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# A note on Turán type and mean inequalities for the Kummer function

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ABSTRACT

Turán-type inequalities for combinations of Kummer functions involving $\Phi(a \pm \nu, c \pm \nu, x)$ and $\Phi(a, c \pm \nu, x)$ have been recently investigated in [Á. Baricz, Functional inequalities involving Bessel and modified Bessel functions of the first kind, Expo. Math. 26 (3) (2008) 279–293; M.E.H. Ismail, A. Laforgia, Monotonicity properties of determinants of special functions, Constr. Approx. 26 (2007) 1–9]. In the current paper, we resolve the corresponding Turán-type and closely related mean inequalities for the additional case involving $\Phi(a \pm \nu, c, x)$. The application to modeling credit risk is also summarized.



## 1. Introduction

The Kummer confluent hypergeometric function is given by

$$\Phi(\alpha, \beta, x) \equiv \sum_{n=0}^{\infty} \frac{(\alpha)_n x^n}{(\beta)_n n!},$$

where $(\alpha)_n$ is the Pochhammer symbol defined by $(\alpha)_n \equiv \Gamma(\alpha+n)/\Gamma(\alpha) = \alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n \in \mathbb{N}$, $(\alpha)_0 = 1$, $(\alpha+1)_{-1} = \frac{1}{\alpha}$, and the Gamma function is $\Gamma(x) \equiv \int_0^\infty t^{x-1}e^{-t}\,dt$, for $x > 0$.

Inequalities involving contiguous Kummer confluent hypergeometric functions of the form $\Phi(a \pm \nu, c \pm \nu, x)$ and $\Phi(a, c \pm \nu, x)$ were presented in Theorem 2 of [4] and Theorem 2.7 of [11]. These inequalities are of the Turán type [15] in the case that $\nu = 1$. In the present note, we resolve the remaining Turán-type case involving $\Phi(a \pm 1, c, x)$ and extend it to include $\Phi(a \pm \nu, c, x)$, $\nu \in \mathbb{N}$. We then establish a closely related mean inequality that provides simultaneous upper and lower bounds for $\Phi(a, c, x)$. Turán-type inequalities, which are of independent interest, also have important applications in Information Theory (as demonstrated by McEliece, Reznick, and Shearer in their paper [12]) and in modeling credit risk, as summarized below.

In particular, Carey and Gordy [9] model a lending relationship in which the bank has an option to foreclose upon the borrower at any time. Following the seminal models of Merton [13] and Black and Cox [6], it is assumed that the value of the firm's assets follows a geometric Brownian motion. It is shown that the bank's optimal foreclosure threshold solves a first order condition involving a ratio of contiguous Kummer functions, which implies that a Turán-type inequality for the Kummer function arises naturally in studying the comparative statics of the model. A proof of this key Turán-type

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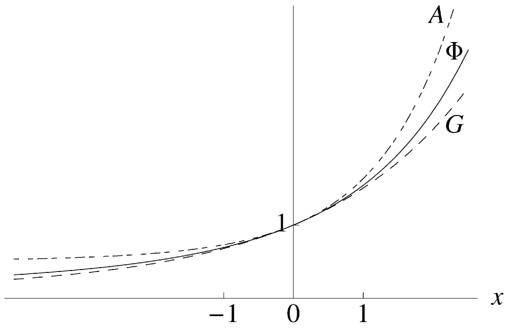


**Fig. 1.** Graphs of $A(x) = A(\Phi(a+1, a+b, x), \Phi(a-1, a+b, x))$; $\Phi(x) = \Phi(a, a+b, x)$; and $G(x) = G(\Phi(a+1, a+b, x), \Phi(a-1, a+b, x))$; with $a = 1$, $b = 1.5$, and $\nu = 1$.

inequality is established in this paper. (For general background on applications of the Kummer function in economic theory and econometrics, see [1].)

