*Acta Math. Hungar.*
**74 (1–2)** (1997), 135–153.

# TAUBERIAN THEOREMS FOR INTEGRAL TRANSFORMS OF DISTRIBUTIONS

S. PILIPOVIĆ and B. STANKOVIĆ (Novi Sad)

## 1. Introduction

Integral transforms of Schwartz's distributions and other generalized functions have been used in the last thirty years as a powerful tool especially in mathematical physics. The monograph of Zemanian [18] is the first one which gives systematic and general approach to different integral transforms of generalized functions. Tauberian type results, with the use of generalized asymptotic behaviours, have been elaborated only for some special integral transforms of distributions. Vladimirov, Drozhinov and Zav'yalov have given in [15] Abelian and Tauberian theorems for the Laplace transform. Various generalized asymptotics as well as Abelian and Tauberian type results for the Stiejtjes transform are investigated in [7]. Wiener type theorems (cf. [17], [10]) were studied in [6] and [8] via integral transforms with appropriate kernels.

We will give in this paper Tauberian theorems for integral transforms which are of Mellin convolution type and whose kernels belong to suitable test function spaces. The results are based on the Wiener–Tauberian theorems for distributions which are proved in [8]. The integral transforms under consideration will be the Laplace, Stieltjes, Weierstrass and Poisson transforms. We will give Tauberian type results for all these transforms.

## 2. Notation and definitions

We denote by $L$ a slowly varying function ([1]). Then $L$ is measurable and positive on $(0,\infty)$ and

$$\frac{L(xh)}{L(x)} \to 1, \quad h \to \infty, \quad \text{for every} \quad x \in \mathbf{R}_+ = (0,\infty).$$

As usual (cf. [11]), $\mathcal{D}(\mathbf{R})$ and $\mathcal{D}'(\mathbf{R})$ denote spaces of test functions and of Schwartz's distributions, respectively, and $\mathcal{D}_{L^1}(\mathbf{R})$ is the space of smooth functions $\varphi$ on $\mathbf{R}$ $\big(\varphi \in C^\infty(\mathbf{R})\big)$ such that

$$\left\| \varphi^{(i)} \right\|_{L^1} < \infty, \quad i \in \mathbf{N}_0 = \{0, 1, \dots\}.$$

The strong dual of $\mathcal{D}_{L^1}(\mathbf{R})$ is denoted by $\mathcal{B}'$. The space of tempered distributions with the support contained in $[0,\infty)$ is denoted by $\mathcal{S}'_+$. It is isomorphic to the strong dual of the space $\mathcal{S}_{[0,\infty)} \equiv \mathcal{S}_+$. Elements of $\mathcal{S}_+$ are smooth functions $\varphi$ on $(0,\infty)$ whose all derivatives admit continuous extensions onto $[0,\infty)$ and all the norms

$$\|\varphi\|_k = \sup\left\{ (1+x^2)^{k/2} \left|\varphi^{(i)}(x)\right| ; x \in [0,\infty), i \leqq k \right\}, \quad k \in \mathbf{N}_0,$$

are finite.

Recall ([15]), an $f \in \mathcal{S}'_+$ has the quasiasymptotics related to some positive measurable function $\rho(k)$, $k > 0$, if there exists $g \in \mathcal{S}'_+$, $g \neq 0$, such that

$$\text{(1)} \qquad \lim_{k\to\infty} \left\langle \frac{f(kx)}{\rho(k)}, \phi(x) \right\rangle = \langle g, \phi \rangle, \quad \phi \in \mathcal{S}.$$

We write it in short: $f \overset{q}{\sim} g$ related to $\rho$.

In this case, it is well known (cf. [15]) that there are $\alpha \in \mathbf{R}$, slowly varying function $L$ and $C \neq 0$ such that $\rho(k) = k^\alpha L(k)$, $k > 0$, and $g = C\theta_{\alpha+1}$, where

$$\theta_{\alpha+1}(x) = \begin{cases} \frac{H(x)x^\alpha}{\Gamma(\alpha+1)}, & \alpha > -1 \\ D^n \theta_{\alpha+n+1}(x), & \alpha + n > -1,\ n \in \mathbf{N},\ x \in \mathbf{R}; \end{cases}$$

$H$ is the Heaviside function and $D$ is the symbol for the derivative.

REMARK. It is enough to take $\phi \in \mathcal{S}_+$ in (1) (cf. [15] p. 50–51).

We denote by $\eta$ a smooth function which is equal to 1 in a neighbourhood of $+\infty$ and to 0 in a neighbourhood of $-\infty$.

Put

$$\text{(2)} \qquad c(x) = \begin{cases} L(\exp x)\exp(\alpha x), & x \geqq 0, \\ \exp(\beta x), & x < 0, \end{cases}$$

with the assumption $\alpha > \beta$.

We need a smooth approximation $c^*$ of $c$. Let

$$\omega \in C^\infty(\mathbf{R}), \quad \operatorname{supp}\omega \subset [-1,1], \quad \omega \geqq 0 \quad \text{and} \quad \int_{-1}^{1} \omega(t)\, dt = 1.$$

Put $L^*(\exp x) = \big(L(\exp t) * \omega(t)\big)(x)$, $x \in \mathbf{R}$. We define $c^*$ to be smooth and positive on $\mathbf{R}$ and

$$c^*(x) = \begin{cases} L^*(\exp x)\exp(\alpha x), & x > 1, \\ \exp \beta x, & x < 0. \end{cases} \tag{3}$$

It is easy to see that for every $x$

$$L^*\big(\exp(x+h)\big)/L(\exp h) \to 1, \quad h \to \infty.$$

We denote by $e$ and $e^*$ the functions

$$e(x) = c(\log x) \quad \text{and} \quad e^*(x) = c^*(\log x), \quad x > 0. \tag{4}$$

The functions $\eta$, $c$, $c^*$, $e$ and $e^*$ will be used in the sequel.

The Fourier transform of an $f \in L^1(\mathbf{R})$ is defined by

$$\mathcal{F}[f](\xi) = \hat{f}(\xi) = \int_{\mathbf{R}} e^{-i\xi t} f(t)\, dt, \quad \xi \in \mathbf{R},$$

and if $f \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$, then the Mellin transform is defined by

$$\mathcal{M}[f](s) = \int_0^\infty f(x) x^{-s-1}\, dx,$$

for those $s \in \mathbf{C}$ for which this integral exists.

## 3. Some relations between the basic spaces on the real line and the half line

We denote by $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$ the space of smooth functions $\phi$ defined on $\mathbf{R}_+$ for which all the seminorms

$$r_k(\phi) = \int_{\mathbf{R}_+} \big|(Dx)^k \phi(x)\big|\, dx, \quad k \in \mathbf{N}_0,$$

are finite, where $(Dx)\phi(x) = \frac{d}{dx}\big(x\phi(x)\big)$.

Note that $r_0$ is a norm. This sequence of seminorms defines the topological structure in $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$.

