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Introduction to the Unified Field Theory of Elementary Particles

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Preface

The present book contains a series of lectures which have been given at the University of Munich during the summer term 1965, with the hope to draw the attention of the younger generation of physicists to the present development of the unified field theory of elementary particles.

The picture in which the elementary particles appear as dynamical systems, comparable to the stationary states of complicated atoms or molecules, and resulting, as in quantum mechanics, from an underlying universal law, has for a long time not appealed to the majority of physicists. It was in fact in obvious disagreement with the term 'elementary' used for these particles and with the resulting hope for a simple understanding of their nature. On the other hand, in the nearly twenty years that have elapsed since its first explicit formulation, this picture has found the support of many more recent experiments. New elementary particles have been discovered in great number. The spectrum of particles and the selection rules in transition processes have been analysed successfully with the help of quantum numbers in a similar way as the optical spectra of atoms. The special mathematical form suggested in 1958 for the underlying natural law has been supported by the later observation of the η -meson, by the determination of coupling constants and, more recently, by the possibility of connecting the photon (and thereby electrodynamics) with the degeneracy of the ground state, the theorem of Goldstone, and the existence of 'strange' particles. All these results seemed to justify the attempt to present the main ideas of the unified field theory in a simple form understandable even for those physicists who are not yet specialists in the field of elementary particle physics.

In all lectures the emphasis has therefore been on the physical ideas of the theory, and only to a lesser degree on the mathematical scheme. In the present state of the theory it would be premature to start from a set of precise mathematical axioms and to proceed by rigorous mathematical deductions. What is required is a mathematical description adapted to the

experimental situation which does not seem to contain contradictions and which therefore may be developed into a rigorous mathematical scheme in the future. The history of physics teaches that as a rule a new theory can be formulated in a precise mathematical language only after all essential physical problems have been solved. The lectures do give a general outline of the expected mathematical scheme without proofs of existence or convergence; besides that they contain a few rather elementary calculations of the simplest mass eigenvalues and coupling constants. A much more elaborate description of the mathematical details will be given in a larger book by H. P. Dürr who has taken the most active part in the development of the theory. The dynamical aspects of the theory have been reviewed recently by Rampacher, Stumpf and Wagner.⁷⁶

The mathematical formalism contains some unconventional features which formerly have rendered its understanding somewhat difficult: the indefinite metric in Hilbert space and the degeneracy of the ground state. But in recent years the indefinite metric has been studied in connexion with the Bleuler-Gupta version of quantum electrodynamics and with the Lee-model, the degeneracy of the ground state plays an important part in modern solid state physics. Therefore these extensions of conventional quantum mechanics should not be too problematic; they have been discussed carefully in the lectures.

The phenomenological theories which have successfully brought some order into the complicated experimental material: dispersion relations, Regge poles, groups of unitary transformations like SU_3 , SU_6 , etc., have not been presented in these lectures in any detail. They can be studied in several good textbooks. The way in which these theories can possibly be connected with the results of the unified field theory has been described in the last part of the present book. At this point again one recognizes that the unified field theory is merely in the first stage of its development and that many difficult problems are open for further investigation. In the space between strong interactions, electromagnetism, weak interactions and gravitation, seen from the viewpoint of this theory, there is still a wide field of research.

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The Important Experimental Facts

1-1 Definition of the elementary particle

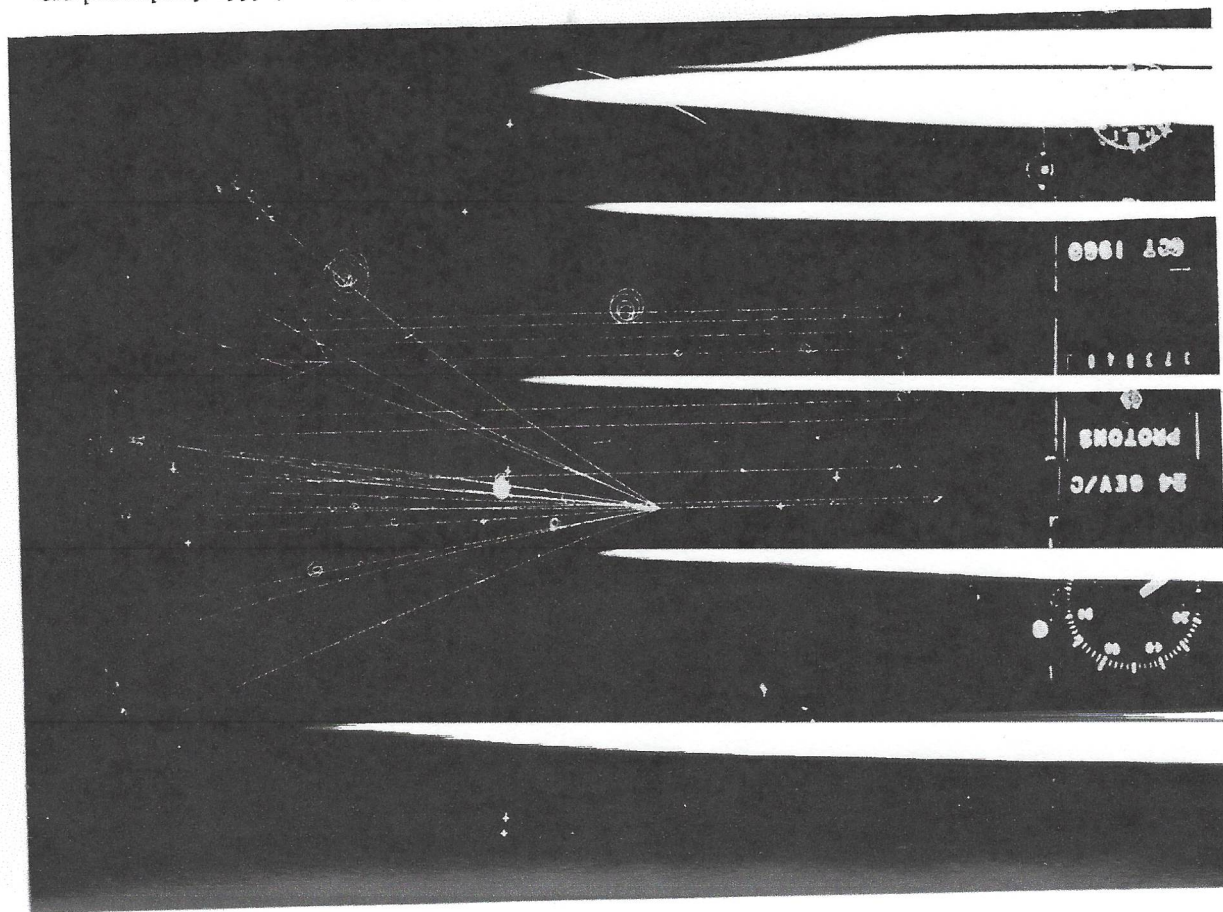
Some fifty years ago only three kinds of 'elementary particles' were known. Electron and proton were considered to be the smallest units of matter, the photon or light quantum the smallest unit of energy in radiation. This concept of the 'smallest unit' had its origin in the problem of the divisibility of matter. A piece of matter can, by mechanical or chemical means, be decomposed into the single atoms. The atom may be separated into its nucleus and the surrounding electrons. The atomic nucleus can be split into protons and neutrons. It would be justified to speak about 'smallest units' only if these units could not be divided again into smaller parts.

