

THE BEURLING-SELBERG BOX MINORANT PROBLEM

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ABSTRACT. In this paper we investigate a high dimensional version of Selberg's minorant problem for the indicator function of an interval. In particular, we study the corresponding problem of minorizing the indicator function of the box $Q_N = [-1, 1]^N$ by a function whose Fourier transform is supported in the same box Q_N . We show that when N is sufficiently large there are no non-trivial minorants (that is, minorants with positive integral). On the other hand, we explicitly construct non-trivial minorants for dimension $N \leq 5$.

Attention: *A new and improved version of this paper will appear shortly and jointly with Noam Elkies. The improvements include explicit upper bounds on $\Delta(2)$ and therefore also the critical dimensions appearing in Theorems 2 and 5.*

1. INTRODUCTION

Let $Q_N = [-1, 1]^N$, let $\mathbf{1}_{Q_N}$ be the indicator function of Q_N , and let $\delta > 0$. A fundamental question in approximation theory asks:

Question 1. *Does there exist a function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ such that:*

- (i) $F(\mathbf{x}) \leq \mathbf{1}_{Q_N}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^N$,
- (ii) the Fourier transform $\widehat{F}(\boldsymbol{\xi})$ of $F(\mathbf{x})$ is supported in the box $[-\delta, \delta]^N$, and
- (iii) $\int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x} > 0$?

Basic considerations will lead the reader to surmise that the existence of such a function depends on the size of δ . If δ is very large, then such a function will surely exist. On the other hand, if δ is very small, then no such function ought to exist. When $N = 1$ the above question was settled by Selberg [36, 38] who showed that there is a positive answer to Question 1 if and only if $\delta > \frac{1}{2}$. From here it is not difficult to show that when $N > 1$, Question 1 has a negative answer whenever $\delta \leq \frac{1}{2}$ (see Lemma 10). When N is large it is unknown how small δ may be for Question 1 to admit a positive answer. The best result in this direction is due to Selberg who proved that when $N > 1$ and $\delta > N - \frac{1}{2}$, then Question 1 has a positive answer. Selberg never published his construction, but he did communicate it to Vaaler and Montgomery (personal communication). His construction has since appeared several times in the literature, see for instance [28, 29, 30]. More

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details about Selberg's (and also Montgomery's) construction can be found in the Appendix.

One of our main theorems, Theorem 3, shows that for sufficiently large dimensions the lower bound $\delta > 1$ is a necessary condition for a positive answer of Question 1. The proof of this, as well as our other main results, are based in a detailed analysis of the following extremal problem (especially the special case $N = 2$) which is the main focus of this paper.

Problem 1. *For every integer $N \geq 1$ determine the value of the quantity*

$$(1.1) \quad \nu(N) = \sup \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x},$$

where the supremum is taken over functions $F \in L^1(\mathbb{R}^N)$ such that:

- (I) $\widehat{F}(\boldsymbol{\xi})$ is supported in Q_N ,
- (II) $F(\mathbf{x}) \leq \mathbf{1}_{Q_N}(\mathbf{x})$ for (almost) every $\mathbf{x} \in \mathbb{R}^N$.

We show that the admissible minorants are given by a Whittaker-Shannon type interpolation formula and we use this formula to demonstrate that the only admissible minorant with non-negative integral that interpolates the indicator function $\mathbf{1}_{Q_N}(\mathbf{x})$ at the integer lattice $\mathbb{Z}^N \setminus \{\mathbf{0}\}$ is the *identically zero function*. We also define an auxiliary quantity $\Delta(N)$ (see (2.2)), similar to $\nu(N)$, and derive a functional inequality, which ultimately implies that $\nu(N)$ *vanishes* for finite N .

As we have remarked, when $N = 1$ Question 1 is settled completely. In fact, even the corresponding *extremal* problem is completely settled (in higher dimensions no extremal results for the box minorant problem are known). Suppose \mathcal{I} is an interval in \mathbb{R} of finite length, $\mathbf{1}_{\mathcal{I}}(x)$ is the indicator of x , and $\delta > 0$. Selberg [36, 38] introduced functions $C(x)$ and $c(x)$ with the properties

- (i) $\widehat{C}(\xi) = \widehat{c}(\xi) = 0$ if $|\xi| > \delta$ (where $\widehat{\cdot}$ denotes the Fourier transform),
- (ii) $c(x) \leq \mathbf{1}_{\mathcal{I}}(x) \leq C(x)$ for each $x \in \mathbb{R}$, and
- (iii) $\int_{-\infty}^{\infty} (C(x) - \mathbf{1}_{\mathcal{I}}(x)) dx = \int_{-\infty}^{\infty} (\mathbf{1}_{\mathcal{I}}(x) - c(x)) dx = \delta^{-1}$.

Furthermore, among all functions that satisfy (i) and (ii) above, Selberg's functions minimize the integrals appearing in (iii) if and only if $\delta \text{length}(\mathcal{I}) \in \mathbb{Z}$. If $\delta \text{length}(\mathcal{I}) \notin \mathbb{Z}$, then the extremal functions have been found by Littmann in [33].

Originally, Selberg was motivated to construct his one dimensional extremal functions to prove a sharp form of the large sieve. His functions and their generalizations have since become part of the standard arsenal in analytic number theory and have a number of applications in fields ranging from probability, dynamical systems, optics, combinatorics, sampling theory, and beyond. For a non-exhaustive

list see [7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 25, 32, 34] and the references therein.

In recent years higher dimensional analogues of Selberg’s extremal function and related constructions have proven to be important in the recent studies of Diophantine inequalities [2, 19, 30, 31], visibility problems and quasicrystals [1, 30], and sphere packings¹ [20, 21, 22, 23, 24, 39]. See also [4] for related constructions recently used in signal processing. Since Selberg’s original construction of his box minorants there has been some progress on the Beurling-Selberg problem in higher dimensions [2, 3, 5, 6, 26]. In particular, Holt and Vaaler [31] consider a variation of the box minorant problem where the boxes are replaced by Euclidean balls. They are actually able to establish extremal results in some cases. The general case was solved in [8].

There seems to be a consensus among experts that despite four decades of progress on Beurling-Selberg problems, box minorants are poorly understood. This sentiment was recently raised in [27]. We hope that the contributions of this paper will help reveal why the box minorant problem is so difficult and move us closer to understanding these enigmatic objects.

2. MAIN RESULTS

In this section we give some definitions and state the main results of the present article. A function $F(\mathbf{x})$ satisfying conditions (I) and (II) of Problem 1 will be called *admissible for $\nu(N)$* (or $\nu(N)$ -admissible) and if it achieves equality in (1.1), then it is said to be *extremal*.

An indispensable tool in our investigation is the Poisson summation formula. If $G : \mathbb{R}^N \rightarrow \mathbb{R}$ is “sufficiently nice,” Λ is a full rank lattice in \mathbb{R}^N of covolume $|\Lambda|$, and Λ^* is the corresponding dual lattice², then the Poisson summation formula is the assertion that

$$\sum_{\lambda \in \Lambda} G(\mathbf{x} + \lambda) = \frac{1}{|\Lambda|} \sum_{\mathbf{u} \in \Lambda^*} \hat{G}(\mathbf{u}) e^{2\pi i \mathbf{u} \cdot \mathbf{x}}$$

for every $\mathbf{x} \in \mathbb{R}^N$.