Define

$$\chi_1 : C^\infty(\mathbf{R}_+) \to C^\infty(\mathbf{R}), \quad \phi(x) \mapsto e^y\phi(e^y), \quad x = e^y, \quad y \in \mathbf{R},$$

$$\chi_2 : C^\infty(\mathbf{R}) \to C^\infty(\mathbf{R}_+), \quad \psi(y) \mapsto \frac{1}{x}\psi(\log x), \quad y = \log x, \quad x \in \mathbf{R}_+.$$

Clearly, they are inverse to each other and thus, they are bijections.

PROPOSITION 1. *The mapping $\chi_1$ is a topological isomorphism of $\mathcal{D}(\mathbf{R}_+)$ onto $\mathcal{D}(\mathbf{R})$, and of $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$ onto $\mathcal{D}_{L^1}(\mathbf{R})$. Its inverse is $\chi_2$.*

PROOF. We shall prove only that $\chi_1 : \tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+) \to \mathcal{D}_{L^1}(\mathbf{R})$ is a topological isomorphism.

The following formulae can be simply proved:

$$(Dx)^k\phi(x) = \sum_{j=0}^{k} a_{k,j}x^j\phi^{(j)}(x), \quad x \in \mathbf{R}_+, \quad k \in \mathbf{N}_0,$$

$$e^{(p+1)y}\phi^{(p)}(e^y) = \sum_{i=0}^{p} b_{p,i}\big(e^y\phi(e^y)\big)^{(i)}, \quad y \in \mathbf{R}, \quad p \in \mathbf{N}_0, \quad \phi \in C^\infty(\mathbf{R}_+),$$

where all the coefficients $a_{k,j}$ and $b_{p,i}$ are different from zero. This implies

$$\int_0^\infty \left|(Dx)^k\phi(x)\right| dx \leqq \sum_{j=0}^{k} c_{k,j} \int_{-\infty}^{\infty} \left|\big(e^y\phi(e^y)\big)^{(j)}\right| dy, \quad k \in \mathbf{N}_0$$

and

$$\int_{-\infty}^{\infty} \left|\big(e^y\phi(e^y)\big)^{(p)}\right| dy \leqq \sum_{i=0}^{p} d_{p,i} \int_0^\infty \left|(Dx)^i\phi(x)\right| dx, \quad p \in \mathbf{N}_0,$$

where $c_{k,j}$ and $d_{p,i}$ are suitable positive constants. This implies that $\chi_1$ is a topological isomorphism. □

Proposition 1 and $\mathcal{D}(\mathbf{R}) \hookrightarrow \mathcal{D}_{L^1}(\mathbf{R})$ imply that $\mathcal{D}(\mathbf{R}_+) \hookrightarrow \tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$, where $\hookrightarrow$ means that the left space is dense in the right one and that the inclusion mapping is continuous. Clearly, $\mathcal{S}_+$ is a subspace of $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$ and since it contains $\mathcal{D}(\mathbf{R}_+)$, it follows $\mathcal{S}_+ \hookrightarrow \tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$. This implies that $\tilde{\mathcal{D}}'_{L^1}(\mathbf{R}_+) \subset \mathcal{D}'(\mathbf{R}_+)$ and that all the elements of $\tilde{\mathcal{D}}'_{L^1}(\mathbf{R}_+)$ are tempered distributions with the support contained in $[0,\infty)$.

Let $\alpha, \beta \in \mathbf{R}$, $\delta > 0$ and $q \in C^\infty(\mathbf{R}_+)$ such that

$$q(x) = \begin{cases} x^{-\alpha-\delta}, & 0 < x \leqq 1/e, \\ x^{-\beta} & x > 1. \end{cases}$$

Denote by $E_{\alpha+\delta,\beta}$ the vector space of smooth functions $\phi$ on $\mathbf{R}_+$ such that all the seminorms

$$\alpha_k(\phi) = \int_{\mathbf{R}_+} q(x)x^{k-1}\left|\phi^{(k)}(x)\right| dx, \quad k \in \mathbf{N}_0, \tag{5}$$

are finite.

PROPOSITION 2. (i) *The mapping* $\chi_3 : \phi(x) \mapsto q(e^{-y})\phi(e^{-y})$, $x = e^{-y}$, $y \in \mathbf{R}$, *is a topological isomorphism of* $E_{\alpha+\delta,\beta}$ *onto* $\mathcal{D}_{L^1}(\mathbf{R})$.

(ii) *If* $k \in E_{\alpha+\delta,\beta}$ *and* $K(t) = k(e^t)$, $t \in \mathcal{R}$, *then* $\eta \check{K} e^{(\alpha+\delta)\cdot}$, $(1-\eta)\check{K}e^{\beta\cdot}$ $\in \mathcal{D}_{L^1}(\mathbf{R})$, *where* $\check{K}$ *denotes the function defined by* $\check{K}(x) = K(-x)$, $x \in \mathbf{R}$.

(iii) *If* $k \in E_{\alpha+\delta,\beta}$ *and* $c^*$ *is given by* (3), *then* $\chi_3(k)c^*(\cdot + h) \in \mathcal{D}_{L^1}(\mathbf{R})$ *for every* $h \in \mathbf{R}$.

PROOF. (i) One can prove by induction that for $m \in \mathbf{N}_0$ and $\phi \in E_{\alpha+\delta,\beta}$,

$$\left(q(e^{-t})\phi(e^{-t})\right)^{(m)} = \sum_{k=0}^{m} a_{k,m} e^{(\alpha+\delta-k)t}\phi^{(k)}(e^{-t}), \quad t > 1$$

and

$$\left(q(e^{-t})\phi(e^{-t})\right)^{(m)} = \sum_{k=0}^{m} b_{k,m} e^{(\beta-k)t}\phi^{(k)}(e^{-t}), \quad t < 0.$$

This implies that there is a constant $C > 0$ such that

$$\left\| \left(q(e^{-t})\phi(e^{-t})\right)^{(m)} \right\|_{L^1} \leqq C \sum_{i=0}^{m} \alpha_i(\phi),$$

where $\alpha_i$ are seminorms (5).

Thus the mapping $E_{\alpha+\delta,\beta} \to \mathcal{D}_{L^1}(\mathbf{R})$ is continuous. Let us prove the continuity of $\mathcal{D}_{L^1}(\mathbf{R}) \to E_{\alpha+\delta,\beta}$.

Let $\psi \in \mathcal{D}_{L^1}(\mathbf{R})$. Then, there exists $\phi \in E_{\alpha+\delta,\beta}$ such that $q(e^{-y})\phi(e^{-y}) = \psi(y)$, $y \in \mathbf{R}$. Indeed, we have to prove that $\phi(x) = \frac{1}{q(x)}\psi(\log x)$, $x \in \mathbf{R}_+$, is in $E_{\alpha+\delta,\beta}$.