## 2. Main results

**Theorem 1.** *Suppose $a, b > 0$. Then for any $\nu \in \mathbb{N}$ with $a, b \geqslant \nu - 1$*

$$\Phi(a, a+b, x)^2 > \Phi(a+\nu, a+b, x)\Phi(a-\nu, a+b, x), \tag{1}$$

*for all nonzero $x \in \mathbb{R}$. Moreover, these expressions coincide with value 1 when $x = 0$ and asymptotically for any $x$ when $b \to \infty$.*

**Corollary 2.** *Suppose $a > 0$ and $c + 1 > 0$ with $c \neq 0$. Then for any $\nu \in \mathbb{N}$ with $a \geqslant \nu - 1$,*

$$\Phi(a, c, x)^2 \geqslant \Phi(a-\nu, c, x)\Phi(a+\nu, c, x), \tag{2}$$

*for all $x > 0$.*

The next result begins with the well-known arithmetic mean-geometric mean inequality,

$$A(\alpha, \beta) \equiv \frac{\alpha+\beta}{2} > \sqrt{\alpha\beta} \equiv G(\alpha, \beta) \quad \text{for } \alpha, \beta \text{ distinct and positive,} \tag{3}$$

which has many interesting refinements and applications (e.g., see [7,8]). Corollary 3 is a refinement of inequality (3) with $\alpha = \Phi(a+\nu, a+b, x)$ and $\beta = \Phi(a-\nu, a+b, x)$ (see illustrated special case in Fig. 1).

**Corollary 3.** *Suppose $\nu \in \mathbb{N}$ and $a, b \geqslant \nu$. Then for all nonzero $x \in \mathbb{R}$*

$$\begin{aligned} &A\big(\Phi(a+\nu, a+b, x), \Phi(a-\nu, a+b, x)\big) \\ &\quad > \Phi(a, a+b, x) \\ &\quad > G\big(\Phi(a+\nu, a+b, x), \Phi(a-\nu, a+b, x)\big). \end{aligned} \tag{4}$$

It is also interesting to compare these results with the elegant Theorem 2.3 and open problems in [5] regarding Turán-type and arithmetic mean-geometric mean inequalities involving the Gaussian hypergeometric function ${}_2F_1$.

## 3. Proofs

**Proof of Theorem 1.** First assume $x > 0$. For $c > -1$, $c \neq 0$, define

$$f_\nu(x) \equiv \Phi(a, c, x)^2 - \Phi(a+\nu, c, x)\Phi(a-\nu, c, x).$$

We will make use of the following contiguous relation (see [10, p. 1013]):

$$\Phi(\alpha+1, \beta, x) - \Phi(\alpha, \beta, x) = \frac{x}{\beta}\Phi(\alpha+1, \beta+1, x). \tag{5}$$

Subtracting and adding a term to $f_{\nu+1}(x) - f_\nu(x)$ and applying this contiguous relation, we have that

$$\begin{aligned}
f_{\nu+1}(x) - f_\nu(x) &= \Phi(a+\nu,c,x)\Phi(a-\nu,c,x) - \Phi(a+\nu+1,c,x)\Phi(a-\nu-1,c,x) \\
&= \Phi(a-\nu,c,x)\big(\Phi(a+\nu,c,x) - \Phi(a+\nu+1,c,x)\big) \\
&\quad + \Phi(a+\nu+1,c,x)\big(\Phi(a-\nu,c,x) - \Phi(a-\nu-1,c,x)\big) \\
&= \Phi(a-\nu,c,x)\left(\frac{-x}{c}\right)\Phi(a+\nu+1,c+1,x) + \Phi(a+\nu+1,c,x)\left(\frac{x}{c}\right)\Phi(a-\nu,c+1,x) \\
&= \frac{x}{c} g_\nu(x),
\end{aligned}$$

where

$$g_\nu(x) \equiv \Phi(a+\nu+1,c,x)\Phi(a-\nu,c+1,x) - \Phi(a-\nu,c,x)\Phi(a+\nu+1,c+1,x).$$