If $F(\mathbf{x})$ is a $\nu(N)$ -admissible function, then it follows from Proposition 9 that the Poisson summation formula may be applied to $F(\mathbf{x})$. That is, $\nu(2)$ -admissible functions are “sufficiently nice.” Thus, upon applying Poisson summation (3.1) to

¹The extremal problems considered for sphere packings differ from the problems that we consider here. Instead of the admissible functions being band-limited, their Fourier transforms are only required to be non-positive outside of a compact set.

²That is, $\Lambda^* = \{\mathbf{u} \in \mathbb{R}^N : \mathbf{u} \cdot \lambda \in \mathbb{Z} \text{ for all } \lambda \in \Lambda\}$.

$F(\mathbf{x})$ and recalling the constraints making it $\nu(2)$ -admissible, we find that

$$\widehat{F}(\mathbf{0}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \widehat{F}(\mathbf{n}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} F(\mathbf{n}) \leq F(\mathbf{0}).$$

Thus we have the fundamental inequality

$$(2.1) \quad \widehat{F}(\mathbf{0}) \leq F(\mathbf{0}).$$

Evidently there is equality in (2.1) if, and only if, $F(\mathbf{n}) = 0$ for each non-zero $\mathbf{n} \in \mathbb{Z}^N$. If $N = 1$ then, by using the interpolation formula (3.2), Selberg was able to show (see [36, 38]) that $\nu(1) = 1$ and that

$$\frac{\sin^2 \pi x}{(\pi x)^2(1-x^2)}$$

is an extremal function (this is not the unique extremal function).

The following is our main result, and Theorem 3 is an immediate corollary. It compiles the basic properties related to the quantity $\nu(N)$, establishing: (1) that extremizers for the quantity $\nu(N)$ do exist, (2) that $\nu(N)$ is a decreasing function of N and, most curiously, (3) that $\nu(N)$ *vanishes for finite* N .

Theorem 2. *The following statements hold.*

(i) *For every $N \geq 2$ there exists a $\nu(N)$ -admissible function $F(\mathbf{x})$ such that*

$$\nu(N) = \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x}.$$

(ii) *If $\nu(N) > 0$ then $\nu(N+1) < \nu(N)$. In particular, $\nu(2) < 1$.*

(iii) *There exists a critical dimension N_c such that $\nu(N_c) > 0$ and $\nu(N) = 0$ for all $N > N_c$. Moreover, we have the following bounds*

$$5 \leq N_c \leq \left\lfloor \frac{2}{1 - \Delta(2)} \right\rfloor.$$

Remarks.

- (1) The quantity $\Delta(2)$ appearing in the above theorem is defined in equation (2.2) and it follows from Theorem 5(iii) that $\Delta(2) < 1$. Unfortunately, we are currently unable to determine an upperbound for $\Delta(2)$ that is strictly less than one. The main reason for this is that our methods are based on a contraction type argument. On the other hand, the lower bound is given via explicit constructions presented in Section 5.
- (2) To put this result in context, note that volume of Q_N is growing exponentially, so there is a lot of volume on both the physical and frequency sides. However, every time another dimension gets added, more constraints also get added so it requires a detailed analysis to determine the behavior of $\nu(N)$. Poisson

summation, which yields the non-intuitive bound $\nu(N) \leq 1$, already detects this tug-of-war.

- (3) Theorem 2 has some parallels in classical asymptotic geometric analysis, and mass concentration in particular. In our first attempts to prove Theorem 2 we tried to employ asymptotic geometric techniques to exploit properties of Q_N but we were not able to uncover a proof. We found it awkward to incorporate the Fourier analytic and one-sided inequality constraints (i.e. (I) and (II) in the definition of $\nu(N)$) with the standard tool kit of asymptotic geometric analysis. It would be very interesting to see a proof of Theorem 2 based on such techniques.

Our next result sheds some light in Question 1.

Theorem 3. *If $\delta \leq 1$, then Question 1 has a negative answer for all sufficiently large N . If $N \leq 5$, then Question 1 has a positive answer if $\delta \geq 1$.*

Our next result shows that Selberg's interpolation strategy to build minorants fails in higher dimensions.

Theorem 4. *Let $N \geq 2$. Let $F(\mathbf{x})$ be an admissible function for $\nu(N)$ and assume that $F(\mathbf{0}) \geq 0$. If $F(\mathbf{n}) = 0$ for every non-zero $\mathbf{n} \in \mathbb{Z}^N$, then $F(\mathbf{x})$ vanishes identically.*

We are also interested in studying an even more “degenerate” problem defined as follows. Let

$$(2.2) \quad \Delta(N) = \sup_F \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x},$$

where the supremum is taken among functions $F(\mathbf{x})$ such that

- (I) $\widehat{F}(\boldsymbol{\xi})$ is supported in Q_N ,
- (II) $F(\mathbf{x}) \leq 0$ for (almost) every $\mathbf{x} \notin Q_N$,
- (III) $F(\mathbf{0}) = 1$.
- (IV) $\widehat{F}(\mathbf{0}) > 0$

The quantity $\Delta(N)$ may not be well defined for some N , in this case we define $\Delta(N) = 0$. Lemma 10 shows that if $\Delta(N_0)$ is well-defined (that is $\Delta(N_0) > 0$), then it is well defined for all $N \leq N_0$. One can also verify that $\Delta(N) > 0$ if and only if $\nu(N) > 0$ and

$$\nu(N) \leq \Delta(N).$$

Thus, they vanish for the first time at the same dimension. Poisson summation shows that $\Delta(N) \leq 1$ for all N and thus $\Delta(1) = 1$. A priori, the existence of extremizers for the $\Delta(N)$ problem is not guaranteed since an extremizing sequence

may blow-up inside the box Q_N . The next theorem shows that $\Delta(N)$ behaves similarly to $\nu(N)$ for $N \geq 2$.

Theorem 5. *The following statements hold.*

- (i) *There exists a constant $B_N \geq 1$, depending only on N , such that if $F(\mathbf{x})$ is admissible for the $\Delta(N)$ problem then $F(\mathbf{x}) \leq B_N$ for all $x \in Q_N$.*
- (ii) *If $\Delta(N) > 0$, then there exists a $\Delta(N)$ -admissible function $F(\mathbf{x})$ such that*

$$\Delta(N) = \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x}.$$

- (iii) *If $\Delta(N) > 0$, then $\Delta(N+1) < \Delta(N)$. In particular, $\Delta(2) < 1$.*
- (iv) *There exists a critical dimension N_c such that $\Delta(N_c) > 0$ and $\Delta(N) = 0$ for all $N > N_c$. Moreover, the same bound holds*

$$5 \leq N_c \leq \left\lfloor \frac{2}{1 - \Delta(2)} \right\rfloor.$$

We now give some explicit lower bounds for the quantity $\nu(N)$ up to dimension $N = 5$. These are constructed explicitly in Section 5.

Theorem 6. *We have the following lower bounds for $\nu(N)$:*

- $\nu(2) \geq \frac{63}{64} = 0.984375$,
- $\nu(3) \geq \frac{119}{128} = 0.9296875$,
- $\nu(4) \geq \frac{95}{128} = 0.7421875$,
- $\nu(5) \geq \frac{31}{256} = 0.12109375$.

As an application of this result, we will prove a sharpened version of the Barton-Montgomery-Vaaler inequality for $N \leq 5$.

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3. PRELIMINARIES

In this section we recall the crucial results needed to demonstrate our main results as well as some basic facts about the theory of Paley-Wiener spaces and extremal functions.