The only processes in which a splitting of elementary particles could possibly be expected, are collisions between particles of extremely high energy. The big accelerators in Berkeley, Dubna, Geneva, Brookhaven have been designed just to study these collision processes. Following their construction many experiments have led to the result that from such a high energy collision between two particles actually many particles may emerge; but the emerging particles are not necessarily smaller than the colliding particles. On the contrary one finds always the same various kinds of particles, whatever the nature of the colliding particles may have been. The phenomena can be described more adequately by saying, that the big kinetic energy of the colliding particles has been transmuted into matter, into the emerging particles ('multiple production of particles'^{1,2}). The photograph on the next page, taken from the bubble-chamber of the Proton-Synchrotron at CERN, Geneva, shows a typical example of multiple production of particles. Energy can become matter by assuming the form of an elementary particle. The different elementary particles may be considered as different forms in which the fundamental substance, matter or energy, can exist.³ They may be called 'smallest units' only in the sense that when they are split, their parts are not smaller units but units of roughly the same size.

This solution of the problem of the 'smallest units of matter' is somewhat surprising and paradoxical, and leads to another question which needs careful consideration. In earlier times the atoms of the chemists or the atomic nuclei were pictured as compound systems, consisting of many elementary particles, while the electron or the proton were taken as indivisible units and therefore 'elementary'. In the present situation such a distinction would be artificial; it would in fact scarcely be possible to find any good definition which could distinguish an 'elementary' particle from a compound system. A pion, for example, could be considered as a system consisting of one or several pairs of nucleon and antinucleon. The nucleon could be built from Λ -hyperon and K -meson, the photon from muon and antimuon, etc. The well-known formula, 'every elementary particle consists of all other elementary particles', seems to be a good description of the paradoxical situation, with which we are confronted in the experiments. For practical purposes it may still be convenient to speak of a compound system, if the energy necessary for dividing the system is small compared with the rest masses of the resulting parts. But this is a rather vague qualitative definition, not a qualitative one. Therefore it is reasonable to drop this distinction altogether.

One of the main reasons for this new situation in elementary particle physics is the process of pair creation, the existence of antiparticles and antimatter.^{4,5} Wave functions as representatives of states cannot be separated completely unless they have different symmetries. Any wave function will as a rule contain contributions from any other wave function of exactly the same symmetry. In unrelativistic quantum mechanics states with different numbers of particles have different symmetries on account of the gauge groups, therefore they can be separated. In relativistic field theory this is no longer true, since we have pair creation.

Another aspect of this situation is the complexity of elementary particle physics. The theory of elementary particles can scarcely be less complicated than quantum chemistry; there will be no counterpart to the optical spectrum of the hydrogen atom and its simple mathematical laws. If in quantum chemistry we want to calculate the binding energy, for example, of the two oxygen atoms in the oxygen molecule, we cannot ignore the fact, that in collisions of higher energy the atoms may be ionized and eventually disintegrated into nucleus and electrons. Therefore we cannot hope for any simple analytical function describing accurately the potential between the two atoms and thereby defining the binding energy. In the same way the mass of the pion cannot be calculated from a simple potential between nucleon and antinucleon, because the analytical behaviour of such a function should reflect the experimental fact, that in high energy collisions between nucleon



A proton of 24 GeV from the Proton-Synchrotron at CERN hits a proton in the hydrogen bubble chamber and produces a number of secondaries, mostly pions.

and antinucleon many particles may be created. Therefore in the theory of elementary particles the problem of mathematical approximation will be as important and as difficult as in quantum chemistry.

If the different elementary particles are taken as different forms of the fundamental substance 'energy' or 'matter', it should be emphasized that they are frequently only transient forms of very short lifetime. A few of them are stable like proton, electron or deuteron, a few others have rather 'long' lifetimes between 10^{-6} and 10^{-16} sec (like pion or Ω^- -hyperon), the vast majority have lifetimes below 10^{-16} sec. (In the context of the present book the terms 'mass' and 'lifetime' refer always to the rest system of the particle concerned.) Hence there will be a continuous transition between states of long lifetime, the properties of which can be determined with great accuracy, and others of very short lifetime, the properties of which are not well defined. This situation is clearly seen in the experiments. A continuous spectrum of states does not belong to definite symmetries, for example, to definite values of the angular momentum or of isospin. A discrete stationary state does belong to definite symmetries. A maximum in the continuous spectrum may, depending on its width, be called a resonance state or just a maximum and may or may not be seen in several symmetries. All intermediate cases should be possible. The term 'elementary particle' will be used quite freely for any of these forms, independent of their stability.

1-2 Classification of elementary particles

From the foregoing discussion it should be clear already, that there is a close analogy between the spectrum of different elementary particles and the spectrum of different stationary states in the quantum theory of the atom. Therefore in the classification of the different elementary particles one should be guided by those principles of classification that have proved useful in the theory of the atom.

The obvious starting point for such a classification is an analysis of those features of the observations and of the underlying natural law that in classical physics have been called 'laws of conservation'; for instance conservation of energy, momentum, angular momentum, charge, etc. The physicists have learned from the mathematicians that conservation laws are connected with symmetries, with the group structure of the underlying natural law. If the underlying law is invariant under a certain transformation, the operator performing this transformation may be used for constructing an observable, which stays constant during the processes described by the natural law. The different states of the system may be considered as different representations

of the group. These representations can, from a mathematical point of view, be characterized by discrete numbers (in case of compact groups) or continuous parameters (in case of non compact groups), the discrete numbers being called quantum numbers from a phenomenological point of view. The conservation law may be stated by saying that the sum of the quantum numbers or the sum of the continuous parameters of the interacting systems stays constant during the collision process, or, in the case of a discrete group, by saying that the product of the quantum numbers (for example, parity) stays constant. All these relations are well known from unrelativistic quantum mechanics; but it is a very important result of all recent observations on elementary particle processes, that in this respect no fundamental difference has been found between elementary particle physics and quantum mechanics. The elementary particles can be classified by quantum numbers, mass-values and lifetimes just like the stationary states in quantum mechanics.

A difference comes in however, by the relevant groups. The most important group in elementary particle physics is undoubtedly the Lorentz group,⁶ which replaces the Galilean group of unrelativistic quantum mechanics and produces the conservation laws of energy, momentum and angular momentum. The conservation laws for baryonic number, leptonic number and electromagnetic charge should be interpreted as the result of corresponding gauge groups. Another important group is connected with the 'isospin' of the particles and with charge symmetry;⁷ this group seems to be isomorphic with the group of rotations in three-dimensional space or with the group SU_2 (group of unitary and unimodular transformations of two complex variables) and produces the isospin-multiplets which will be discussed in more detail below. The empirical fact that electromagnetic charge is composed of two parts, isospin charge and hypercharge,⁸ and that the latter is connected with the concept of strangeness,⁹ suggests a connexion between the isospin group and the gauge group producing strangeness. These two groups should therefore probably be combined to the group U_2 (group of unitary transformations of two complex variables), which is just the direct product of SU_2 and one gauge group. Finally the observations show the relevance of several discrete groups, space reflection parity (P),¹⁰ charge conjugation (C), time reversal (T) and G -conjugation¹¹ (product of C and isospin reversal).