It is easy to prove that

$$q(x)x^{k-1}\phi^{(k)}(x) = \sum_{i=0}^{k} c_{i,k} x^{-1}\psi^{(i)}(\log x) \quad \text{for} \quad x > 1/e$$

which implies

$$\int_1^\infty q(x)x^{k-1}\left|\phi^{(k)}(x)\right| dx \leqq \sum_{i=0}^{k} |c_{i,k}| \int_1^\infty \left|\psi^{(i)}(\log x)\right| \frac{dx}{x}$$

$$\leqq \sum_{i=0}^{k} |c_{i,k}| \int_0^\infty \left|\psi^{(i)}(y)\right| dy.$$

In the same way we can prove that

$$\int_0^1 q(x)x^{k-1}\left|\phi^{(k)}(x)\right| dx < C\sum_{i=0}^{k}\left\|\psi^{(i)}\right\|_{L^1},$$

which completes the proof of (i).

(ii) It follows by direct computation of $\mathcal{D}_{L^1}$-seminorms for given functions.

(iii) Let $i, j \in \mathbf{N}_0$, $h \in \mathbf{R}$ be fixed and $x_0 = \max(1-h, h)$. Then $x+h > 1$ when $x > x_0$ and $x + h < 0$ when $x < -x_0$.

Let $K(t) = k(e^t)$, $t \in \mathbf{R}$. By using the estimate

$$\left|c^{*\,(i)}(x+h)\check{K}^{(j)}(x)\right| \leqq \begin{cases} c_{i,j}\exp(\alpha x + \delta x)\left|\check{K}^{(j)}(x)\right|, & x+h>1 \\ c_{i,j}\exp(\beta x)\left|\check{K}^{(j)}(x)\right|, & x+h<0 \end{cases}$$

for a fixed $h$, which is proved in [8], p. 512, it follows that $c^*(\cdot + h)\check{K} \in \mathcal{D}_{L^1}(\mathbf{R})$.

## 4. Tauberian theorems for the Mellin convolution type transforms

First, we recall the results from [8] (Theorems A and B and Proposition A) which will be used in the sequel. For the notation see Section 2.

THEOREM A. *Let $f \in \mathcal{B}'$ and $K \in \mathcal{D}_{L^1}(\mathbf{R})$ such that $\mathcal{F}[K](\xi) \neq 0$, $\xi \in \mathbf{R}$. If*

$$\lim_{x\to\infty}(f * K)(x) = a\int_{\mathbf{R}} K(t)\,dt, \quad a \in \mathbf{R},$$

*then for every $\psi \in \mathcal{D}_{L^1}(\mathbf{R})$*

$$\lim_{x\to\infty}(f * \psi)(x) = a\int_{\mathbf{R}} \psi(t)\,dt.$$

THEOREM B. *For $f \in \mathcal{D}'(\mathbf{R})$ and $K \in C^\infty(\mathbf{R})$ we assume:*

(i) *The set $\{f(\cdot + h)/c(h);\, h \in \mathbf{R}\}$ is bounded in $\mathcal{D}'(\mathbf{R})$.*

(ii) *There exists $\delta > 0$ such that*

$$\eta\check{K}\exp\big((\alpha+\delta)\,\cdot\,\big), (1-\eta)\check{K}\exp(\beta\cdot) \in \mathcal{D}_{L^1}(\mathbf{R}).$$

*Then:*

a) *$\mathcal{F}\big[K\exp(-\alpha\cdot)\big](\xi) = \mathcal{F}[K](\xi - i\alpha)$, $\xi \in \mathbf{R}$, exists.*

b) *The convolution $f * K$ exists.*

*Moreover, if we assume:*

(iii) $\mathcal{F}[K](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$, *and*

(iv) $\lim_{x\to\infty} \frac{(f*K)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbf{R}} K(t)\exp(-\alpha t)\,dt$, $a \in \mathbf{R}$,

*then*

c) *for every* $\psi \in C^\infty(\mathbf{R})$ *for which* $\eta\check{\psi}\exp\big((\alpha+\delta)\cdot\big), (1-\eta)\check{\psi}\exp(\beta\cdot) \in \mathcal{D}_{L^1}(\mathbf{R})$, *there holds*

$$\lim_{x\to\infty} \frac{(f*\psi)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbf{R}} \psi(t)\exp(-\alpha t)\,dt.$$

PROPOSITION A. *Let $f$ and $k$ be in $L^1_{\mathrm{loc}}(\mathbf{R})$; $\alpha, \beta \in \mathbf{R}$, $\alpha > \beta$ such that*

a) $f/c \in L^\infty(\mathbf{R})$,

b) $\check{k}\exp\big((\alpha+\delta)\cdot\big) \in L^1\big((a,\infty)\big)$ *and* $\check{k}\exp(\beta\cdot) \in L^1\big((-\infty,b)\big)$ *for some* $a, b \in \mathbf{R}$, *and some* $\delta > 0$.

*Then* $(f*k)(x)$, $x \in \mathbf{R}$, *exists and* $\check{k}\exp(\alpha\cdot) \in L^1(\mathbf{R})$. *If* $\mathcal{F}[k\exp(-\alpha\,\cdot)](y) \neq 0$, $y \in \mathbf{R}$, *and*

$$\lim_{x\to\infty} \frac{(f*k)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbf{R}} k(t)\exp(-\alpha t)\,dt, \quad a \in \mathbf{R},$$

*then for every* $\psi \in L^1_{\mathrm{loc}}(\mathbf{R})$ *such that* $\check{\psi}\exp\big((\alpha+\delta)\cdot\big) \in L^1\big((a,\infty)\big)$ *and* $\check{\psi}\exp(\beta\cdot) \in L^1\big((-\infty,b)\big)$, *there holds*

$$\lim_{x\to\infty} \frac{(f*\psi)(x)}{L(\exp x)\exp(\alpha x)} = a\int_{\mathbf{R}} \psi(t)\exp(-\alpha t)\,dt.$$

*Particularly, if* $\beta + 1 > 0$, *then*

$$\int_0^x f(\log t)\,dt \sim \frac{a}{\alpha+1}x^{\alpha+1}L(x), \quad x \to \infty.$$

Suppose that $k \in C^\infty(\mathbf{R}_+)$ and that the function

$$t \mapsto \frac{1}{t}e^*(t)k\left(\frac{x}{t}\right), \quad t \in \mathbf{R}_+,$$

belongs to $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$ for every $x \in \mathbf{R}_+$. Let $F \in \mathcal{D}'(\mathbf{R}_+)$ such that $F/e^* \in \tilde{\mathcal{D}}'_{L^1}(\mathbf{R}_+)$. We define the Mellin convolution by

$$(6)\qquad (F \underset{M}{*} k)(x) = \left\langle F(t)/e^*(t), \frac{1}{t}e^*(t)k\left(\frac{x}{t}\right)\right\rangle$$

$$= \left\langle F(xt)/e^*(xt), \frac{1}{t}e^*(xt)k\left(\frac{1}{t}\right)\right\rangle, \quad x \in \mathbf{R}_+.$$

We refer to [18], Chapter 4, for this convolution. Note that several important integral transforms of distributions are of this form.