For a given function $F : \mathbb{R}^N \rightarrow \mathbb{R}$ we define its Fourier transform as

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^N} F(\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x}.$$

In this paper we will almost always deal with functions $F(\mathbf{x})$ that are integrable and whose Fourier transforms are supported in the box

$$Q_N = [-1, 1]^N.$$

For this reason, given $p \in [1, 2]$ we define $PW^p(Q_N)$ as the set of functions $F \in L^p(\mathbb{R}^N)$ such that their Fourier transform is supported in Q_N . By Fourier inversion these functions can be identified with analytic functions that extend to \mathbb{C}^N as entire functions. The following is a special case of Stein's generalization of the Paley-Wiener theorem.

Theorem 7 (Stein, [37]). *Let $p \in [1, 2]$ and $F \in L^p(\mathbb{R}^N)$. The following statements are equivalent.*

- (i) $\text{supp}(F) \subset Q_N$.
- (ii) $F(\mathbf{x})$ is a restriction to \mathbb{R}^N of an analytic function defined in \mathbb{C}^N with the property that there exists a constant $C > 0$ such that

$$|F(\mathbf{x} + i\mathbf{y})| \leq C \exp \left[2\pi \sum_{n=1}^N |y_n| \right]$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.

Remark. In particular this theorem implies that every function $F \in PW^1(Q_N)$ is bounded on \mathbb{R}^N , hence $PW^1(Q_N) \subset PW^2(Q_N)$.

Theorem 8 (Pólya-Plancherel, [35]). *If $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \dots$ is a sequence in \mathbb{R}^N satisfying that $|\boldsymbol{\xi}_n - \boldsymbol{\xi}_m|_{\ell^\infty} \geq \varepsilon$ for all $m \neq n$ for some $\varepsilon > 0$ then*

$$\sum_n |F(\boldsymbol{\xi}_n)|^p \leq C(p, \varepsilon) \int_{\mathbb{R}^N} |F(\boldsymbol{\xi})|^p d\boldsymbol{\xi}$$

for every $F \in PW^p(Q_N)$.

Proposition 9 (Poisson Summation for $PW^1(Q_N)$). *For all $F \in PW^1(Q_N)$ and for any $\mathbf{t} \in \mathbb{R}^N$ we have*

$$(3.1) \quad \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x} = \sum_{\mathbf{n} \in \mathbb{Z}^N} F(\mathbf{n} + \mathbf{t}).$$

where the convergence takes place uniformly on compact subsets of \mathbb{R}^N .

Let $F \in PW^2(Q_N)$. If $\mathbf{t} \in \mathbb{C}^{N-k}$, then the function $\mathbf{y} \in \mathbb{R}^k \mapsto G_{\mathbf{t}}(\mathbf{y}) = F(\mathbf{y}, \mathbf{t})$ is the inverse Fourier transform of the following function

$$\boldsymbol{\xi} \in \mathbb{R}^k \mapsto \int_{Q_{N-k}} \widehat{F}(\boldsymbol{\xi}, \mathbf{u}) e^{2\pi i \mathbf{t} \cdot \mathbf{u}} d\mathbf{u}.$$

Since $\widehat{F} \in L^2(\mathbb{R}^N)$, we conclude that the above function has finite $L^2(\mathbb{R}^k)$ -norm and as a consequence $G_{\mathbf{t}} \in PW^2(Q_k)$. A similar result is valid for $p = 1$, but only for $\nu(N)$ -admissible functions.

Lemma 10. *Let $N > k > 0$ be integers. If $F(\mathbf{x})$ is $\nu(N)$ -admissible then the function $\mathbf{y} \in \mathbb{R}^k \mapsto F(\mathbf{y}, \mathbf{0})$ with $\mathbf{0} \in \mathbb{R}^{N-k}$ is $\nu(k)$ -admissible and*

$$\int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x} \leq \int_{\mathbb{R}^k} F(\mathbf{y}, \mathbf{0}) d\mathbf{y}.$$

Proof. We give a proof only for the case $N = 2$ since it will be clear that the general case follows by an adaption of the following argument.

Let $F(x, y)$ be a function admissible for $\nu(2)$ and define $G(x) = F(x, 0)$. By Fourier inversion we obtain that

$$G(x) = \int_{-1}^1 \left(\int_{-1}^1 \widehat{F}(s, t) dt \right) e^{2\pi i s x} ds.$$

This shows that $G \in PW^2(Q_1)$. Now, for every $a \in (0, 1)$ define the functions

$$G_a(x) = G((1-a)x) \left(\frac{\sin(a\pi x)}{a\pi x} \right)^2.$$

and

$$F_a(x, y) = F((1-a)x, y) \left(\frac{\sin(a\pi x)}{a\pi x} \right)^2.$$

By an application of Holder's inequality and Theorem 7, we deduce that $G_a \in PW^1(Q_1)$ and $F_a \in PW^1(Q_2)$ for all $a \in (0, 1)$. Hence, we can apply Poisson summation to conclude that

$$\begin{aligned} \int_{\mathbb{R}} G_a(x) dx &= \sum_{n \in \mathbb{Z}} G((1-a)n) \left(\frac{\sin(a\pi n)}{a\pi n} \right)^2 \\ &\geq \sum_{(n,m) \in \mathbb{Z}^2} F((1-a)n, m) \left(\frac{\sin(a\pi n)}{a\pi n} \right)^2 \\ &= \int_{\mathbb{R}^2} F((1-a)x, y) \left(\frac{\sin(a\pi x)}{a\pi x} \right)^2 dx dy, \end{aligned}$$

where the above inequality is valid because the function $F(x, y)$ is a minorant of the box Q_2 . Observing that $G_a(x) \leq \mathbf{1}_{Q_1/(1-a)}(x)$ for every $x \in \mathbb{R}$, we can apply Fatou's lemma to conclude that

$$\int_{\mathbb{R}} [\mathbf{1}_{Q_1}(x) - G(x)] dx \leq \liminf_{a \rightarrow 0} \int_{\mathbb{R}} [\mathbf{1}_{Q_1/(1-a)}(x) - G_a(x)] dx$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x) dx - \limsup_{a \rightarrow 0} \int_{\mathbb{R}^2} F((1-a)x, y) \left[\frac{\sin(a\pi x)}{a\pi x} \right]^2 dx dy \\
&= \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x) dx - \int_{\mathbb{R}^2} F(x, y) dx dy < \infty.
\end{aligned}$$

This concludes the proof. \square

We now introduce an interpolation theorem which has proven indispensable throughout our investigations.

Theorem 11. *For all $F \in PW^2(Q_N)$ we have*

$$(3.2) \quad F(\mathbf{x}) = \prod_{n=1}^N \left(\frac{\sin \pi x_n}{\pi} \right)^2 \sum_{\mathbf{n} \in \mathbb{Z}^N} \sum_{j \in \{0,1\}^N} \frac{\partial_j F(\mathbf{n})}{(\mathbf{x} - \mathbf{n})^{2-j}}$$

where $\partial_j = \partial_{j_1 \dots j_N}$ and $(\mathbf{x} - \mathbf{n})^{2-j} = (x_1 - n_1)^{2-j_1} \dots (x_N - n_N)^{2-j_N}$ and the right hand side of (3.2) converges uniformly on compact subsets of \mathbb{R}^N .

Proof. This theorem is proven by induction using Vaaler's result [38, Theorem 9] as the base case and Theorem 8 (Pólya-Plancherel), which guarantees that the sequence $\{F(\mathbf{n}) : \mathbf{n} \in \mathbb{Z}^N\}$ is square summable for any $F \in PW^2(Q_N)$. Also note that, by Fourier inversion, these spaces are closed under partial differentiation. \square

Finally, the next lemma demonstrates that extremal functions always exist for $\nu(N)$ and other minorization problems.