While the existence of these groups contains the most important information concerning the structure of the underlying natural law, it is somewhat disturbing, that some of these symmetries seem to be only approximately valid. In fact, the existence of multiplets in the spectrum of elementary particles shows already that there are weaker interactions which are less

symmetrical. In case of complete symmetry any n -dimensional representation of the group should lead to n identical eigenvalues for the mass. A splitting of these eigenvalues can be produced only by an interaction which is less symmetrical. In quantum mechanics it is well known that, for example, an outer electric field destroys the rotational symmetry of the original system and thereby leads to a splitting of the optical lines (Stark effect). The symmetry can be reduced by actions from outside. On the other hand the multiplet structure of the optical spectra rests upon the fact, that the introduction of the magnetic spin-orbit interaction destroys the independence of rotational symmetry of the spin and the orbital motion. In this case it is not an action from outside which breaks the symmetry; by neglecting the magnetic forces one had artificially produced a higher symmetry (the independence of rotation of spin and orbit), which is broken afterwards. One learns from these examples, that one should distinguish between two kinds of symmetries or invariances. Symmetries of the first kind are inherent in the underlying natural law (for example, invariance under rotation in quantum mechanics). Symmetries of the second kind rest upon approximations which may be more or less justified in the different states of the system concerned. Since symmetries of the first kind may be broken by action from outside, the term 'outside' needs some specification. In many cases the outside action can be arranged arbitrarily by the experimenting physicist; the system can be put into an electric or magnetic field, enclosed in a box, etc. In other cases the action may come from the world as a whole and cannot be influenced by the physicist. In Einstein's theory of gravitation the centrifugal force results from the outside action of the groundstate of the world or from the very distant masses. It cannot be changed arbitrarily.

Therefore the observation of mass-multiplets in the spectrum of elementary particles, not caused by artificial outer fields, leaves the choice between two different interpretations. The symmetries may be inherent, i.e. of the first kind, but broken by the ground state 'world'; or they may belong to the second kind, caused by suitable approximations, i.e. approximate by nature. The decision between these two possibilities is obviously very important for the formulation of the theory.

Besides the groups that have been mentioned so far, the mass spectrum seems to contain some rather wide multiplets of a higher order suggesting the relevance of the group SU_3 ,¹² with the U_2 of isospin and strangeness as a subgroup. Two octets of bosons, one octet and one decuplet of fermions correspond to this assumption. The mass differences between different components of the multiplets are large (roughly a hundred times larger than in the case of the isospin-multiplets), but follow rather closely a formula resulting from a broken SU_3 -group.¹³ The mass of the Ω -hyperon had been

predicted (in 1962) from this formula.¹⁴ Again the question arises whether the SU_3 -symmetry belongs to the first or the second kind.

1-3 The different types of interactions

The appearance of approximate symmetries in the mass spectrum and in the interactions between elementary particles suggests a division of the interactions into different types according to the symmetry which they obey. Historically the 'strong interactions' were defined as those which are responsible for the binding between nucleons in the atomic nucleus and for the scattering of pions from nucleons. The electromagnetic interactions were known already from classical physics. The 'weak interactions' were observed in the radioactive decay. With the present knowledge it is probably more convenient to define as 'strong interactions' that part of the total interaction which is strictly symmetrical under all groups mentioned above, except SU_3 ; as 'electromagnetic' interactions those which violate the isospin invariance and G -conjugation, but which are symmetrical under the other groups; as weak (radioactive) interactions those which violate conservation of strangeness and iso-charge, P , C and G , but are still invariant under the proper Lorentz group, PC and PCT and obey conservation of baryonic and leptonic number and electric charge. There may be other weak interactions, which are less symmetrical. Recent experiments suggest the existence of a small interaction violating PC . Finally there is gravitation which plays a very important rôle for the structure of the world in large dimensions, but seems to have practically no immediate influence on the structure of the elementary particles.

The strong interactions could possibly again be divided into two parts, those which are symmetrical under SU_3 and those which are not. But the mass splitting caused by the second part is only slightly smaller than that caused by the first part. (1 GeV may be a characteristic energy for the first part, 0.2 to 0.4 GeV for the second part.) The mass formulas within the SU_3 -multiplets however seem to hold with a higher accuracy than what would be expected from this ratio.

The electromagnetic forces which cause the splitting of the isospin-multiplets (U_2 -multiplets) are very much weaker. The corresponding mass differences are of the order of 1 MeV, i.e. roughly the electrostatic energy of a sphere with the electronic charge and a radius of the order 10^{-13} cm. The violation of the isospin symmetry is seen immediately from the fact, that the electromagnetic properties of protons and neutrons are quite different, while for strong interactions they are the same particles, seen from a different frame of reference, they differ just by the 'direction of isospin'.

The strength of the weak (radioactive) interactions can be estimated by comparing, for example, the lifetime for the radioactive decay of charged pions ($\sim 10^{-8}$ sec) with the electromagnetic decay of neutral pions ($\sim 10^{-16}$ sec). The weak interactions are by a factor $\sim 10^{-8}$ smaller than the electromagnetic ones. The mass splitting caused by weak interactions is negligible compared with the present accuracy of mass measurement. The breakdown of space-reflection parity in weak interactions can be recognized immediately from the polarization of particles emitted in radioactive decay.

Finally the gravitation between electron and proton is roughly by a factor 10^{-39} smaller than the electrostatic force between them.

1-4 The spectrum of elementary particles

The discussions concerning the different symmetries or groups relevant in elementary particle physics and the different types of interaction define the general framework within which the complicated spectrum of elementary particles can be analysed. In Fig. 1 the observed stationary states are given as functions of their charge as abscissa and their mass as ordinate. To each state there exists a corresponding antistate of equal mass and opposite charge (which in a few cases of charge zero is identical with the state). These antistates are as a rule not listed separately in the figure. Only states with baryonic numbers 1, 0, -1 and leptonic numbers 0, or leptonic numbers 1, 0, -1 and baryonic number 0 are taken as 'elementary particles'.

The first division of the spectrum is into the two types: fermions and bosons, i.e. particles which obey Fermi statistics or Bose statistics. On account of Lorentz- and PCT -invariance all particles with half integer Dirac-spin must be fermions, all particles with integer Dirac-spin bosons. The bosons belong to baryonic and leptonic number zero. The term 'meson' has frequently been used for particles of intermediate mass (100 to 800 MeV), mostly bosons, but also for muons (μ -mesons) and is not very convenient for classification.