THEOREM 1. *Let* $F \in \mathcal{D}'(\mathbf{R}_+), k \in C^\infty(\mathbf{R}_+)$ *and* $\alpha > \beta$ *such that*

(i)′ *the set* $\left\{ \frac{F(u\cdot)}{e(u)};\ u \in \mathbf{R}_+ \right\}$ *is bounded in* $\mathcal{D}'(\mathbf{R}_+)$ *and*

(ii)′ *the kernel* $k$ *belongs to* $E_{\alpha+\delta,\beta}$ *for some* $\delta > 0$.

*Then,*

a)′ $\mathcal{M}[x^{-\alpha}k(x)](i\xi), \xi \in \mathbf{R}$, *exists and*

b)′ $(F \underset{M}{*} k)$, $x \in \mathbf{R}_+$, *exists as well.*

*Moreover, if we assume that*

(iii)′ $\mathcal{M}[x^{-\alpha}k(x)](i\xi) \neq 0, \xi \in \mathbf{R}$, *and*

(iv)′ $\lim_{x\to\infty}(F \underset{M}{*} k)(x)/e(x) =$

$$\lim_{x\to\infty} \frac{1}{e(x)} \left\langle \frac{F(t)}{e^*(t)}, \frac{e^*(t)}{t} k\left(\frac{x}{t}\right) \right\rangle = A \int_{\mathbf{R}_+} t^{-\alpha-1} k(t)\, dt, \quad A \in \mathbf{R},$$

*then,*

c)′ *the Mellin convolution* $(F \underset{M}{*} g)(x)$, $x \in \mathbf{R}_+$, *exists for every* $g \in E_{\alpha+\delta,\beta}$ *and*

$$\frac{(F \underset{M}{*} g)(x)}{e(x)} \to A \int_{\mathbf{R}_+} t^{-\alpha-1} g(t)\, dt, \quad x \to \infty \tag{7}$$

*(e and* $e^*$ *are defined in* (4)*).*

PROOF. We will show that the assumptions of Theorem 1 imply those of Theorem B. Then we will prove the assertions of Theorem 1 by using the conclusions of Theorem B.

Put $K(t) = k(e^t), t \in \mathbf{R}$. By Proposition 2 (ii), assumption (ii)′ implies that

$$\eta \check{K} e^{(\alpha+\delta)\cdot}, (1-\eta)\check{K} e^{\beta\cdot} \in \mathcal{D}_{L^1}(\mathbf{R}).$$

Assumptions $\alpha > \beta$ and $k \in E_{\alpha+\delta,\beta}$ imply that

$$\int_0^\infty x^{-\alpha} k(x) x^{-i\xi-1}\, dx$$

is finite for every $\xi \in \mathbf{R}$. Thus we proved a)′.

Let us prove the existence of $F \underset{M}{*} k$.

Since $k \in E_{\alpha+\delta,\beta}$, it follows that for every $x \in \mathbf{R}_+$ the function

$$t \mapsto \frac{1}{t} e^*(t) k\left(\frac{x}{t}\right), \quad t \in \mathbf{R}_+,$$

belongs to $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$. Proposition 2 (iii) gives that $c^*(\cdot+h)\check{K} \in \mathcal{D}_{L^1}(\mathbf{R})$ for every $h \in \mathbf{R}$.

Since

$$\chi_2\left[c^*(\cdot+h)\check{K}\right](t) = \frac{1}{t}c^*(\log t + \log x)k(e^{-\log t}) = \frac{1}{t}e^*(tx)k\left(\frac{1}{t}\right),$$

$t \in \mathbf{R}_+$, $x = e^h$, $h \in \mathbf{R}$, it follows that for every $x \in \mathbf{R}_+$

$$t \mapsto \frac{1}{t}e^*(tx)k\left(\frac{1}{t}\right), \quad x \in \mathbf{R}_+,$$

belongs to $\tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+)$.

Next, we prove that $F/e^*$ belongs to $\tilde{\mathcal{D}}'_{L^1}(\mathbf{R}_+)$. Proposition 1 implies that

$$\phi \mapsto \left\langle F(x), \frac{1}{x}\phi(\log x)\right\rangle, \quad \phi \in \mathcal{D}(\mathbf{R}),$$

defines a distribution $f \in \mathcal{D}'(\mathbf{R})$. Assumption (i)$'$ and

$$\left\langle \frac{f(t+h)}{c(h)}, \phi(t)\right\rangle = \left\langle \frac{F(xp)}{e(p)}, \frac{1}{x}\phi(\log x)\right\rangle, \quad h = \log p, \quad p > 0$$

imply that for every $\phi \in \mathcal{D}(\mathbf{R})$

$$\left\{\left\langle \frac{f(t+h)}{c(h)}, \phi(t)\right\rangle; \quad h \in \mathbf{R}\right\}$$

is a bounded set of continuous functions on $\mathbf{R}$ ($c$ is defined in (2)). This implies that $f/c^* \in \mathcal{B}'$. In fact, this is proved in [8] p. 512. Let $\varphi \in \mathcal{D}(\mathbf{R}_+)$. We have

$$\left\langle \frac{F(x)}{e^*(x)}, \varphi(x)\right\rangle = \left\langle \frac{f(y)}{c^*(y)}, e^y\varphi(e^y)\right\rangle.$$

Proposition 1 implies that

$$\frac{F}{e^*} = \frac{f}{c^*} \circ \chi_1 \quad \text{on} \quad \mathcal{D}(\mathbf{R}_+).$$

Since $\mathcal{D}(\mathbf{R}_+)$ is dense in $\tilde{\mathcal{D}}_{L^1}(\mathbf{R})$, we have

$$\frac{F}{e^*} = \frac{f}{e^*} \circ \chi_1 \quad \text{on} \quad \tilde{\mathcal{D}}_{L^1}(\mathbf{R}_+).$$

Hence, the Mellin convolution $F \underset{M}{*} k$ exists and

$$(f * K)(h) = (F \underset{M}{*} k)(p), \quad p = e^h, \quad h \in \mathbf{R}. \tag{8}$$

Thus, we have proved b)$'$. Note that assumption (iii)$'$ is equivalent to the assumption $\mathcal{F}[K](\xi - i\alpha) \neq 0$, $\xi \in \mathbf{R}$. By (8) we have that (iv)$'$ implies (iv) in Theorem B. Now, Theorem B c) and (8), with $g$ instead of $K$, imply the assertions in Theorem 1 c)$'$. □

COROLLARY TO THEOREM 1. *If in Theorem* 1, $F \in \mathcal{S}'_+$, $A \neq 0$ *and* $\alpha > \beta > -1$, *then the assumptions* (i)$'$–(iv)$'$ *imply that* $F$ *has the quasiasymptotics related to* $e$ *with the limit* $A\Gamma(\alpha+1)\theta_{\alpha+1}$.

PROOF. Formally, we have

$$\frac{(F \underset{M}{*} g)(x)}{e(x)} = \frac{1}{e(x)} \left\langle F(t), \frac{1}{t} g\left(\frac{x}{t}\right) \right\rangle = \frac{1}{e(x)} \left\langle F(tx), \psi(x) \right\rangle,$$

where $\psi(x) = \dfrac{1}{x} g\left(\dfrac{1}{x}\right)$, $x \in \mathbf{R}_+$.