Lemma 12. *Suppose $G \in L^1(\mathbb{R}^N)$ is a real valued function. Let $F_1(\mathbf{x}), F_2(\mathbf{x}), \dots$ be a sequence in $PW^1(Q_N)$ such that $F_\ell(\mathbf{x}) \leq G(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^N$ and each ℓ . Assume that there exists $A > 0$ such that $\widehat{F}_\ell(\mathbf{0}) \geq -A$ for each ℓ . Then there exists a subsequence $F_{\ell_k}(\mathbf{x})$ and a function $F \in PW^1(Q_N)$ such that $F_{\ell_k}(\mathbf{x})$ converges to $F(\mathbf{x})$ uniformly on compact sets as k tends to infinity. In particular, we deduce that $F(\mathbf{x}) \leq G(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^N$ and $\limsup_{k \rightarrow \infty} \widehat{F}_{\ell_k}(\mathbf{0}) \leq \widehat{F}(\mathbf{0})$.*

Proof. By the remark after Theorem 7, each $F_\ell \in PW^2(Q_N)$ and we can bound their $L^2(\mathbb{R}^N)$ -norm in the following way

$$\|F_\ell\|_2 = \|\widehat{F}_\ell\|_2 \leq \text{vol}_N(Q_N)^{1/2} \|\widehat{F}_\ell\|_\infty \leq 2^{N/2} \|F_\ell\|_1$$

and

$$(3.3) \quad \|F_\ell\|_1 \leq \|G - F_\ell\|_1 + \|G\|_1 \leq 2\|G\|_1 + A.$$

Hence the sequence $F_1(\mathbf{x}), F_2(\mathbf{x}), \dots$ is uniformly bounded in $L^2(\mathbb{R}^N)$ and, by the Banach-Alaoglu theorem, we may extract a subsequence (that we still denote by $F_\ell(\mathbf{x})$) that converges weakly to a function $F \in PW^2(Q_N)$. By Theorem 7 we can

assume that $F(\mathbf{x})$ is continuous. By using Fourier inversion we have

$$F_\ell(\mathbf{x}) = \int_{Q_N} \widehat{F}_\ell(\xi) e(\xi \cdot \mathbf{x}) d\xi.$$

Thus, the weak convergence implies that $F_\ell(\mathbf{x}) \rightarrow F(\mathbf{x})$ point-wise for all $x \in \mathbb{R}^N$. Fourier inversion also shows that $\|F_\ell\|_\infty \leq 2^N \|F_\ell\|_1$. However, we also have

$$|\partial_j F_\ell(\mathbf{x})| = 2\pi \left| \int_{Q_N} \xi_j \widehat{F}_\ell(\xi) e(\xi \cdot \mathbf{x}) d\xi \right| \leq 2\pi \|F_\ell\|_1 2^N.$$

We can use (3.3) to conclude that $|F_\ell(\mathbf{x})| + |\nabla F_\ell(\mathbf{x})|$ is uniformly bounded in \mathbb{R}^N . We can apply the Ascoli-Arzelà theorem to conclude that, by possibly extracting a further subsequence, $F_\ell(\mathbf{x})$ converges to $F(\mathbf{x})$ uniformly on compact sets of \mathbb{R}^N .

We conclude that $G(\mathbf{x}) \geq F(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^N$. By applying Fatou's lemma to the sequence of functions $G(\mathbf{x}) - F_1(\mathbf{x}), G(\mathbf{x}) - F_2(\mathbf{x}), \dots$ we find that $F \in L^1(\mathbb{R}^N)$ and

$$\limsup_{\ell \rightarrow \infty} \int_{\mathbb{R}^N} F_\ell(\mathbf{x}) d\mathbf{x} \leq \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x}.$$

This concludes the lemma. \square

4. PROOFS OF THE MAIN RESULTS

The next theorem is the cornerstone in the proof of our main results. This theorem is in stark contrast with the one dimensional case. In the one dimensional case Selberg's function interpolates at all lattice points, and is therefore extremal. In two dimensions, on the other hand, if a minorant interpolates everywhere except for possibly the origin, then it is identically zero. This theorem is therefore troublesome because it seems to disallow the possibility of using interpolation (in conjunction with Poisson summation) to prove an extremality result.

Theorem 13. *Let $F(x, y)$ be admissible for $\nu(2)$ such that $F(n, m) = 0$ for each non-zero $(n, m) \in \mathbb{Z}^2$ and $F(0, 0) \geq 0$, then $F(x, y)$ vanishes identically.*

Proof. Step 1. First we assume that the function $F(x, y)$ is invariant under the symmetries of the square, that is,

$$(4.1) \quad F(x, y) = F(y, x) = F(|x|, |y|)$$

for all $x, y \in \mathbb{R}$. We claim that for any $(m, n) \in \mathbb{Z}^2$ we have

- (a) $\partial_x F(m, n) = 0$ if $(m, n) \neq (\pm 1, 0)$ and $\partial_y F(m, n) = 0$ if $(m, n) \neq (0, \pm 1)$;
- (b) $\partial_{xx} F(m, n) = 0$ if $n \neq 0$ and $\partial_{yy} F(m, n) = 0$ if $m \neq 0$;
- (c) $\partial_{xy} F(m, n) = 0$ if $n \neq \pm 1$ or $m \neq \pm 1$.

We can apply Theorem 10 to deduce that, for each fixed non-zero integer n , the function $x \in \mathbb{R} \mapsto F(x, n)$ is a non-positive function belonging to $PW^1(Q_1)$

that vanishes in the integers, hence identically zero by formula (3.2). Also note that the points $(m, 0)$ for $m \in \mathbb{Z}$ with $|m| > 1$ are local maximums of the function $x \in \mathbb{R} \mapsto F(x, 0)$. These facts in conjunction with the invariance property (4.1) imply items (a) and (b).

Finally, note that a point (m, n) with $|n| > 1$ has to be a local maximum of the function $F(x, y)$. Thus, the Hessian determinant of $F(x, y)$ at such a point has to be non-negative, that is,

$$\text{Hess}_F(m, n) := \partial_{xx}F(m, n)\partial_{yy}F(m, n) - [\partial_{xy}F(m, n)]^2 \geq 0$$

However, by item (b), $\partial_{xx}F(m, n) = 0$ and we conclude that $\partial_{xy}F(m, n) = 0$. This proves item (c) after using again the property (4.1).

Step 2. We can now apply formula (3.2) and deduce that $F(x, y)$ has to have the following form

$$F(x, y) = \left(\frac{\sin \pi x}{\pi x}\right)^2 \left(\frac{\sin \pi y}{\pi y}\right)^2 \left\{ F(0, 0) + a \frac{x^2}{x^2 - 1} + a \frac{y^2}{y^2 - 1} + b \frac{x^2 y^2}{(x^2 - 1)(y^2 - 1)} \right\},$$

where $a = 2\partial_x F(1, 0)$ and $b = 4\partial_{xy}F(1, 1)$. Denote by $B(x, y)$ the expression in the brackets above and note that it should be non-positive if $|x| \geq 1$ or $|y| \geq 1$. We deduce that

$$F(0, 0) + a + (a + b) \frac{x^2}{x^2 - 1} = B(x, \infty) \leq 0$$

for all real x . We conclude that $a + b = 0$, $F(0, 0) \leq -a$ and

$$B(x, y) = F(0, 0) + a \left[1 - \frac{1}{(x^2 - 1)(y^2 - 1)} \right].$$

For each $t > 0$, the set of points $(x, y) \in \mathbb{R}^2 \setminus Q_2$ such that $(x^2 - 1)(y^2 - 1) = 1/t$ is non-empty and $B(x, y) = F(0, 0) + a - at$ at such a point. Therefore $a \geq 0$ and we deduce that $F(0, 0) \leq 0$. We conclude that $F(0, 0) = 0$, which in turn implies that $a = 0$. Thus $F(x, y)$ vanishes identically.