The fermions can be divided again into two very different classes of particles, the baryons and the leptons. The lepton group comprises only a few different particles (electrons, muons and two kinds of neutrinos) with an interesting common property: the absence of any strong interaction. The strongest interaction for the leptons is electromagnetic (for electrons and muons) or 'weak' interaction (for neutrinos). All other fermions belong to the baryon group. It is only within the baryon group that one can recognize isospin-multiplets and SU_3 -multiplets. The symmetries relevant in the main interactions are responsible for the structure of the mass spectrum as well. In Fig. 1 the lowest SU_3 -octet and the decuplet for Dirac spin $3/2$ are

should exceed several GeV. At present one cannot exclude the possibility that high energy scattering of leptons could lead to a distribution of large transversal momenta totally different from $\exp(-p_{\perp}^2/p_0^2)$. The complete absence of strong interactions may create a new situation, and the mathematical scheme of quantum electrodynamics may be taken as an indication in this direction. With the existing experimental evidence however, this problem must be left completely open.

Experimental results on high-energy scattering, for example, of electrons from nuclei, may also be interpreted as informations concerning the charge distribution, or mass distribution, or more generally the structure of the particles concerned. This interpretation is somewhat problematic, since it requires assumptions about the mechanism of interaction, which are correct within conventional quantum mechanics, but may not be equally correct in the final relativistic theory. Still, it gives a reasonable picture, for example, of the nucleon which should be as good as any picture of these smallest objects can be. The nucleon appears as a rather soft cloud of matter, with a radius of roughly 10^{-13} cm. The density has a maximum but apparently no strong singularity in the centre and falls off rapidly at radii $\sim 10^{-13}$ cm. A similar picture should be true for all particles taking part in the strong interactions. For leptons however the picture could be quite different.

2

The Mathematical Tools

2-1 Mathematical description of free particles

Since most experiments on elementary particles begin with and end up with free particles moving through a cloud-chamber, bubble-chamber or emulsion, possibly interacting still with an outer electric or magnetic field, a mathematical description of the phenomena will conveniently start with a description of the free particles. For this purpose the well-known methods of the conventional linear field theory are sufficient.¹⁶ For each kind of elementary particle a field operator $\psi_{\alpha}(x)$ is introduced, obeying the linear differential equation

$$(\square + \kappa^2) \psi_{\alpha}(x) = 0, \quad 2(1)$$

where κ represents the mass of the particle, x the space-time co-ordinates. The index α is used for characterizing the transformation properties of the particle under the different groups and thereby stating its spin, isospin and other quantum numbers. Even x may simply be considered as a continuous parameter necessary for defining the transformation properties under the inhomogeneous part of the Lorentz group, the translations. Mathematically the free particle states may be considered as special representations of the underlying groups. A free particle state can then be expressed by applying the field operator $\psi_{\alpha}(x)$ on the vacuum state:

$$|\psi_f\rangle = \int d^4x f_{\alpha}(x) \psi_{\alpha}^*(x) |0\rangle, \quad 2(2)$$

2(2) may be called the 'contravariant representation' of the state $|\psi_f\rangle$ by means of the function $f_{\alpha}(x)$. The function $g_{\alpha}(x)$ defined by

$$g_{\alpha}(x) = \langle 0 | \psi_{\alpha}(x) | \psi_f \rangle \quad 2(3)$$

will be interpreted as 'covariant representation' of $|\psi_f\rangle$. As one sees from 2(1), the function $g_{\alpha}(x)$ fulfills the same wave equation as $\psi_{\alpha}(x)$:

$$(\square + \kappa^2) g_{\alpha}(x) = 0.$$

This is generally not true for the function $f_{\alpha}(x)$.

An important mathematical quantity is the so called 2-point function, the vacuum expectation value of the product of the free field operator and its conjugate

$$\langle 0 | \psi_\alpha(x) \psi_\beta^\dagger(y) | 0 \rangle = \int d^4p e^{ip(x-y)} \frac{c_{\alpha\beta}(p)}{p^2 - \kappa^2}, \quad 2(4)$$

where the numerator $c_{\alpha\beta}(p)$ depends on the special representation concerned. It is convenient on the right-hand side of 2(4) to integrate on the real axis in p_1, p_2, p_3 , but in the complex p_0 -plane on a path of integration depending on the order of factors on the left-hand side. For a different order of factors, for example, for the time ordered product or for the commutator or anticommutator a different path of integration must be chosen. The details of these relations, for which the existence of a pole at $p^2 - \kappa^2 = 0$ is essential, are described in the textbooks on quantum field theory and are briefly summarized in Appendix A I.

When the particles move under the influence of an outer electromagnetic field, their behaviour can be described by a linear wave equation differing from 2(1) by supplementary terms which depend on the charge, the magnetic moment and other electromagnetic properties of the particles.

The many different field operators for the many different free particles and their corresponding wave equations do in this way contain a great number of important informations concerning the properties of the particles. They are suited for defining the state of affairs at the beginning and at the end of an experiment, for describing the incoming and the outgoing particles. Therefore they are frequently called the 'in' or 'out' operators or asymptotic operators.

On the other hand they are not very useful as mathematical tools for formulating a theory of the elementary particles. They are secondary quantities, derived from the asymptotic behaviour of other more fundamental mathematical expressions describing the inner mechanism of the world of elementary particles. The asymptotic operators can be constructed only after the solutions of the mathematical problems of elementary particle physics have been found.

The initial or the final states are, by definition, states in which the interaction between the particles can be neglected. Such states, and only such states, can be constructed by applying sums of products of asymptotic operators on the vacuum. They are usually supposed to form a complete set of states in the sense that any process of interaction will finally, when the interaction has become negligible, lead to a state within this set. Even if this is true, the state of interaction itself may belong to a wider group of states which cannot be represented by the asymptotic states alone. Besides that

this idea of a complete set of asymptotic states encounters the difficulty that any such state will as a rule contain an unknown number of infra-red light quanta¹⁷ which cannot be observed. Therefore the actual state is not a 'pure case' in the quantum theoretical sense but rather a 'mixture'. These difficulties are, however, of a more formal nature and shall not be discussed here in detail. In many cases the infra-red light quanta are not important and can simply be neglected.

2-2 The S-matrix

The S-matrix or scattering matrix relates the initial to the final states, the incoming cluster of particles to the outgoing cluster.¹⁸ Therefore it is a first and very important mathematical tool for describing—at least phenomenologically—the interaction. Mathematically the S-matrix is a unitary transformation matrix, which transforms from the initial to the final asymptotic states:

$$|in\rangle = S|out\rangle, \quad |out\rangle = S^{-1}|in\rangle \quad 2(5)$$

or, in slightly different notation

$$S = |in\rangle \langle out|. \quad 2(6)$$

The unitarity of the S-matrix is essential for its physical interpretation. From

$$S^* S = S S^* = 1 \quad 2(7)$$

one gets for the diagonal elements

$$\sum_{out} |\langle out|in\rangle|^2 = \sum_{in} |\langle out|in\rangle|^2 = 1. \quad 2(8)$$

This equation allows to assume, that $|\langle out|in\rangle|^2$ is the probability for finding the state $|out\rangle$ in an experiment which starts with the state $|in\rangle$, or for finding the initial state $|in\rangle$ in an experiment, where we know that the final state is the special state $|out\rangle$. In this way the S-matrix can be compared immediately with the observations in a given experiment; in fact it seems to comprise everything that can actually be observed.

