in the weak sense (i.e., it implies no new number-theoretical theorem). Whether, for a satisfactory concept of "standard definition", this is true for *all* definable $\aleph_\tau$ not excluded by König's theorem is an open question. An affirmative answer would require the solution of the difficult problem of making the concept of standard definition, or some wider concept, precise. Cohen's work, which

[41]Unless one is satisfied with inductive (probable) decisions, such as verifying the theorem up to very great numbers, or more indirect inductive procedures (see pp. 265, 272).

[42]I.e., universal propositions about integers which can be decided in each individual instance.

no doubt is the greatest advance in the foundations of set theory since its axiomatization, has been used to settle several other important independence questions. In particular, it seems to follow that the axioms of infinity mentioned in footnote 20, to the extent to which they have so far been precisely formulated, are not sufficient to answer the question of the truth or falsehood of Cantor's continuum hypothesis.]

# On an extension of finitary mathematics which has not yet been used[a] (*1972*)

[The introductory note to *1972*, as well as to related items, is found page 217, immediately preceding *1958*.]

## Abstract

P. Bernays has pointed out that, even in order to prove only the consistency classical number theory, it is necessary to extend Hilbert's finitary standpoint. suggested admitting certain abstract concepts in addition to the combinatorial conce referring to symbols. The abstract concepts that so far have been used for this purp are those of the constructive theory of ordinals and those of intuitionistic logic. is shown that a certain concept of computable function of finite simple type over natural numbers can be used instead, where no other procedures of constructing s functions are necessary except primitive recursion by a number variable and definit of a function by an equality with a term containing only variables and/or previou introduced functions beginning with the function +1.

P. Bernays has pointed out[1] on several occasions that, in view of the f that the consistency of a formal system cannot be proved by any ded tion procedures available in the system itself, it is necessary to go bey the framework of finitary mathematics in Hilbert's sense in order to pr the consistency of classical mathematics or even of classical number t

[1]See: *Bernays 1941a*, pp. 144, 147, 150, 152; *Hilbert and Bernays 1939*, pp. 347– 357–360; *Bernays 1954*, p. 9; cf. also *Bernays 1935*, pp. 62, 69.

[a]The present paper is not a literal translation of the German original publis in *Dialectica* (*1958*). In revising the translation by Leo F. Boron, I have rephr many passages. But the meaning has nowhere been substantially changed. Some m inaccuracies have been corrected and a number of notes have been added, to which letters (**a**)–(**n**) refer. I wish to express my best thanks to Professor Dana Scott supervising the typing of this and the subsequent paper [*1972a*] while I was ill, an Professor Paul Bernays for reading the proof sheets and calling my attention to s oversights in the manuscript.