Step 3. Now we finish the proof. Let $F(x, y)$ be a $\nu(2)$ -admissible function such that $F(0, 0) = \widehat{F}(0, 0) \geq 0$. Define the function

$$G_1(x, y) = \frac{F(x, y) + F(-x, y) + F(x, -y) + F(-x, -y)}{4}.$$

Clearly the following function

$$G_0(x, y) = \frac{G_1(x, y) + G_1(y, x)}{2}$$

is also $\nu(2)$ -admissible and $G_0(0, 0) = \widehat{G}_0(0, 0) = F(0, 0) \geq 0$. Moreover, $G_0(x, y)$ satisfies the symmetry property (4.1). By Steps 1 and 2 the function $G_0(x, y)$ must vanish identically. Thus, we obtain that

$$G_1(x, y) = -G_1(y, x).$$

However, since $G_1(x, y)$ is also $\nu(2)$ -admissible we conclude that $G_1(x, y)$ is identically zero outside the box Q_2 , hence it vanishes identically. An analogous argument can be applied to the function $G_2(x, y) = [F(x, y) + F(-x, y)]/2$ to conclude that this function is identically zero outside the box Q_2 , hence it vanishes identically. Using the same procedure again we finally conclude that $F(x, y)$ vanishes identically and the proof of the theorem is complete. \square

4.1. Proof of Theorem 4. The proof is done via induction and the base case is Theorem 13. Assume that the theorem is proven for some dimension $N \geq 2$. Let $F(\mathbf{x}, x_{N+1})$ be a $\nu(N+1)$ -admissible function such that $F(\mathbf{n}, m) = 0$ for all non-zero $(\mathbf{n}, m) \in \mathbb{Z}^{N+1}$. Now, for every fixed $t \in \mathbb{R}$ define $G_t(\mathbf{x}) = F(\mathbf{x}, t)$. An application of Lemma 10 shows that $G_t \in PW^1(Q_N)$ for all $t \in \mathbb{R}$ and is $\nu(N)$ -admissible if $|t| < 1$ and non-positive if $|t| \geq 1$. Moreover, for any fixed non-zero $m \in \mathbb{Z}$ we have $G_m(\mathbf{n}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^N$, thus by induction we have $G_m \equiv 0$ for every non-zero $m \in \mathbb{Z}$. That is, $F(\mathbf{x}, x_{N+1}) = 0$ if one of its entries is a non-zero integer. We conclude that the $\nu(N)$ -admissible function $G_t(\mathbf{x})$ satisfies $G_t(\mathbf{n}) = 0$ for every non-zero $\mathbf{n} \in \mathbb{Z}^N$. By induction again, $G_t \equiv 0$ for all real t . This implies that $F \equiv 0$ and this finishes the proof.

4.2. Proof of Theorem 2. The item (i) is a direct consequence of Lemma 12 while item (iii) is a consequence of Theorem 5 item (iv). It remains to show item (ii).

Clearly by Lemma 10, we have $\nu(N) \geq \nu(N+1)$. Suppose by contradiction that $\nu(N) = \nu(N+1)$. Let $(\mathbf{x}, t) \in \mathbb{R}^N \times \mathbb{R} \mapsto F(\mathbf{x}, t)$ be an extremal function for $\nu(N+1)$. Let $G_m(\mathbf{x}) = F(\mathbf{x}, m)$ for each $m \in \mathbb{Z}$. Lemma 10 implies that $G_m(\mathbf{x})$ is also admissible for $\nu(N)$ (if $m \neq 0$ then the function is non-positive). By the Poisson summation formula we have for each non-zero $m \in \mathbb{Z}$

$$(4.2) \quad \widehat{F}(\mathbf{0}) = \sum_{\mathbf{n} \in \mathbb{Z}^N} \sum_{k \in \mathbb{Z}} F(\mathbf{n}, k) \leq \sum_{\mathbf{n} \in \mathbb{Z}^N} (F(\mathbf{n}, m) + F(\mathbf{n}, 0)) = \widehat{G}_m(0) + \widehat{G}_0(0).$$

By assumption

$$(4.3) \quad \widehat{G}_0(0) \leq \nu(N) = \nu(N+1) = \widehat{F}(\mathbf{0})$$

Combining (4.2) and (4.3) yields $0 \leq \widehat{G}_m(0)$ for each $m \neq 0$. However, the function $G_m(\mathbf{x}) \leq 0$ for each $\mathbf{x} \in \mathbb{R}^N$ whenever m is a non-zero integer. Consequently, $G_m(\mathbf{x})$ vanishes identically. It follows that $F(\mathbf{n}) = 0$ for each non-zero $\mathbf{n} \in \mathbb{Z}^{N+1}$. By Theorem 4, $F(\mathbf{x})$ vanishes identically. Therefore $\nu(N+1) = \nu(N) = 0$, a contradiction. The theorem is finished.

4.3. Proof of Theorem 5. First we prove item (i). Assume by contradiction that there exists a sequence of $\Delta(N)$ -admissible functions $F_\ell(\mathbf{x})$ $\ell = 1, 2, \dots$ such that $M_\ell = \max_{\mathbf{x} \in Q_N} \{F_\ell(\mathbf{x})\}$ converges to ∞ when $\ell \rightarrow \infty$. Let $G_\ell(\mathbf{x}) = F_\ell(\mathbf{x})/M_\ell$,

and note that $G_\ell(\mathbf{x})$ is $\nu(N)$ -admissible for all ℓ . Also let $\mathbf{x}_\ell \in Q_N$ be such that $F_\ell(\mathbf{x}_\ell) = M_\ell$. We can assume by compactness that $\mathbf{x}_\ell \rightarrow \mathbf{x}_0$. By Lemma 12 we may also assume that there exists a function $G(\mathbf{x})$, $\nu(N)$ -admissible such that $G_\ell(\mathbf{x})$ converges uniformly on compact sets to $G(\mathbf{x})$. We also have by Lemma 12 that

$$0 \leq \limsup_\ell \int_{\mathbb{R}^N} G_\ell(\mathbf{x}) d\mathbf{x} \leq \int_{\mathbb{R}^N} G(\mathbf{x}) d\mathbf{x}.$$

However, $G_\ell(\mathbf{0}) = 1/M_\ell \rightarrow 0$ and thus $G(\mathbf{0}) = 0$. By the Poisson summation formula, for any fixed non-zero $\mathbf{n} \in \mathbb{Z}^N$ we have

$$0 \leq \widehat{G}_\ell(\mathbf{0}) \leq 1/M_\ell + G_\ell(\mathbf{n}).$$

Thus, we conclude that $G_\ell(\mathbf{n}) \rightarrow 0$ as $\ell \rightarrow \infty$. This, implies that $G(\mathbf{n}) = 0$ for all $\mathbf{n} \in \mathbb{Z}^N$. By Theorem 4 we conclude that $G(\mathbf{x})$ vanishes identically. However, by uniform convergence we have $G(\mathbf{x}_0) = 1$, a contradiction. This proves item (i)

Item (ii) is a consequence of Lemma 12 in conjunction with item (i). Item (iii) can be proven exactly as in Theorem 2 item (ii), since now we know that extremizers exist. It remains to show the upper bound of item (iv). For this we will show a stronger result.