Besides the postulate of unitarity the S-matrix has to fulfill other conditions resulting from the fundamental group structure of the underlying natural law. If this law is invariant under a certain group, the S-matrix must be invariant as well. The operators O_i performing given transformations within the group can be represented as unitary operators in the space

of the asymptotic states. For these operators the invariance of the S-matrix can be formulated by

$$O_i^{-1} S O_i = S \quad 2(9)$$

or

$$O_i S - S O_i = 0. \quad 2(10)$$

The generating operators must commute with the S-matrix.

If the initial and final states are influenced by outer fields, for example, electromagnetic fields, the invariance of the S-matrix will be broken, unless either these fields themselves are invariant under the transformation, or they take part in the transformation. In the second case the invariance of the S-matrix can be verified experimentally only when the fields can be chosen arbitrarily. A field, however, which is produced by the world as a whole, or by very distant masses (like the centrifugal force in the theory of general relativity) cannot be changed arbitrarily. Therefore in this case the invariance of the S-matrix is practically broken; its invariance, which would still be true 'in principle', and the corresponding law of conservation cannot be tested experimentally. This is the reason why, as has been pointed out in §1-2, one cannot distinguish experimentally in any simple way between an intrinsic lack of symmetry in the fundamental natural law and a perturbation of this symmetry by an outer field which comes from the whole world or from very distant masses.

The invariance of S produces immediately the corresponding laws of conservation. If, for example, the initial state belongs to an eigenvalue O_i of the transformation operator O_i :

$$O_i |\text{in}\rangle = O_i' |\text{in}\rangle, \quad 2(11)$$

the final state belongs to the same eigenvalue O_i' :

$$\begin{aligned} O_i |\text{out}\rangle &= O_i S^{-1} |\text{in}\rangle = O_i S^{-1} O_i^{-1} O_i |\text{in}\rangle \\ &= S^{-1} O_i' |\text{in}\rangle = O_i' |\text{out}\rangle. \end{aligned} \quad 2(12)$$

Therefore the quantity O_i' has been 'conserved' during the interaction process. This conservation would not be true, if there were outer fields in the initial or final state, which are not invariant under O_i . In this case one might say, that some of the quantity O_i' has been exchanged between the outer field and the elementary particles. The conservation law may then be restored formally by including in the balance the quantity connected with the outer fields and their sources. Practically this conservation could however not be verified experimentally when the outer fields come from the whole world or from very distant masses.

Besides unitarity and invariance under the fundamental groups, the

S-matrix must in some way contain a mathematical formulation of causality, i.e., of that relation between cause and effect which follows from special relativity (compare §1-5). This is necessary since the S-matrix permits a complete construction of the outgoing states from the incoming states. Many investigations carried out in recent years have given a limited insight into the mathematical representation of causality in the analytical behaviour of S-matrix elements.¹⁹ The simplest example is the elastic scattering of two interacting particles. In this case the analytical behaviour of the matrix elements corresponding to local causality has been analysed thoroughly. It leads to various 'dispersion relations', which seem to be a very valuable tool for a theoretical analysis of the experimental data. For more complicated cases however, like the production of new particles in a collision, only little is known about the analytical structure of the S-matrix elements representing causality. A first survey can probably be obtained by treating problems of this kind by a first-order perturbation calculation in a local field theory. Whether the analytical behaviour derived in this way should be generally valid in an exact theory, is however an open question. On the one hand one cannot exclude the possibility that the real interaction has some features of non-locality, on the other hand a completely local field theory with interaction could possibly contain mathematical inconsistencies. Therefore, with the present knowledge of quantum field theory a definite answer to the problem of causality cannot be given.

The postulate of causality in the relativistic sense has still another interesting consequence for the asymptotic operators and the S-matrix. If interaction can be carried only from a point to a neighbouring point, it is expected that for the asymptotic operators such solutions of 2(1) will play a rôle, which for a very short time interval are concentrated in a very small space region. The Fourier transform of such solutions (of the type of Schwinger's Δ -function) contains positive and negative frequencies. Therefore the asymptotic operators are conventionally interpreted as both creation and annihilation operators, i.e. as operators which create anti-particles and annihilate particles, while their conjugate operators create particles and annihilate antiparticles. An S-matrix element may be constructed formally by a certain number of in- and out-operators:

$$\langle \text{out} | \text{in} \rangle = \langle 0 | \psi_{\text{out}}^{(1)} \psi_{\text{out}}^{(2)} \dots \psi_{\text{in}}^{*(1)} \psi_{\text{in}}^{*(2)} \dots | 0 \rangle. \quad 2(13)$$

By changing one of these asymptotic operators from a creation to an annihilation operator or vice versa, and thereby changing it from an in- to an out-operator, one arrives at a different S-matrix element containing the same total number of asymptotic operators, i.e. referring to the same total number of in- and out-going particles. In case of strict local causality

these two matrix elements, considered as analytical functions of the co-ordinates in the asymptotic operators, should be connected by analytical continuation. This relation is usually called 'crossing symmetry'²⁰ and seems to agree well with the existing observations. Its mathematical analysis can be given only after establishing a mathematical scheme for calculating S-matrix elements from a more general theoretical basis. Eqn. 2(13) is just a formal definition, not a basis for a theoretical calculation.

In this way the postulate of causality finds an adequate mathematical representation in the analytical character of the S-matrix elements, considered as functions of the co-ordinates of the incoming and outgoing particles. On the other hand the theory of analytical functions of many complex variables is very complicated. Therefore it would probably be difficult to reach in this way general mathematical statements concerning the behaviour of S-matrix elements depending on many variables.

The three main features of the S-matrix: unitarity, invariance under the fundamental groups and analytical representation of causality, are possibly sufficient in principle, to determine its structure completely. Still, on account of its mathematical complexity, the S-matrix can scarcely be a good mathematical tool for defining the natural law underlying elementary particle physics. A (possibly incomplete) description of the phenomena by means of an S-matrix will certainly be possible. But the S-matrix should be the result, not the definition of the theory, it should be the outcome of a mathematical process starting from axioms which can be formulated in a much simpler way. This is certainly the rôle which the S-matrix plays in unrelativistic quantum mechanics; and since quantum mechanics is just a special part of elementary particle physics, one should expect that the rôle of the S-matrix in this more general field is essentially the same.

2-3 The field operators

The next mathematical tool which possibly may be used for a mathematical description of elementary particle phenomena is a local field operator $\chi(x)$. There are mainly two arguments for an applicability of such operators. In unrelativistic quantum mechanics Jordan, Klein and Wigner²¹ have shown that the fundamental equation can be formulated by means of a field operator. Besides that, the success of the postulate of local causality (dispersion relations and crossing symmetry) in elementary particle physics suggests the existence of some fundamental field operator and a corresponding differential equation ('field equation') which could possibly provide the definition of the mathematical structure of the theory.