Let $\delta > 0$. If we prove that for every $\psi \in \mathcal{S}_+$ there exists $g \in E_{\alpha+\delta,\beta}$ such that

$$x^{-1} g(x^{-1}) = \psi(x), \quad x \in \mathbf{R}_+,$$

the above formal calculation holds (since $F \in \mathcal{S}_+$) and the assertion of the corollary follows from (7) because it implies that for every $\psi \in \mathcal{S}_+$ $\lim\limits_{x \to \infty} \dfrac{1}{e(x)} \langle F(tx), \psi(x) \rangle$ exists.

So, we have to prove that $g(x) = (1/x)\psi(1/x)$, $x \in \mathbf{R}$, belongs to $E_{\alpha+\delta,\beta}$. Since for $k \in \mathbf{N}$ and $x \in \mathbf{R}_+$,

$$g^{(k)}(x) = \frac{1}{x^{k+1}} \sum_{i=0}^{k} a_{i,k} \frac{1}{x^i} \psi^{(i)}\left(\frac{1}{x}\right); \quad a_{i,k} \in \mathbf{R}, \quad k \in \mathbf{N}_0,$$

where $a_{i,k}$ are suitable numbers, we have

$$\left| q(x) x^{k-1} g^{(k)}(x) \right| \leqq \sum_{i=0}^{k} |a_{i,k}| x^{-2-i-\alpha-\delta} \left| \psi^{(i)}\left(\frac{1}{x}\right) \right| < C, \quad 0 < x < e^{-1},$$

and

$$\left| q(x) x^{k-1} g^{(k)}(x) \right| \leqq \sum_{i=0}^{k} |a_{i,k}| x^{-\beta-i-2} \left| \psi^{(i)}\left(\frac{1}{x}\right) \right| \leqq C x^{-\beta-2}, \quad x > 1.$$

This proves that $g \in E_{\alpha+\delta,\beta}$, if $\alpha > \beta > -1$. □

Let $\{\theta_\alpha; \alpha \in \mathbf{R}\}$ be the family of distributions given in Section 2 and $F \in \mathcal{S}'_+$. Let

$$F^{(-\alpha)} = \theta_\alpha * F, \quad \alpha \in \mathbf{R}.$$

If $\varphi \in \mathcal{S}_+$, and $m \in \mathbf{N}$, then

$$u^{-m} \left\langle F^{(-m)}(u\xi), \varphi(\xi) \right\rangle = \left\langle F(u\xi), \left\langle \theta_m(t), \varphi(\xi + t) \right\rangle \right\rangle, \tag{9}$$

where the mapping $\varphi \mapsto \langle \theta_m(t), \varphi(\cdot + t) \rangle$ is an automorphism on $\mathcal{S}_+$ (cf. [15], p. 52).

The next theorem is accomodated to later use.

THEOREM 2. *Let $f \in \mathcal{S}'_+$, $m \in \mathbf{N}_0$ and $\alpha > \beta$. Assume:*

(i″) *The set $\left\{ \frac{f(.u)}{u^{-m}e(u)}; u \in \mathbf{R}_+ \right\}$ is bounded in $\mathcal{S}'_+$.*

(ii″) *The kernel $k_m$ belongs to $E_{\alpha+\delta,\beta}$ for some $\delta > 0$.*

*Then,*

a″) *$\mathcal{M}[x^{-\alpha}k_m(x)](i\xi)$, $\xi \in \mathbf{R}$ exists and*

b″) *$(f^{(-m)} \underset{M}{*} k_m)(x)$, $x \in \mathbf{R}_+$, exists, as well.*

*Moreover, if*

(iii″) *$\mathcal{M}[x^{-\alpha}k_m(x)](i\xi) \neq 0, \xi \in \mathbf{R}$, and*

(iv″) *$\lim_{x\to\infty}(f^{(-m)} \underset{M}{*} k_m)(x)/e(x) = A(m)\int_{\mathbf{R}_+} \frac{1}{x}k_m\left(\frac{1}{x}\right)x^\alpha\,dx$, where $A(m) \in \mathbf{R}$, then:*

c″) *For every $g \in E_{\alpha+\delta,\beta}$, the Mellin convolution $(f^{(-m)} \underset{M}{*} g)(x), x \in \mathbf{R}_+$, exists and*

d″) *$\lim_{x\to\infty} \frac{(f^{(-m)} \underset{M}{*} g)(x)}{e(x)} = A(m)\int_{\mathbf{R}_+} \frac{1}{t}g\left(\frac{1}{t}\right)t^\alpha\,dt.$*

*Particularly, if $\beta > -1$ and $A(m) \neq 0$, then $F \overset{q}{\sim} A(m)\Gamma(\alpha+1)\theta_{\alpha+1-m}$ related to $u^{-m}e(u)$.*

PROOF. If $f$ satisfies (i″), then (9) implies that $\left\{ \frac{f^{(-m)}(u.)}{e(u)}; u \in \mathbf{R}_+ \right\}$ is a bounded set in $\mathcal{D}'(\mathbf{R}_+)$, as well. Thus, $f^{(-m)}$ satisfies condition (i′) in Theorem 1. By applying Theorem 1 to $f^{(-m)}$ and $k_m$ we obtain a″), b″), c″) and d″) in Theorem 2. By the Corollary of Theorem 1 it follows that

$$f^{(-m)} \overset{q}{\sim} A(m)\Gamma(\alpha+1)\theta_{\alpha+1} \quad \text{related to} \quad e(u).$$

Theorem 1, Section 3.1 in [15], implies

$$f \overset{q}{\sim} A(m)\Gamma(\alpha+1)\theta_{\alpha+1-m} \quad \text{related to} \quad u^{-m}e(u).$$

Proposition A, and the change of variables $t = e^x$, $x \in \mathbf{R}$, imply the next theorem.

THEOREM 3. *Let $F$ and $k \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ and $\alpha > \beta$, $\delta > 0$, be such that*
i) $F/e \in L^\infty(\mathbf{R}_+)$,
ii) $k(t)t^{-(\alpha+\delta+1)} \in L^1\big((0,1)\big)$ *and* $k(t)t^{-(\beta+1)} \in L^1\big((1,\infty)\big)$.
*Then, there exist*

$$(F \underset{M}{*} k)(x) = \int_0^\infty F(t)k\left(\frac{x}{t}\right)\frac{dt}{t}, \quad x \in \mathbf{R}_+,$$

*and* $\mathcal{M}[x^{-\alpha}k(x)](i\xi)$, $\xi \in \mathbf{R}$. *If*
iii) $\mathcal{M}[x^{-\alpha}k(x)](i\xi) \neq 0$, $\xi \in \mathbf{R}$, *and*
iv) $\lim_{x\to\infty}(F \underset{M}{*} k)(x)/e(x) = A\int_0^\infty k(t)t^{-\alpha-1}\,dt$,
*then*

$$\lim_{x\to\infty}(F \underset{M}{*} \psi)(x)/e(x) = A\int_0^\infty \psi(t)t^{-\alpha-1}\,dt, \quad A \in \mathbf{R},$$

*for every $\psi \in L^1_{\mathrm{loc}}(\mathbf{R}_+)$ such that $\psi(t)t^{-(\alpha+\delta+1)} \in L^1\big((0,1)\big)$ and $\psi(t)t^{-\beta-1}$ $\in L^1\big((1,\infty)\big)$ hold. In particular, if $\beta > -1$, then*

$$\int_0^x F(t)\,dt \sim \frac{A}{\alpha+1}x^{\alpha+1}L(x), \quad x \to \infty.$$

REMARK. Theorem 1.7.5 in [1], gives conditions which together with the last relation imply that $F(x) \sim Ax^\alpha L(x)$, $x \to \infty$.