Lemma 14. *The function*

$$\delta(N) = \frac{1 - \Delta(N)}{N}$$

is non-decreasing. That is, if $\Delta(N) > 0$ and $M < N$ then $\delta(M) \leq \delta(N)$.

Proof. For a given $\mathbf{n} \in \mathbb{Z}^N$ let $\sigma(\mathbf{n})$ denote the quantity of distinct numbers in \mathbb{Z}^N that can be constructed by only permuting the entries in \mathbf{n} . It is simple to see that if $M < N$, $\mathbf{m} \in \mathbb{Z}^M$ is non-zero and $(\mathbf{m}, \mathbf{0}) \in \mathbb{Z}^N$ then

$$\sigma(\mathbf{m}, \mathbf{0}) \geq (N/M)\sigma(\mathbf{m}),$$

and equality is attained if \mathbf{m} has only one entry different than zero. Let Γ^N be the subset of non-zero $\mathbf{n} = (n_1, \dots, n_N) \in \mathbb{Z}_+^N$ such that $n_1 \geq n_2 \geq \dots \geq n_N \geq 0$ ($\mathbb{Z}_+ = \{0, 1, 2, \dots\}$). Note that $(\Gamma^M, \mathbf{0}) \subset \Gamma^N$ if $M < N$. Also let $\varepsilon(\mathbf{n})$ be the number of non-zero entries in a vector $\mathbf{n} \in \Gamma^N$. Now, for a given N , let $F_N(\mathbf{x})$ be an extremal function for the $\Delta(N)$ problem. We can assume that it is invariant under the symmetries of Q_N . Define $G_N(\mathbf{y}) = F_N(\mathbf{y}, \mathbf{0})$ for every $\mathbf{y} \in \mathbb{R}^M$, $M < N$.

By Poisson summation we obtain

$$\begin{aligned}
\Delta(N) &= \widehat{F}_N(\mathbf{0}) = 1 + \sum_{\mathbf{n} \in \Gamma^N} 2^{\varepsilon(\mathbf{n})} \sigma(\mathbf{n}) F_N(\mathbf{n}) \\
&\leq 1 + \sum_{\mathbf{m} \in \Gamma^M} 2^{\varepsilon(\mathbf{m}, \mathbf{0})} \sigma(\mathbf{m}, \mathbf{0}) F_N(\mathbf{m}, \mathbf{0}) \\
&= 1 + \sum_{\mathbf{m} \in \Gamma^M} 2^{\varepsilon(\mathbf{m})} \sigma(\mathbf{m}, \mathbf{0}) G_N(\mathbf{m}) \\
&\leq 1 + (N/M) \sum_{\mathbf{m} \in \Gamma^M} 2^{\varepsilon(\mathbf{m})} \sigma(\mathbf{m}) G_N(\mathbf{m}) = 1 + (N/M)(\widehat{G}_N(0) - 1) \\
&\leq 1 + (N/M)(\Delta(M) - 1).
\end{aligned}$$

We conclude that

$$\frac{1 - \Delta(N)}{N} \geq \frac{1 - \Delta(M)}{M},$$

and this finishes the lemma. \square

Proof of Theorem 5 continued. The previous Lemma implies that if $\Delta(N) > 0$ then

$$\frac{1 - \Delta(N)}{N} = \delta(N) \geq \delta(2) = \frac{1 - \Delta(2)}{2}.$$

We conclude that

$$\frac{2}{1 - \Delta(2)} > N,$$

and this finishes the proof of the theorem. \square

Remarks.

(1) We note that, by giving any explicit upper bound strictly less than one for some $\Delta(N)$, we would obtain an explicit upper bound for the critical dimension N_c .

(2) Let $M < N \leq N_c$, then

$$\Delta(M) \geq (1 - M/N) + (M/N)\Delta(N).$$

This inequality produces lower bounds for lower dimensions once a lower bound is given for a higher dimension.

5. EXPLICIT MINORANTS IN LOW DIMENSIONS

We define an auxiliary variational quantity $\lambda(N)$ over a more restrictive set of admissible functions than $\nu(N)$. Let

$$\lambda(N) = \sup \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x}$$

where the supremum is taken over functions $F(\mathbf{x})$ that are admissible for $\nu(N)$ and, in addition, $F(\mathbf{0}) = 1$, and

$$F(\mathbf{n}) = 0$$

for each non-zero $\mathbf{n} \in \mathbb{Z}^N$ unless \mathbf{n} is a corner of the box Q_N . Here, a corner of the box Q_N is a vector $\mathbf{n} \in \partial Q_N \cap \mathbb{Z}^N$ such that there exist at most $N - 2$ zero entries in \mathbf{n} and all the non-zero entries are equal to ± 1 . This definition makes any k -dimensional slice of an admissible function for $\lambda(N)$ ($k < N$) admissible for $\lambda(k)$, which in turn implies that

$$\lambda(N + 1) \leq \lambda(N)$$

for all N . We note that Selberg's functions (see Appendix) are always admissible for $\lambda(N)$ but have negative integral. Our aim is to mimic Selberg's construction but to incorporate a correction term so that our minorants have positive integral. Notice that by Theorem 2 it is impossible to do this in sufficiently high dimensions.

Making use of the interpolation formula (3.2) we conclude that every function $F(\mathbf{x})$ admissible for $\lambda(N)$ has the following useful representation

$$(5.1) \quad F(\mathbf{x}) = S(\mathbf{x})P(\mathbf{x})$$

where

$$S(\mathbf{x}) = \prod_{n=1}^N \left(\frac{\sin(\pi x_n)}{\pi x_n (x_n^2 - 1)} \right)^2$$

and $P(\mathbf{x})$ is a polynomial such that each variable x_n appearing in its expression has an exponent not greater than 4. Notice that, by Poisson summation, if $F(\mathbf{x})$ is admissible for $\lambda(N)$ and is invariant under the symmetries of Q_N then

$$(5.2) \quad \int_{\mathbb{R}^N} F(\mathbf{x}) d\mathbf{x} = 1 + \sum_{k=2}^N \binom{N}{k} 2^{-k} P(\mathbf{u}_k)$$

where $\mathbf{u}_k = (\overbrace{1, 1, 1, \dots, 1}^{k \text{ times}}, 0, \dots, 0)$.

In what follows will be useful to use a particular family of symmetric functions. For given integers $N \geq k \geq 1$ we define

$$\sigma_{N,k}(\mathbf{x}) = \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq N} x_{n_1}^2 x_{n_2}^2 \dots x_{n_k}^2$$

and

$$\tilde{\sigma}_{N,k}(\mathbf{x}) = \sum_{1 \leq n_1 < n_2 < \dots < n_k \leq N} x_{n_1}^4 x_{n_2}^4 \dots x_{n_k}^4.$$

Theorem 15. *Define the functions $\mathcal{F}_2(x_1, x_2)$, $\mathcal{F}_3(x_1, x_2, x_3)$, $\mathcal{F}_4(x_1, \dots, x_4)$ and $\mathcal{F}_5(x_1, \dots, x_5)$ by using representation (5.1) and the following polynomials respectively*

- $P_2(x_1, x_2) = (1 - x_1^2)(1 - x_2^2) - \frac{1}{16} \tilde{\sigma}_{2,2}(x_1, x_2)$

- $P_3(x_1, x_2, x_3) = \prod_{n=1}^3 (1 - x_n^2) - \frac{1}{16} \tilde{\sigma}_{3,2}(x_1, x_2, x_3)$
- $P_4(x_1, \dots, x_4) = \prod_{n=1}^4 (1 - x_n^2) - \frac{3}{4} \sigma_{4,4}(x_1, \dots, x_4) - \frac{1}{16} \tilde{\sigma}_{4,2}(x_1, \dots, x_4)$
- $P_5(x_1, \dots, x_5) = \prod_{n=1}^5 (1 - x_n^2) - \frac{3}{4} \sigma_{5,4}(x_1, \dots, x_5) - \frac{1}{16} \tilde{\sigma}_{5,2}(x_1, \dots, x_5)$.