Field operators which can be used in this way for defining the theory are

of a very different nature from the asymptotic field operators of free particles discussed in §2-1. Such fundamental field operators are not connected with only one special type of particle; they might be interpreted as creating and annihilating matter in general, not only special stationary states of matter.

If the fundamental field operators $\chi_\alpha(x)$ constitute a simple non-trivial representation of the underlying symmetry groups, one may, by building expressions of the type

$$O_\lambda(x) = \int \chi_\alpha(x) \chi_\beta(x + \xi_1) \chi_\gamma^*(x + \xi_2) \dots f_{\alpha\beta\gamma\dots\lambda}(\xi_1, \xi_2, \dots) d\xi_1 d\xi_2 \dots \quad 2(14)$$

construct many other more general representations of the same groups. Operators of this general type may again be interpreted as creating and annihilating matter. Their connexion with the 'asymptotic' operators (free particle operators) can be seen in the following way. The 2-point functions constructed from $O_\lambda(x)$ may contain (in their momentum representation) poles at certain mass values:

$$\langle 0 | O_\lambda(x) O_\mu^*(y) | 0 \rangle = \int d^4p e^{ip(x-y)} \left(\frac{c_{\lambda\mu}(p)}{p^2 - \kappa^2} + \dots \right). \quad 2(15)$$

In this case an asymptotic operator $\psi_\lambda(x)$ could be defined, which in its 2-point function gives just the same pole term as $O_\lambda(x)$, but no other pole or cut. Then one may say, that the operator $O_\lambda(x)$ 'contains' the free particle operator $\psi_\lambda(x)$, that $O_\lambda(x)$ creates or annihilates particles of mass κ , characterized with respect to their quantum numbers by the transformation properties of $O_\lambda(x)$. In this sense $O_\lambda(x)$ will generally contain several free particle operators; its 2-point function will, besides several pole terms, generally also contain cuts representing continuous spectra belonging to this symmetry. The knowledge of the analytical properties of the 2-point function 2(15) would already imply the complete knowledge of all particles of this symmetry. For a given symmetry many operators $O_\lambda(x)$ may be constructed, containing different numbers of $\chi(x)$ and $\chi^*(x)$ operators; still the poles in their 2-point functions 2(15) will be identical (not the residues), since the pole belongs to a definite symmetry. The different operators $O_\lambda(x)$ belonging to the same symmetry will presumably be connected by some fundamental field equation, which defines interaction and, by its transformation properties, the symmetries of the underlying natural law.

If fundamental field operators can in this way be applied for a mathematical description of the phenomena, the postulate of local causality should be expressed by the simple assumption, that the commutator (or,

in case of spinor field, the anticommutator) vanishes for space like distances of the co-ordinates. This assumption is probably sufficient for the definition of local causality, if it is combined with the axiom, that the field operators taken within the arbitrarily small (but finite) time interval between t and $t + \Delta t$ should be sufficient²² for constructing the complete Hilbert space from the ground state $|0\rangle$. We will not go into the mathematical details of these assumptions, since a complete analysis of the axiomatic situation in this kind of field theory has not yet been given; nor will we try to specify too closely the Hilbert space of states, in which the operators $\chi_a(x)$ act. The structure and the metric of the Hilbert space should finally be decided upon by the requirement of consistency of the whole mathematical scheme. The Hilbert space must definitely contain the asymptotic states, which can be constructed from the groundstate by the asymptotic operators; because these are the physical states which can be immediately interpreted in terms of observations. But it may contain other states as well,²³ as a kind of mathematical supplement necessary—or at least useful—for a simple formulation of the theory.

If the postulate of local causality holds and if the operators of the time interval between t and $t + \Delta t$ are sufficient for constructing the complete Hilbert space, the operators taken at other times must be related to the operators of the time interval by a mathematical process compatible with the assumption of commutability (or anticommutability) at space like distances. The simplest definition of such a process could be given by a differential equation for the field operators. The differential equation should in the case of local causality be invariant under Lorentz-transformations and would therefore be hyperbolic; hence it would be compatible with commutability of observables at space-like distances. Still a differential equation for the field operators raises problems which do not exist in a classical field theory.

This equation should define interaction, therefore it cannot be a linear equation. The interaction must be local, hence the corresponding term in the equation should contain the product of field operators at the same space time point. As a rule, however, such a product is not well defined. It would certainly not be defined if the field operators would obey a canonical commutation rule. The commutator (or anticommutator) is probably singular near the origin. The problem of this singularity cannot be separated from the other problem of defining what the interaction term actually means.²³ In fact the singularity of the commutator would be largely determined by the interaction term, since the 2-point function and the commutator must be compatible with the differential equation.

Therefore, any attempt of defining a quantum field theory with interaction

by a differential equation requires a careful study of the singularity of the 2-point function near the origin.

A differential equation for the field operators constitutes an infinite set of differential equations for infinitely many matrix elements of products of field operators. The solutions of this set are not sufficiently defined by the equations. The differential equations must be supplemented by boundary conditions in order to construct a Hilbert space from their solutions. In conventional quantum mechanics there are essentially two kinds of boundaries: infinite space-like distances, and very small distances at singular points. The behaviour of the wave functions at these boundaries is restricted by the condition, that the norm of the state represented by the wave function should be finite. In a relativistic quantum field theory there may be a third boundary.²⁴ The interaction defined by the differential equation should allow the creation of pairs. In this case the infinite set of matrix elements connected by differential equations contains matrix elements with an arbitrarily high number of variables. The third boundary therefore refers to an infinitely high number of variables. The condition of finite norm requires first of all a definition of the norm in this case and raises difficult mathematical problems, which will be discussed later only in connexion with the special field theory described in the present book.

2-4 Connexion between field operators and S-matrix

The general procedure 2(14) for constructing an operator $O_\lambda(x)$ from the field operator $\chi(x)$ permits—as has been stated before—the construction of many different operators $O_\lambda(x)$ belonging to the same symmetry and also of different operators belonging to different symmetries. If any of these operators of equal symmetry leads to a pole in the corresponding 2-point function 2(15), the same pole will as a rule occur in the 2-point function of any of the other operators. The existence of the pole in 2(15) will in fact prove, that there exists a discrete stationary state $|\psi\rangle$ of mass κ and of the symmetry given by $O_\lambda(x)$, for which the matrix element

$$\phi_\lambda(x) = \langle 0 | O_\lambda(x) | \psi \rangle \quad 2(16)$$

obeys the wave equation

$$(\square + \kappa^2) \phi_\lambda(x) = 0. \quad 2(17)$$

$\phi_\lambda(x)$ may, except for a constant factor, be identified with the wave function $\psi_\lambda(x)$ of 2(3) for the 'free particle' state $|\psi\rangle$. In the same way any of the other operators $O_\lambda(x)$ of equal symmetry will lead to similar matrix elements which as a rule are different from zero and differ only by the constant

factor from $g_\lambda(x)$. This was meant by the statement, that the operators $O_\lambda(x)$ 'contain the free particle field' $\psi_\lambda(x)$.