The Cauchy product reveals

$$\begin{aligned}
g_\nu(x) &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(a+\nu+1)_k(a-\nu)_{n-k}}{k!(n-k)!}\left(\frac{1}{(c)_k(c+1)_{n-k}} - \frac{1}{(c)_{n-k}(c+1)_k}\right)x^n \\
&= \sum_{n=1}^{\infty}\sum_{k=0}^{n} \frac{(a+\nu+1)_k(a-\nu)_{n-k}}{k!(n-k)!}\left(\frac{(c+k)-(c+n-k)}{c(c+1)_{n-k}(c+1)_k}\right)x^n \\
&= \frac{1}{c}\sum_{n=1}^{\infty}\sum_{k=0}^{n} T_{n,k}(2k-n)x^n,
\end{aligned}$$

where $T_{n,k} \equiv \frac{(a+\nu+1)_k(a-\nu)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_k}$. If $n$ is even, then

$$\begin{aligned}
\sum_{k=0}^{n} T_{n,k}(2k-n) &= \sum_{k=0}^{n/2-1} T_{n,k}(2k-n) + \sum_{k=n/2+1}^{n} T_{n,k}(2k-n) \\
&= \sum_{k=0}^{n/2-1} T_{n,k}(2k-n) + \sum_{k=0}^{n/2-1} T_{n,n-k}\big(2(n-k)-n\big) \\
&= \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n-2k),
\end{aligned}$$

where $[\cdot]$ denotes the greatest integer function. Similarly, if $n$ is odd, then

$$\sum_{k=0}^{n} T_{n,k}(2k-n) = \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n-2k).$$

Therefore,

$$f_{\nu+1}(x) - f_\nu(x) = \frac{x}{c} g_\nu(x) = \frac{x}{c^2}\sum_{n=1}^{\infty}\sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n-2k)x^n. \tag{6}$$

Simplifying, we find that

$$\begin{aligned}
T_{n,n-k} - T_{n,k} &= \frac{(a+\nu+1)_{n-k}(a-\nu)_k - (a+\nu+1)_k(a-\nu)_{n-k}}{k!(n-k)!(c+1)_{n-k}(c+1)_k} \\
&= \frac{(a+\nu+1)_k(a-\nu)_k}{k!(n-k)!(c+1)_{n-k}(c+1)_k}\left(\frac{(a+\nu+1)_{n-k}}{(a+\nu+1)_k} - \frac{(a-\nu)_{n-k}}{(a-\nu)_k}\right) \\
&= \frac{(a+\nu+1)_k(a-\nu)_k}{k!(n-k)!(c+1)_{n-k}(c+1)_k}\big(h(a+\nu+1) - h(a-\nu)\big),
\end{aligned} \tag{7}$$

where $h(\beta) \equiv \frac{(\beta)_{n-k}}{(\beta)_k}$. For $\beta > 0$ and $n-k > k$ (i.e., for $[(n-1)/2] \geqslant k$), the logarithmic derivative of $h$ satisfies

$$\frac{h'(\beta)}{h(\beta)} = \Psi(\beta+n-k) - \Psi(\beta+k) > 0,$$

where $\Psi \equiv \Gamma'/\Gamma$ is the digamma function. Hence, $h$ is increasing under the conditions stated. This fact together with (6) and (7) yield

$$f_{\nu+1}(x) - f_\nu(x) = \frac{x}{c^2} \sum_{n=1}^{\infty} \sum_{k=0}^{[(n-1)/2]} (T_{n,n-k} - T_{n,k})(n-2k)x^n > 0 \tag{8}$$

when $a \geqslant \nu \geqslant 0$, since $x > 0$ and $c+1 > 0$, $c \neq 0$.

Thus, (8) implies that

$$f_{\nu+1}(x) = \big(f_{\nu+1}(x) - f_\nu(x)\big) + \big(f_\nu(x) - f_{\nu-1}(x)\big) + \cdots + \big(f_1(x) - f_0(x)\big) > 0,$$

for $a \geqslant \nu > \nu - 1 \geqslant \cdots \geqslant 0$ and $f_0(x) = 0$. Replacing $\nu$ by $\nu - 1$, we conclude that, for $x > 0$,

$$f_\nu(x) > 0, \quad \text{for each } \nu \in \mathbb{N} \text{ satisfying } a \geqslant \nu - 1. \tag{9}$$

Moreover, under these conditions, $f_\nu$ is absolutely monotonic on $(0, \infty)$ (i.e., $f_\nu^{(n)}(x) > 0$ for $n = 0, 1, 2, \ldots$). With $c = a + b$, this proves Theorem 1 for the case that $x > 0$.