These functions are admissible for $\lambda(2)$, $\lambda(3)$, $\lambda(4)$ and $\lambda(5)$ respectively and their respective integrals are equal to: $63/64 = 0.984375$, $119/128 = 0.9296875$, $95/128 = 0.7421975$ and $31/256 = 0.12109375$.

Proof. The integrals of these functions can be easily calculated using formula (5.2), we prove only their admissibility. We start with $\mathcal{F}_2(\mathbf{x})$. Clearly, if $|x_1| > 1 > |x_2|$ then $P_2(x_1, x_2) < 0$. Also, writing $t = |x_1 x_2|$ we obtain

$$\begin{aligned} P_2(x_1, x_2) &= 1 + x_1^2 x_2^2 - x_1^2 - x_2^2 - x_1^4 x_2^4 / 16 \\ &\leq 1 + x_1^2 x_2^2 - 2|x_1 x_2| - x_1^4 x_2^4 / 16 \\ &= 1 + t^2 - 2t - t^4 / 16. \end{aligned}$$

On the other hand, we have

$$(5.3) \quad 1 + t^2 - 2t - t^4 / 16 = (1 - t)^2 - t^4 / 16$$

and

$$(5.4) \quad 1 + t^2 - 2t - t^4 / 16 = (t - 2)^2 (4 - 4t - t^2) / 16.$$

If $|x_1|, |x_2| < 1$ then $0 \leq t < 1$, and by (5.3) we deduce that $P_2(x_1, x_2) < 1$. If $|x_1|, |x_2| > 1$ then $t > 1$, and by (5.4) we deduce that $P_2(x_1, x_2) \leq 0$. This proves that $\mathcal{F}_2(\mathbf{x})$ is $\lambda(2)$ -admissible.

Now, observe that $P_3(x_1, x_2, x_3) < 1$ inside the box Q_3 and $P_3(x_1, x_2, x_3) < 0$ if exactly one or three variables have modulus greater than one. If exactly two variables have modulus greater than one, suppose for instance that $|x_1|, |x_2| > 1 > |x_3|$, then

$$P_3(x_1, x_2, x_3) \leq P_2(x_1, x_2) \leq 0.$$

This proves that $\mathcal{F}_3(\mathbf{x})$ is admissible for $\lambda(3)$.

In the same way, clearly $P_4(x_1, \dots, x_4) < 1$ if all the variables have modulus less than one. If an odd number of variables have modulus greater than one then the function is trivially negative. If $|x_1|, |x_2| > 1 > |x_3|, |x_4|$ then

$$P_4(x_1, x_2, x_3, x_4) \leq P_2(x_1, x_2) \leq 0.$$

On the other hand, if $|x_1|, |x_2|, |x_3|, |x_4| > 1$ then, suppressing the variables, we have

$$P_4 = 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} + \frac{1}{4}\sigma_{4,4} - \frac{1}{16}\tilde{\sigma}_{4,2}.$$

Observing that

$$\sigma_{4,2} - \sigma_{4,3} \leq x_1^2 x_2^2 + x_3^2 x_4^2,$$

we obtain

$$\begin{aligned} 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} - \frac{1}{16}\tilde{\sigma}_{4,2} &\leq -1 + P_2(x_1, x_2) + P_2(x_3, x_4) \\ &\quad - \frac{1}{16}[x_1^4 x_3^4 + x_1^4 x_4^4 + x_2^4 x_3^4 + x_2^4 x_4^4]. \end{aligned}$$

Since $P_2(x_1, x_2) \leq 0$ and $P_2(x_3, x_4) \leq 0$, we deduce that

$$\begin{aligned} P_3(x_1, \dots, x_4) &\leq -1 + \frac{1}{4}x_1^2 x_2^2 x_3^2 x_4^2 - \frac{1}{16}[x_1^4 x_3^4 + x_1^4 x_4^4 + x_2^4 x_3^4 + x_2^4 x_4^4] \\ &\leq -1 + \frac{1}{16}(x_1^4 + x_2^4)(x_3^4 + x_4^4) - \frac{1}{16}[x_1^4 x_3^4 + x_1^4 x_4^4 + x_2^4 x_3^4 + x_2^4 x_4^4] \\ &= -1. \end{aligned}$$

This proves that $\mathcal{F}_4(\mathbf{x})$ is admissible for $\lambda(4)$. By a similar argument one can prove that $\mathcal{F}_5(\mathbf{x})$ is admissible for $\lambda(5)$. \square

6. AN APPLICATION

Let $\|\cdot\|$ denote the distance to the nearest integer lattice point. The following theorem is from [2].

Theorem 16 (Barton-Montgomery-Vaaler). *Let $\varepsilon > 0$ and $\xi_1, \dots, \xi_M \in \mathbb{R}^N / \mathbb{Z}^N$ such that $\|\xi_n\| > \varepsilon$ for each $n = 1, \dots, M$. Then*

$$M \leq 3 \sum_{\substack{\mathbf{n} \in \mathbb{Z}^N \\ 0 < \|\mathbf{n}\|_\infty \leq N\varepsilon^{-1}}} \left| \sum_{m=1}^M e(\mathbf{n} \cdot \xi_m) \right|$$

We can improve the above estimate by using admissible minorants for $\nu(N)$.

Theorem 17. *Suppose that $\varepsilon > 0$, that $F(\mathbf{x})$ is admissible for $\nu(N)$, and that $\widehat{F}(0) > 0$. If $\xi_1, \dots, \xi_M \in \mathbb{R}^N / \mathbb{Z}^N$ such that $\|\xi_n\| > \varepsilon$ for each $n = 1, \dots, M$, then*

$$M \leq \frac{\|\widehat{F}\|_\infty}{\widehat{F}(0)} \sum_{\substack{\mathbf{n} \in \mathbb{Z}^N \\ 0 < \|\mathbf{n}\|_\infty \leq \varepsilon^{-1}}} \left| \sum_{m=1}^M e(\mathbf{n} \cdot \xi_m) \right|.$$

Proof. Let $U_\varepsilon = \varepsilon Q_N + \mathbb{Z}^N \subset \mathbb{R}^N / \mathbb{Z}^N$ and let

$$\Psi_\varepsilon(\mathbf{x}) = \varepsilon^N \sum_{\substack{\mathbf{n} \in \mathbb{Z}^N \\ \|\mathbf{n}\|_\infty < \varepsilon^{-1}}} \widehat{F}(\varepsilon \mathbf{n}) e(\mathbf{n} \cdot \mathbf{x}).$$

If $\varepsilon < \frac{1}{2}$, then $\Psi_\varepsilon(\mathbf{x})$ is a minorant of U_ε . Observe

$$0 = \sum_{m=1}^M \mathbf{1}_{U_\varepsilon}(\boldsymbol{\xi}_m) \geq \sum_{m=1}^M \Psi_\varepsilon(\boldsymbol{\xi}_m)$$

and that

$$\varepsilon^{-N} \sum_{m=1}^M \Psi_\varepsilon(\boldsymbol{\xi}_m) = \widehat{F}(0)M + \sum_{\substack{\mathbf{n} \in \mathbb{Z}^N \\ 0 < \|\mathbf{n}\|_\infty < \varepsilon^{-1}}} \widehat{F}(\varepsilon \mathbf{n}) \sum_{m=1}^M e(\mathbf{n} \cdot \boldsymbol{\xi}_m).$$

The proof is complete upon rearranging terms and applying the triangle inequality. \square

Remark. It can be verified numerically that the minorants of Theorem 6 satisfy the property that maximum of their Fourier transforms is achieved at the origin. That is, there are $\nu(N)$ admissible functions for $N \leq 5$ for which

$$\frac{\|\widehat{F}\|_\infty}{\widehat{F}(0)} = 1.$$

However, we remark this is not a property shared by any admissible minorant, since we have also constructed explicit admissible minorants which fail to satisfy this property.