The vacuum expectation value

$$\langle 0 | T O_\lambda^{(1)}(x_1) O_\lambda^{(2)}(x_2) \dots O_\lambda^{(n)}(x_n) | 0 \rangle \quad 2(18)$$

(where the operators $O_\lambda^{(i)}(x_i)$ may belong to different or equal symmetries) is now closely related to an S-matrix element. (The letter T means the time ordered product in the conventional sense, in which operators with larger time values are on the left-hand side of those with smaller time values.) If some of the time variables in 2(18) go to very large positive values ($t_1, t_2, \dots, t_s \rightarrow +\infty$), the others to very large negative values ($t_{s+1}, t_{s+2}, \dots, t_n \rightarrow -\infty$), the matrix element may, in this limit, be interpreted as an S-matrix element $\langle \text{out} | \text{in} \rangle$ in 2(6). The operators $O^{(s+1)}, O^{(s+2)} \dots O^{(n)}$ create the incoming particles at $t \rightarrow -\infty$, while $O^{(1)}, O^{(2)} \dots O^{(s)}$ annihilate at $t \rightarrow +\infty$ the outgoing particles. Therefore this asymptotic value of 2(18) defines an S-matrix element.

In momentum space 2(18) will have the general form

$$\delta(p_1 + p_2 + \dots + p_n) \frac{f(p_1, p_2, \dots, p_n)}{(p_1^2 - \kappa_1^2)(p_2^2 - \kappa_2^2) \dots (p_n^2 - \kappa_n^2)} \quad 2(19)$$

For the asymptotic value of 2(18) in ordinary space ($t \rightarrow \pm\infty$) only the infinitesimal surrounding of the poles in 2(19) is important. This is true, independent of the choice between $t \rightarrow +\infty$ or $t \rightarrow -\infty$ in any one of the operators. If in a special operator one goes over from the limit $t \rightarrow -\infty$ to $t \rightarrow +\infty$, one thereby goes over from the first S-matrix element to a different one referring to the same number of particles. The two S-matrix elements are connected by analytical continuation in 2(18). This connexion is the immediate expression of 'crossing symmetry'.²⁰

2-5 The axiomatic formulation

The S-matrix description of elementary particle phenomena (§2-2) seems to be a rather wide frame, which, judging from previous mathematical analysis, should be consistent, unless very special assumptions would be added concerning the analytical representation of causality. The concept of a local field operator (§2-3) is much more problematic. The condition of (anti-) commutativity at space-like distances and the assumption of a differential equation for the field operator impose severe restrictions on the mathematical scheme. If they can be fulfilled, the differential equation should allow a simple formulation of the natural law underlying elementary particle physics. Even this frame is considerably wider than the axiomatic

frame which has frequently been discussed in connexion with quantum electrodynamics and has been constructed in close analogy to the axiomatic frame of conventional unrelativistic quantum mechanics.

This rather narrow frame, which has been studied in detail by Wightman,²⁵ Lehmann, Symanzik and Zimmermann²⁶ imposes two further restrictions (beyond the assumptions of §2-3) on the mathematical scheme. It assumes that the metric in the Hilbert space on which the field operators act should be definite and positive, and that the asymptotic field operators alone should be sufficient for constructing the complete Hilbert space. The second condition gives the decisive restriction. Together with the unitarity of the S-matrix it enforces already the definite metric for all those subspaces of the total Hilbert space, which are separated from the rest by a superselection rule. The definite metric sets sharp limits to the possible singularities of the 2-point functions in the neighbourhood of the origin and the light-cone. The singularity must be at least of the type of a δ -function. If the interaction does not allow a singularity of this type, the conditions cannot be fulfilled. Hitherto it has not been possible to give any example of a theory with local interaction fitting consistently into this narrow axiomatic frame.

This result is somewhat strange since—given the unitarity of the S-matrix—the postulate of positive metric for the Hilbert space does not seem to be a very severe restriction. One cannot see easily why the addition of unphysical states to the (asymptotic) Hilbert space would make it much simpler to define a local field operator and to construct a non-linear differential equation for its change in time. (As a special example of this situation we mention the Bleuler-Gupta²⁷ version of quantum electrodynamics.) Therefore it is important to notice that from a group theoretical point of view the unitarity of the S-matrix is a very natural postulate, while the assumption of a definite metric in the complete Hilbert space is not. This can be seen in the following way.

The group of all transformations which can be represented in the complete Hilbert space is necessarily a non-compact group. This follows from the fact, that the Lorentz group must belong to this group as a subgroup, and is non-compact. A finite representation of a non-compact group can be given only in space with indefinite metric. It is true that representations of a non-compact group are possible in a space of an infinite number of dimensions with definite metric. But the finite representations which could refer to a subspace of the total infinite Hilbert space are simpler, and in so far, more natural, and they require the indefinite metric. The S-matrix, on the other hand, refers to specified values of energy and momentum, it is defined 'on the energy-shell'. When the total energy and momentum are given, the group of remaining transformations is a compact group. Compact

groups can be represented in a space with positive definite metric and a finite number of dimensions. Hence for the S-matrix unitarity is a natural requirement.

From the point of view of physical interpretation only the unitarity of the S-matrix is necessary. This unitarity is sufficient for permitting the conventional probability interpretation of the S-matrix elements. But a corresponding interpretation of local events within space regions of 10^{-13} cm is not required; one may well doubt whether it is possible. Therefore, the reduction of the total Hilbert space to the space of the asymptotic states and the postulate of definite metric seem to be an unnecessary constraint which may rather complicate than simplify the mathematical structure of field theory with interaction.

If one tries to formulate a field theory within the wider frame of §2-3, the concept of 'asymptotic states' may become problematic. Even in the limit $t \rightarrow \pm \infty$, states may remain which cannot be composed of eigenstates of energy or momentum. Still the transformation from the incoming to the outgoing states, expressed by the S-matrix, should exist. By a further transformation within the incoming and the outgoing states the S-matrix can be brought into its diagonal form. Among the diagonal elements belonging to eigenstates of energy and momentum some will be of the form $\exp i\alpha$ (α real), the others will not. Only the states belonging to the first group should be called physical states; they form a subspace of the total Hilbert space, and the process of interaction can never lead from a state within this subspace to a state outside of it. Within this subspace the conventional definition of probability can be taken over from ordinary quantum mechanics.

2-6 Hilbert space with indefinite metric

A quantum field theory of the general type §2-3 seems to require a Hilbert space with indefinite metric. For such a space the mathematical formalism is somewhat different from the conventional formalism in the textbooks, which refers to definite metric. Therefore the essential concepts concerning the Hilbert space with indefinite metric should be outlined briefly.²⁸

If the Hilbert space is constructed by a number of fundamental state vectors $|\Phi_n\rangle$, then every state vector $|\psi\rangle$ in this space can be defined by two sets of components.

We can either write

$$|\psi\rangle = |\Phi_i\rangle \cdot \psi^i, \quad 2(20)$$

where it is understood that the summation shall be carried out if the same

index occurs as lower and upper index. Then the ψ^i are called the contravariant components of $|\psi\rangle$.