Now suppose $x < 0$, $a, b > 0$, and $\nu \in \mathbb{N}$ with $a, b \geqslant \nu - 1$. Taking advantage of the available symmetry with $c = a + b$, we can interchange $a$ and $b$ in (1) to arrive at

$$\Phi(b, a+b, -x)^2 - \Phi(b+\nu, a+b, -x)\Phi(b-\nu, a+b, -x) > 0.$$

Kummer's transformation [2, p. 191], $\Phi(\alpha, \beta, -x) = e^{-x}\Phi(\beta - \alpha, \beta, x)$, yields

$$e^{-2x}\Phi(a, a+b, x)^2 - e^{-2x}\Phi(a-\nu, a+b, x)\Phi(a+\nu, a+b, x) > 0.$$

Thus, (1) also holds for $x < 0$. □

**Proof of Corollary 2.** It follows from the proof of Theorem 1 that (9) holds for $x > 0$ under the conditions that $c + 1 > 0$, $c \neq 0$. □

**Proof of Corollary 3.** First suppose $x \geqslant 0$ and let $a, b \geqslant \nu$, $\nu \in \mathbb{N}$. The first inequality in (4) is a direct consequence of the fact that $A((a+\nu)_n, (a-\nu)_n) = (a)_n$ for $n = 0, 1$ and

$$A\big((a+\nu)_n, (a-\nu)_n\big) > (a)_n \quad \text{for all } n \geqslant 2,$$

which follows by induction. Thus,

$$\begin{aligned} A\big(\Phi(a+\nu, a+b, x), \Phi(a-\nu, a+b, x)\big) &= \sum_{n=0}^{\infty} \frac{A((a+\nu)_n, (a-\nu)_n)x^n}{(a+b)_n n!} \\ &> \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(a+b)_n n!} = \Phi(a, a+b, x). \end{aligned}$$

For $x \geqslant 0$, the second inequality in (4) follows by taking the square-root across (1), which is allowed since the right-hand side of (1) is nonnegative when $a, b \geqslant \nu$.

Now suppose $x < 0$ with $a, b \geqslant \nu$. Interchanging $a$ and $b$ in (4), we have

$$\begin{aligned} &A\big(\Phi(b+\nu, a+b, -x), \Phi(b-\nu, a+b, -x)\big) \\ &\quad > \Phi(b, a+b, -x) > G\big(\Phi(b+\nu, a+b, -x), \Phi(b-\nu, a+b, -x)\big). \end{aligned}$$

Kummer's transformation and the homogeneity of $A$ and $G$ yield

$$\begin{aligned} &e^{-x}A\big(\Phi(a-\nu, a+b, x), \Phi(a+\nu, a+b, x)\big) \\ &\quad > e^{-x}\Phi(a, a+b, x) > e^{-x}G\big(\Phi(a-\nu, a+b, x), \Phi(a+\nu, a+b, x)\big). \end{aligned}$$

Thus, (4) also holds for $x < 0$. □

## 4. Concluding remarks

The proof of Theorem 1 can also be used to verify cases when the Turán-type inequality reverses. For example, if $a < 0$ and $c + 1 < 0$ with $[a] = [c+1]$ and $c$ not a negative integer, then

$$\Phi(a, c, x)^2 < \Phi(a+1, c, x)\Phi(a-1, c, x) \quad \text{for all } x > 0.$$