In the context of Theorem 3 see also [4] and Theorem 4.8.3 in [1].

## 5. Applications on special integral transforms

Theorems A and B and in Proposition A are concerned with the so-called convolution transforms. Integral transforms, to which Theorems 2 and 3 can be applied, are of Mellin convolution type.

**5.1.** *Integral transforms with kernels in the Zemanian spaces $\mathcal{L}_{a,b}$ and $\mathcal{M}_{a,b}$.* Zemanian introduced in [18] the spaces $\mathcal{L}_{a,b}$ and $\mathcal{M}_{a,b}$ in order to define the Laplace and Mellin integral transforms of generalized functions. Recall

$$\mathcal{L}_{a,b} = \left\{\varphi \in C^\infty(\mathbf{R});\ \gamma_k(\varphi) = \sup_{t\in\mathbf{R}}\left|\mathcal{H}_{a,b}(t)\varphi^{(k)}(t)\right| < \infty, \quad k \in \mathbf{N}_0\right\},$$

where $\mathcal{H}_{a,b}(t) = e^{at}$, $t \geqq 0$ and $\mathcal{H}_{a,b}(t) = e^{bt}$, $t < 0$.

Note that $\gamma_k$; $k \in \mathbf{N}$ are seminorms and $\gamma_0$ is a norm in $\mathcal{L}_{a,b}$.

If the kernel $\check{K}$ belongs to $\mathcal{L}_{a,b}$ then it satisfies condition (ii) of Theorem B for $a > \alpha > \beta > b$. This follows from

$$\left(\eta(x)\exp\left((\alpha+\delta)x\right)\check{K}(x)\right)^{(m)}$$
$$= \sum_{i=0}^{m} \binom{m}{i} (\alpha+\delta)^{m-j} e^{-(a-\alpha-\delta)x} e^{ax} \check{K}^{(i)}(x), \quad x > \omega,$$

and

$$\left(\left(1-\eta(x)\right)\exp(\beta x)\check{K}(x)\right)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} \beta^{m-j} e^{(\beta-b)x} e^{bx} \check{K}^{(i)}(x), \quad x < -\omega,$$

where $0 < \delta < a - \alpha$ and $\omega$ is a large enough positive number.

*In such a way Theorem* B *can be applied to all transforms of convolution type whose kernels belong to* $\mathcal{L}_{a,b}$ *with* $a > \alpha > \beta > b$, $0 < \delta < a - \alpha$.

The space $\mathcal{M}_{a,b}$ is defined as follows (cf. [18]). Let

$$\xi_{a,b}(x) = x^{-a}, \quad 0 < x \leqq 1; \quad \xi_{a,b}(x) = x^{-b}; \quad 1 < x < \infty.$$

Then

$$\mathcal{M}_{a,b} = \left\{ \psi \in C^\infty(\mathbf{R}_+);\ \sup_{0<x<\infty} \left| \xi_{a,b}(x) x^{k+1} \psi^{(k)}(x) \right| < \infty,\ k \in \mathbf{N}_0 \right\}.$$

The mapping: $\theta(x) \to e^{-t}\theta(e^{-t})$, $x = e^{-t}$, is an isomorphism of $\mathcal{M}_{a,b}$ onto $\mathcal{L}_{a,b}$. If $\psi \in \mathcal{M}_{a,b}$ and $a > \alpha + 1 > \beta + 1 > b$, $0 < \delta < a - \alpha - 1$, then $\psi \in E_{\alpha+\delta,\beta}$. This follows from

$$\left| x^{k-1} q(x) \psi^{(k)}(x) \right| \leqq x^{-(\alpha-a+\delta+2)} \left| x^{k+1-a} \psi^{(k)}(x) \right|$$
$$\leqq C_1 x^{-(\alpha-a+\delta+2)}, \quad 0 < x < e^{-1},$$
$$\left| x^{k-1} q(x) \psi^{(k)}(x) \right| \leqq x^{-(\beta-b+2)} \left| x^{k+1-b} \psi^{(k)}(x) \right|$$
$$\leqq C_2 x^{-(\beta-b+2)}, \quad x > e^{-1}.$$

*Hence, Theorem* 2 *can be applied to transforms which are of Mellin convolution type with kernels in* $\mathcal{M}_{a,b}$, $a > \alpha + 1 > \beta + 1 > b$, $0 < \delta < a - \alpha - 1$.

**5.2.** *Laplace transform.* The most precise Tauberian theorem for the Laplace transform of distributions can be found in [15]. We refer to [13] and [14] for the Tauberian theorems concerning positive measures.

Recall that the Laplace transform of an $S \in \mathcal{S}'_+$ is defined by

$$\tilde{S}(z) = \langle S(t), e^{izt} \rangle, \quad z = x + iy \in \mathbf{R} + i\mathbf{R}_+.$$

THEOREM 4. *Let $f \in \mathcal{S}'_+$, $m \in \mathbf{N}_0$, $\alpha > \beta > -1$. Assume:*

a) *The set $\left\{ \frac{f(u.)}{u^{-m} e(u)};\ u \in \mathbf{R}_+ \right\}$ is bounded in $\mathcal{S}'_+$.*

b) $\lim\limits_{y \to 0+} \dfrac{y}{y^m e\left(\frac{1}{y}\right)} \langle f(t), e^{-yt} \rangle = A \displaystyle\int_0^\infty e^{-t} t^\alpha \, dt, \quad A \in \mathbf{R}.$

*Then $f \overset{q}{\sim} A\Gamma(\alpha+1)\theta_{\alpha-m+1}$ related to $e$.*

PROOF. If $y > 0$, then $e^{-y\cdot} \in \mathcal{S}_+$ and

$$\langle f(t), e^{-yt} \rangle = y^m \langle f^{(-m)}(t), e^{-yt} \rangle, \quad y > 0.$$

Put $k_m(t) = t^{-1} \exp(-1/t)$, $t \in \mathbf{R}_+$. Then $k_m \in E_{\alpha+\delta,\beta}$ for any $\alpha > \beta$, $\beta > -1$ and $\delta > 0$. Therefore,

$$(f^{(-m)} \underset{M}{*} k_m) \left( \frac{1}{y} \right) = y^{-m+1} \langle f(t), e^{-yt} \rangle$$

and

$$\mathcal{M}\left(t^{-\alpha} k_m(t)\right)(i\xi) = \int_0^\infty e^{-t} t^{\alpha+i\xi} \, dt = \Gamma(\alpha+1+i\xi) \neq 0,$$
$$\xi \in \mathbf{R}, \quad \alpha + 1 > 0.$$

Now Theorem 4 follows from Theorem 2 (with $k = k_m$.)