Or one can put

$$\psi_i = \langle \Phi_i | \psi \rangle. \quad 2(21)$$

The ψ_i are the covariant components of $|\psi\rangle$.

The metric is defined by the relation

$$g_{ik} = \langle \Phi_i | \Phi_k \rangle. \quad 2(22)$$

In connexion with 2(22) we require

$$g_{ik} = g_{ki}^* \quad 2(23)$$

and from 2(20,21,22):

$$\psi_i = g_{ik} \psi^k. \quad 2(24)$$

From the $|\Phi_i\rangle$ one can generally construct the 'inverse' system of state vectors Φ_i^k , defined by

$$\langle \Phi_i^k | \Phi_l \rangle = \delta_l^k. \quad 2(25)$$

From 2(20) and 2(25) one gets

$$\psi^i = \langle \Phi_i^k | \psi \rangle. \quad 2(26)$$

The norm of $|\psi\rangle$ can be calculated from the components

$$\langle \psi | \psi \rangle = \psi_i^* \psi^i = \psi_i^{*l} \psi_l = \psi_i^{*l} \psi^k g_{lk} = \psi_l^* \psi_k g^{lk}. \quad 2(27)$$

The matrix elements of an operator A can be introduced by

$$\langle \Phi_i | A | \Phi_k \rangle = {}_i A_k; \quad \langle \Phi_i | A | \Phi^k \rangle = {}_i A^k, \text{ etc.}$$

and matrix multiplication then follows the rule

$${}_k (AB)_l = {}_k A_m {}^m B_l = {}_k A^m {}_m B_l = {}_k A_m g^{mn} {}_n B_l.$$

A selfadjoint operator is defined by

$$\langle \Phi_k | A | \Phi_l \rangle = \langle \Phi_l | A | \Phi_k \rangle^* \quad \text{or} \quad {}_k A_l = {}_l A^k.$$

In the indefinite metric one may also use the term 'pseudohermitian' for this property. In this sense the adjoint or pseudohermitian conjugate B^* to the operator B is given by

$$\langle \Phi_k | B^* | \Phi_l \rangle = \langle \Phi_l | B | \Phi_k \rangle^*$$

or

$${}_k B_l^* = ({}_l B_k)^*. \quad 2(28)$$

The expectation value of an operator B in the state $|\psi\rangle$ is defined by

$$\bar{B} = \frac{\langle\psi|B|\psi\rangle}{\langle\psi|\psi\rangle}; \quad 2(29)$$

it is real for a selfadjoint operator.

Observable quantities are generally represented by selfadjoint operators. The eigenvalues of these operators need not necessarily be real. Let $|\psi\rangle$ be an eigenstate of the selfadjoint operator O , and O' the corresponding eigenvalue:

$$O|\psi\rangle = O'|\psi\rangle, \quad 2(30)$$

then multiplication from the left by $\langle\psi|$ leads to

$$\langle\psi|O|\psi\rangle = O'\langle\psi|\psi\rangle. \quad 2(31)$$

The expression on the left-hand side is real according to 2(28). If the norm $\langle\psi|\psi\rangle$ is different from zero, then O' must be real and is in fact identical with the expectation value 2(29) of O in the state $|\psi\rangle$. If O' is imaginary or complex, the norm $\langle\psi|\psi\rangle$ must necessarily vanish. Since physical states are states with a finite norm, the eigenvalues and the expectation values of selfadjoint operators, representing observables, are always real in these states. Complex eigenvalues of such operators can belong only to 'ghost'-states of norm zero.

If $|\psi_1\rangle$ and $|\psi_2\rangle$ are two different eigenstates of O , belonging to the two eigenvalues O'_1 and O'_2 , then from

$$O|\psi_1\rangle = O'_1|\psi_1\rangle \quad \text{and} \quad O|\psi_2\rangle = O'_2|\psi_2\rangle \quad 2(32)$$

or

$$\langle\psi_2|O = O'_2^*\langle\psi_2| \quad 2(33)$$

one concludes

$$\left. \begin{aligned} \langle\psi_2|O|\psi_1\rangle &= \langle\psi_2|\psi_1\rangle O'_1 = O'_2^*\langle\psi_2|\psi_1\rangle, \\ \text{or} \\ (O'_1 - O'_2^*)\langle\psi_2|\psi_1\rangle &= 0. \end{aligned} \right\} \quad 2(34)$$

Therefore the two states must be orthogonal to each other, unless $O'_1 = O'_2^*$. Two physical eigenstates with the different real eigenvalues O'_1 and O'_2 are necessarily orthogonal.

The eigenstates of an operator O will as a rule not be sufficient for constructing the complete Hilbert space.

These relations holding for a Hilbert space with indefinite metric replace the corresponding relations known from conventional quantum mechanics. They may be sufficient as an explanation for the use which is made of the Hilbert space with indefinite metric in the later chapters.

3

The Fundamental Field Equation

3-1 The mathematical form of the field equation

In the attempt to formulate a unified field theory of elementary particles the guiding principles discussed in §1 are the empirical group structure, i.e. the laws of conservation, and relativistic causality. The empirical fact that some of the groups are only approximately valid (compare §1-2), requires a distinction between fundamental groups, characterizing the underlying natural law, and approximate symmetries arising from the dynamics of the system. The non-linear spinor theory described in the present book takes the Lorentz group and, for the isospin, the group U_2 as fundamental, but the group SU_3 or higher groups as approximate dynamical symmetries*. The arguments for this choice will be discussed in §3-2 and §3-5.

Starting from these assumptions there seems to be only one simple equation which can possibly represent the observed elementary particles and shall be taken as the basis of the present book:²⁹

$$i\sigma^\nu \frac{\partial \chi(x)}{\partial x^\nu} + l^2 \sigma^\nu : \chi(x) (\chi^*(x) \sigma_\nu \chi(x)) : = 0. \quad 3(1)$$

l is an arbitrary constant of the dimension of a length, if the 2-point function $\langle 0|\chi(x)\chi^*(y)|0\rangle$ is in the conventional way taken as the minus third power of a length. l may simply be taken as the unit of length.

The field operator $\chi(x)$ is defined as a 2-component spinor (Weyl-spinor) under Lorentz transformations and a 2-component spinor in isospace. If the first index of $\chi(x) = \chi_{\alpha\beta}(x)$ refers to Lorentz space, the second to isospace, the explicit form of 3(1) should be

$$i\sigma^\nu_{\alpha\beta} \frac{\partial \chi_{\beta\gamma}(x)}{\partial x^\nu} + l^2 \sigma^\nu_{\alpha\beta} : \chi_{\beta\gamma}(x) \chi^*_{\delta\epsilon}(x) \sigma_{\nu,\delta\eta} \chi_{\eta\epsilon}(x) : = 0. \quad 3(2)$$

The $\sigma^\nu = (1, \boldsymbol{\sigma})$ are the conventional Pauli matrices.

*Different fundamental assumptions have been discussed in various papers⁸⁰