To see this, take $\nu = 0$ in (6) and then simplify to find that

$$\Phi(a, c, x)^2 - \Phi(a+1, c, x)\Phi(a-1, c, x) = \frac{x}{ac^2} \sum_{n=1}^{\infty} \left( \sum_{k=0}^{[(n-1)/2]} \frac{(a)_k (a)_{n-k} (n-2k)^2}{(c+1)_k (c+1)_{n-k} k!(n-k)!} \right) x^n. \tag{10}$$

The result follows by noting that $(a)_m$ and $(c+1)_m$ will have the same signs under the stated conditions (unless $(a)_m = 0$ for some $m \geqslant 2$). Hence

$$\frac{1}{ac^2} \sum_{k=0}^{[(n-1)/2]} \frac{(a)_k (a)_{n-k} (n-2k)^2}{(c+1)_k (c+1)_{n-k} k!(n-k)!} \leqslant 0$$

for all $n \in \mathbb{N}$. Moreover, the first nonzero term in the series in (10) simplifies to $\frac{x^2}{c^2(c+1)} < 0$.

Finally, we note that the techniques of proof presented here can be used to obtain a result similar to (4) with $\Phi = {}_1F_1$ replaced by the generalized hypergeometric function ${}_pF_q$, where

$${}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; x) \equiv \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n x^n}{(b_1)_n \cdots (b_q)_n n!}.$$

It can be shown that if $p \leqslant q+1$, $\alpha > 1$, $b_i > 0$ for $i = 1, \ldots, q$, and $a_i > b_i$ for $i = 1, \ldots, p-1$, then for all $x > 0$ in the interval of convergence,

$$\begin{aligned} &A\big({}_pF_q(\alpha+1, a_i; b_j; x), {}_pF_q(\alpha-1, a_i; b_j; x)\big) \\ &\quad > {}_pF_q(\alpha, a_i; b_j; x) > G\big({}_pF_q(\alpha+1, a_i; b_j; x), {}_pF_q(\alpha-1, a_i; b_j; x)\big) \end{aligned} \tag{11}$$

where ${}_pF_q(\alpha, a_i; b_j; x) \equiv {}_pF_q(\alpha, a_1, \ldots, a_{p-1}; b_1, \ldots, b_q; x)$.

Of particular interest is the case that $p = 2$ and $q = 1$. In this case, inequality (11) completes the results of M.E.H. Ismail and A. Laforgia [11] and of Á. Baricz [3,5] regarding the Gaussian hypergeometric function ${}_2F_1$. See for example Theorems 2.13 and 2.14 in [11] and Theorem 2.17 in [3].

The first inequality in (11) follows as in Theorem 1. The second inequality in (11) follows by using a generalized version of (5) (see [14, Identity 30, p. 440]) to reveal that

$$\begin{aligned} F(x) &\equiv {}_pF_q(\alpha, a_i; b_j; x)^2 - {}_pF_q(\alpha+1, a_i; b_j; x)\, {}_pF_q(\alpha-1, a_i; b_j; x) \\ &= \frac{x \prod_{i=1}^{p-1} a_i^2}{\alpha \prod_{i=1}^{q} b_i^2} \sum_{n=1}^{\infty} \left( \sum_{k=0}^{[(n-1)/2]} \frac{(\alpha)_k (\alpha)_{n-k} \prod_{i=1}^{p-1} ((a_i+1)_k (a_i+1)_{n-k})}{k!(n-k)! \prod_{i=1}^{q} ((b_i+1)_k (b_i+1)_{n-k})} R_{n,k} \right) x^n, \end{aligned}$$

where

$$R_{n,k} = \left( \frac{\prod_{i=1}^{q} (b_i + n - k)}{\prod_{i=1}^{p-1} (a_i + n - k)} - \frac{\prod_{i=1}^{q} (b_i + k)}{\prod_{i=1}^{p-1} (a_i + k)} \right)(n - 2k).$$

For $n - k > k$, the positivity of $R_{n,k}$ (and hence $F$) follows when $r \mapsto \frac{\prod_{i=1}^{q}(b_i+r)}{\prod_{i=1}^{p-1}(a_i+r)}$ is increasing, which is the case under the stated conditions on the $a_i$'s and $b_i$'s.

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