REMARK. In [15] , Section 7, Theorem 2, the authors proved the assertion of Theorem 4 under the assumptions b) and, instead of a) the following one:

There are $M > 0$, $\sigma > 0$ and $y_0 > 0$ such that

$$\left| \frac{y^{n-m+1}}{e\left(\frac{1}{y}\right)} \langle f(t), e^{-yt} \rangle \right| \leqq M n! n^\sigma, \quad 0 < y \leqq y_0, \quad n \in \mathbf{N}. \tag{10}$$

**5.3.** *Stieltjes transform.* Lavoine and Misra [3] defined the Stieltjes transform by introducing the space $\mathcal{J}'(r)$, $r \in \mathbf{R} \setminus (-\mathbf{N})$. It is the space of all distributions $f \in \mathcal{S}'_+$ such that there exist $m \in \mathbf{N}_0$ and a locally integrable function $F$, $\operatorname{supp} F \subset [0, \infty)$ such that $f = D^m F$ and

$$\int_{\mathbf{R}_+} |F(t)| (t+\beta)^{-r-1-m} \, dt < \infty \quad \text{for some} \quad \beta > 0.$$

The Stieltjes transform of an $f \in \mathcal{J}'(r)$ is defined by

$$(11) \qquad S_r[f](s) = (r+1)_m \int_0^\infty F(t)(t+s)^{-r-m-1}\,dt, \quad s \in \mathbf{C} \setminus (-\infty, 0],$$

where

$$(r+1)_m = (r+1)\dots(r+m) = \frac{\Gamma(r+m+1)}{\Gamma(r+1)}, \quad (r+1)_0 = 1.$$

Relation (11) can be written in the form

$$S_r[f](s) = (r+1)_m \langle f^{(-m)}(t), (t+s)^{-r-m-1} \rangle, \quad s \in \mathbf{C} \setminus (-\infty, 0].$$

We shall use the identity

$$(12) \qquad \frac{s^{r+m+1}}{(r+1)_m} S_r[f](s) = \left\langle f^{(-m)}(t), \left(\frac{t}{s}+1\right)^{-r-m-1} \right\rangle, \quad s \in \mathbf{R}_+.$$

The next theorem follows from Theorem 2.

THEOREM 5. *Let $f \in \mathcal{S}'_+$, $m \in N_0$, $r+m+1 > \alpha+1 > \beta+1 > 0$ and $\delta > 0$. Assume:*

a) *The set $\left\{ \frac{f(.u)}{u^{-m}e(u)};\, u > 0 \right\}$ is bounded in $\mathcal{S}'_+$.*

*Then, $S_r[f](u)$, $u \in \mathbf{R}_+$, exists. If moreover,*

b) $\lim_{u\to\infty} \frac{1}{u^{-(r+m)}e(u)} S_r[f](u) = C_m$, $C_m \in \mathbf{R}$,

*then for every $g \in E_{\alpha+\delta,\beta}$,*

(13)
$$\lim_{u\to\infty} \left\langle \frac{f^{(-m)}(\xi u)}{e(u)}, \frac{1}{\xi} g\left(\frac{1}{\xi}\right) \right\rangle = C_m \frac{\Gamma(r+1)}{\Gamma(r+m-\alpha)} \left\langle \theta_{\alpha+1}(\xi), \frac{1}{\xi} g\left(\frac{1}{\xi}\right) \right\rangle.$$

*Hence,*

$$(14) \qquad f^{(-m)} \overset{q}{\sim} C_m \frac{\Gamma(r+1)}{\Gamma(r+m-\alpha)} \theta_{\alpha+1} \quad \textit{related to} \quad e(u)$$

*and*

$$(15) \qquad f \overset{q}{\sim} C_m \big(\Gamma(r+1)/\Gamma(r+m-\alpha)\big)\, \theta_{\alpha+1-m} \quad \textit{related to} \quad u^{-m}e(u).$$

PROOF. Let $F = f^{(-m)}$ and

$$k_m(t) = \frac{1}{t}\left(\frac{1}{t}+1\right)^{-(r+m+1)}, \quad t \in \mathbf{R}_+.$$

Since $r+m+1 > \alpha+1 > \beta+1 > 0$ and

$$q(x)x^{p-1}k_m^{(p)}(x) = \sum_{i=0}^{p} a_{i,p}q(x)x^{-i-2}\left(\frac{1}{x}+1\right)^{-(r+m+i+1)}, \quad x \in \mathbf{R}_+,$$

it follows that $k_m \in E_{\alpha+\delta,\beta}$. Note that $t^{-1}k_m(t^{-1}) = (t+1)^{-(r+m+1)}$, $t \in \mathbf{R}_+$. This and (12) imply

$$(16) \qquad (f^{(-m)} \underset{M}{*} k_m)(u) = \langle f^{(-m)}(\xi u), (\xi+1)^{-(r+m+1)} \rangle$$

$$= \frac{1}{u}\left\langle f^{(-m)}(t), \left(\frac{t}{u}+1\right)^{-(r+m+1)}\right\rangle = \frac{u^{r+m}}{(r+1)_m} S_r[f^{(-m)}](u), \quad u \in \mathbf{R}_+.$$

We shall show that the conditions of Theorem 2 hold for $f$ and $k = k_m$. In fact (i) is a) and (iii$''$) follows from

$$\mathcal{M}[x^{-\alpha}k_m(x)](i\xi) = \frac{\Gamma(\alpha+1+i\xi)\Gamma(r+m-\alpha-i\xi)}{\Gamma(r+m+1)} \neq 0, \quad \xi \in \mathbf{R}.$$

Condition b) and (16) imply (iv$''$) of Theorem 2:

$$\lim_{u\to\infty} \frac{(f^{(-m)} \underset{M}{*} k_m)(u)}{e(u)} = \lim_{u\to\infty} \frac{u^{r+m}}{(r+1)_m e(u)} S_r[f](u)$$

$$= \frac{C_m}{(r+1)_m} = A_m \int_0^\infty \frac{t^\alpha}{(t+1)^{r+m+1}}\,dt = A_m \int_0^\infty \frac{1}{x}k_m\left(\frac{1}{x}\right)x^\alpha\,dx$$

where $A_m = \frac{C_m\Gamma(r+1)}{\Gamma(r+m-\alpha)}$.

Theorem 2 implies (13) and (14) since $\mathcal{S}_+ \subset E_{\alpha+\delta,\beta}$. As in the proof of Theorem 2 we have that the quasiasymptotic behaviour of $f^{(-m)}$ implies the appropriate quasiasymptotic behaviour of $\left(f^{(-m)}\right)^{(m)} = f$ and this is given in (15).

**5.4.** *Weierstrass transform.* The Weierstrass transform is related to the space $\mathcal{K}_1'$ ([2]). Recall

$$\mathcal{K}_1 \equiv \Big\{\phi \in C^\infty(\mathbf{R});$$

$$\|\phi\|_k \equiv \sup_{x\in\mathbf{R},0\leqq\alpha\leqq k} \Big\{ \big|\phi^{(\alpha)}(x)\big| \exp\big(k|x|\big) \Big\} < \infty,\ k\in \mathbf{N}_0\Big\}.$$

Let

$$k(s,t) = \frac{1}{(4\pi t)^{1/2}} \exp(-s^2/4t), \quad s\in\mathbf{C}, \quad t>0.$$

The Weierstrass transform, $W_t(f)$, $t>0$, of $f\in\mathcal{K}_1'$ is defined by

$$W_t(f)(s) = \big\langle f(x), k(s-x,t)\big\rangle = \big(f * k(\cdot,t)\big)(s), \quad s\in\mathbf{C}, \quad |\arg s| < \pi/4.$$

A Tauberian type theorem for this transform, in the case when $c(h)\equiv 1$, is given in [9]. The next one is more general.

THEOREM 6. *Let $f\in\mathcal{D}'(\mathbf{R})$, $\alpha>\beta$, and let the set $\left\{\frac{f(\cdot+h)}{c(h)};\ h\in\mathbf{R}\right\}$ be bounded in $\mathcal{D}'(\mathbf{R})$. Then $W_t(f)(s)$ exists for $s\in\mathbf{C}$, $|\arg s|\leqq \pi/4$ and $t>0$. Moreover, if we assume*

$$\lim_{h\to\infty}\frac{1}{c(h)}W_{t_0}[f](h) = \frac{A(t_0)}{(4\pi t_0)^{1/2}}\int_{-\infty}^{\infty}\exp(-x^2/4t_0)\exp(-\alpha x)\,dx$$

$$= A(t_0)\exp(\alpha^2 t_0) = B(t_0),$$

*then*

$$\lim_{h\to\infty}\left\langle \frac{f(x+h)}{c(h)},\psi(x)\right\rangle = B(t_0)\exp(-\alpha^2 t_0)\big\langle e^{\alpha x},\psi(x)\big\rangle \tag{17}$$

*for every $\psi\in C^\infty$ such that $\eta\check{\psi}\exp\big((\alpha+\delta)\cdot\big)$ and $(1-\eta)\check{\psi}\exp(\beta\cdot)$ belong to $\mathcal{D}_L(\mathbf{R})$.*

The proof follows by showing that the conditions of Theorem B are satisfied. We omit the details.

REMARK. If $A(t_0)\neq 0$, then (17) implies that $f$ has the $S$-asymptotics related to $c$ with the limit $B(t_0)\exp(-\alpha^2 t_0)\exp(\alpha x)$ in $\mathcal{D}'(\mathbf{R})$ and in $\mathcal{K}_1'$. We refer to [7] for the $S$-asymptotic behaviour of distributions.

**5.5.** *The Poisson transform.* The Poisson transform of a distribution $f \in \mathcal{D}'(\mathbf{R})$ is defined by

$$P[f](x,y) = \left(f(t) * \frac{y}{t^2+y^2}\right)(x) = \left\langle f(t), \frac{y}{(x-t)^2+y^2}\right\rangle, \quad y>0, \quad x \in \mathbf{R}$$

(cf. [16], p. 179). The equality

$$\int_{-\infty}^{\infty} e^{-i\xi t}\frac{y}{t^2+y^2}\,dt = e^{-|y\xi|}, \quad y>0, \quad \xi \in \mathbf{R},$$

and Theorem 1 imply:

THEOREM 7. *Let $f \in \mathcal{B}'$. If for some $y > 0$*

$$\lim_{x\to\infty} P[f](x,y) = a\int_{-\infty}^{\infty}\frac{y}{t^2+y^2}\,dt, \quad a \in \mathbf{R},$$

*then*

$$\lim_{x\to\infty}\langle f(t+x), \psi(t)\rangle = \langle a, \psi(t)\rangle$$

*for every $\psi \in \mathcal{D}_{L^1}(\mathbf{R})$.*

REMARK. If $a \neq 0$, then the last equality means that $f$ has the $S$-asymptotics in $\mathcal{B}'$ related to $c = 1$.

## References


[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular Variation,* Cambridge University Press (Cambridge, 1989).

[2] M. Hasumi, Note on the $n$-dimensional tempered ultra-distributions, *Tohoku Math. J.,* **13** (1961), 94–104.

[3] J. Lavoine et O. P. Misra, Sur la transformation de Stieltjes des distributions et son inversion au moyen de la transformation de Laplace, *C. R. Acad. Sc. Paris Série A,* **290** (1980), 139–142.

[4] J. Karamata, Neuer Beweis und Verallgemeinerung der Tauberschen Sätze, welche die Laplacesche und Stieltjessche Transformation betreffen, *J. Reine Angew. Math.,* **164** (1931), 27–39.

[5] R. S. Pathak and S. K. Mishra, Asymptotic behaviours of a class of integral transforms in complex domains, *Proc. Edinburgh Math. Soc.,* **32** (1989), 231–247.

[6] J. Peetre, On the value of a distribution at a point, *Portug. Math.,* **27** (1968), 149–159.

[7] S. Pilipović, B. Stanković and A. Takači, *Asymptotic Behaviour and Stieltjes Transformation of Distributions,* Teubner Verlagsgeesellschaft (Leipzig, 1990).

[8] S. Pilipović and B. Stanković, Wiener Tauberian theorems for distributions, *J. London Math. Soc.,* **47** (1993), 507–515.

[9] S. Pilipović, Asymptotic behaviour of the distributional Weierstrass transform, *Appl. Anal.*, **25** (1987), 171–179.

[10] H. R. Pitt, General Tauberian Theorems, *Proc. London Math. Soc.*, **44** (1938), 243–288.

[11] L. Schwartz, *Théorie des distributions*, Hermann (Paris, 1966).

[12] B. Stanković, Abelian and Tauberian theorems for Stieltjes transforms of distributions, *Russian Math. Surveys*, **40** (1985), 99–113.

[13] B. Stanković, Theorems of the generalized Tauberian type for measures, *Rev. Research Fac. Sc. Novi Sad*, **13** (1983), 1–14.

[14] B. Stanković, Theorems of Tauberian type for measures, *Glas. Mat. Ser. III*, **20**(40) (1985), 383–390.

[15] V. S. Vladimirov, Yu. N. Drožžinov and B. I. Zavialov, *Tauberian Theorems for Generalized Functions*, Kluwer Academic Publishers (Maine, 1988).

[16] V. S. Vladimirov, *Generalized Functions in Mathematical Physics*, Mir Publishers (Moscow, 1979).

[17] N. Wiener, Tauberian theorems, *Ann. of Math.*, **33** (1932), 1–100.

[18] A. H. Zemanian, *Generalized Integral Transformations*, Interscience Publishers (New York, 1968).

*(Received June 19, 1995)*

INSTITUTE OF MATHEMATICS
UNIVERSITY OF NOVI SAD
21 000 NOVI SAD
YUGOSLAVIA