Therefore we have the following corollary.

Corollary 18. *Let $N = 1, 2, 3, 4$, or 5 and $\varepsilon > 0$. If $\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_M \in \mathbb{R}^N / \mathbb{Z}^N$ such that $\|\boldsymbol{\xi}_n\| > \varepsilon$ for each $n = 1, \dots, M$, then*

$$M \leq \sum_{\substack{\mathbf{n} \in \mathbb{Z}^N \\ 0 < \|\mathbf{n}\|_\infty \leq \varepsilon^{-1}}} \left| \sum_{m=1}^M e(\mathbf{n} \cdot \boldsymbol{\xi}_m) \right|.$$

APPENDIX: SELBERG AND MONTOMGERY'S CONSTRUCTIONS

In this appendix we will present the box minorant constructions of Selberg and Montgomery and we will perform some asymptotic analysis on their integrals. In particular we will show in which regimes Selberg's minorant is a better approximate than Montgomery's and visa-versa. The interested readers are encouraged to consult [30, 36, 38] for more on Selberg's functions and [2, 19] for more on Montgomery's functions. Our treatment is by no means exhaustive.

Both constructions begin with the following entire functions

$$K(z) = \left(\frac{\sin \pi z}{\pi z} \right)^2$$

and

$$H(z) = \left\{ \frac{\sin^2 \pi z}{\pi^2} \right\} \left(\sum_{n=-\infty}^{\infty} \frac{\operatorname{sgn}(n)}{(z-n)^2} + \frac{2}{z} \right)$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Let $[a_1, b_1], \dots, [a_N, b_N] \subset \mathbb{R}$ where $b_n > a_n$ for each $n = 1, \dots, N$, and set $B = \prod [a_i, b_i]$. For each $i = 1, \dots, N$ define

$$\begin{aligned} V_i(z) &= \frac{1}{2}H(z - a_i) + \frac{1}{2}H(b_i - z) \\ E_i(z) &= \frac{1}{2}K(z - a_i) + \frac{1}{2}K(b_i - z) \\ C_i(z) &= V_i(z) + E_i(z) \\ c_i(z) &= V_i(z) - E_i(z). \end{aligned}$$

The following theorem can be deduced from [30, 36, 38].

Theorem 19 (Selberg). *The function*

$$\mathfrak{C}_B(\mathbf{x}) = -(N-1) \prod_{i=1}^N C_i(x) + \sum_{n=1}^N c_n(x) \prod_{m \neq n} C_m(x)$$

satisfies

- (i) $\hat{\mathfrak{C}}_B(\boldsymbol{\xi}) = 0$ for each $\|\boldsymbol{\xi}\|_\infty > 1$;
- (ii) $\mathfrak{C}_B \leq \mathbf{1}_B(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^N$; and
- (iii)

$$\int_{\mathbb{R}^N} \mathfrak{C}_B(\mathbf{x}) d\mathbf{x} = -(N-1) \prod_{i=1}^N (b_i - a_i + 1)$$

$$+ \sum_{n=1}^N (b_n - a_n - 1) \prod_{m \neq n} (b_m - a_m + 1).$$

Corollary 20. *Let $B = [-\delta, \delta]^N$. We have*

$$\int_{\mathbb{R}^N} \mathcal{C}_B(\mathbf{x}) d\mathbf{x} > 0$$

if and only if

$$\delta > N - \frac{1}{2}.$$

On the other hand, if N is fixed, then

$$\int_{\mathbb{R}^N} \mathcal{C}_B(\mathbf{x}) d\mathbf{x} = (2\delta)^N - (N-1)(2\delta)^{N-1} + O(\delta^{N-2})$$

as $\delta \rightarrow \infty$.

Proof. Setting $a_n = -\delta$ and $b_n = \delta$ it follows from Theorem 19 (iii) that

$$\int_{\mathbb{R}^N} \mathcal{C}_B(\mathbf{x}) d\mathbf{x} = (2\delta + 1)^{N-1} (2\delta - (2N-1)).$$

This quantity is positive if and only if $2\delta - (2N-1) > 0$, which occurs if and only if $\delta > N - \frac{1}{2}$. On the other hand,

$$(2\delta + 1)^{N-1} (2\delta - (2N-1)) = (2\delta)^N - (2N-1)(2\delta)^{N-1} + O_N(\delta^{N-2})$$

as $\delta \rightarrow \infty$. □

The following theorem can be deduced from [19].

Theorem 21 (Montgomery). *The function*

$$\mathcal{G}_B(\mathbf{x}) = \prod_{i=1}^N V_i(x) - \prod_{i=1}^N (V_i(x) + 2E_i(x)) + \prod_{i=1}^N (V_i(x) + E_i(x))$$

satisfies

- (i) $\hat{\mathcal{G}}_B(\boldsymbol{\xi}) = 0$ for each $\|\boldsymbol{\xi}\|_\infty > 1$;
- (ii) $\mathcal{G}_B \leq \mathbf{1}_B(\mathbf{x})$ for each $\mathbf{x} \in \mathbb{R}^N$; and
- (iii)

$$\int_{\mathbb{R}^N} \mathcal{G}_B(\mathbf{x}) d\mathbf{x} = \prod_{n=1}^N (b_n - a_n) - \prod_{n=1}^N (b_n - a_n + 2) + \prod_{n=1}^N (b_n - a_n + 1)$$

Corollary 22. *Let $B = [-\delta, \delta]^N$, $\epsilon > 0$, and $\phi = (1 + \sqrt{5})/2$. We have*

$$\int_{\mathbb{R}^N} \mathcal{G}_B(\mathbf{x}) d\mathbf{x} < 0$$

if

$$\delta < \left(\frac{1}{2 \log(\phi)} - \epsilon \right) N = (1.039\dots - \epsilon) N$$

and

$$\int_{\mathbb{R}^N} \mathcal{G}_{rQ_N}(\mathbf{x}) d\mathbf{x} > 0$$

if

$$\delta > \left(\frac{1}{2 \log(\phi)} + \epsilon \right) N = (1.039\dots + \epsilon) N$$

when N is sufficiently large. When N is fixed and $\delta \rightarrow \infty$ we have

$$\int_{\mathbb{R}^N} \mathcal{G}_B(\mathbf{x}) d\mathbf{x} = (2\delta)^N - (2\delta)^{N-1} + O(\delta^{N-2})$$

Proof. We will only prove the first statement of the corollary since the second statement is straightforward. Setting $a_n = -\delta$ and $b_n = \delta$ we have by Theorem 21

$$\int_{\mathbb{R}^N} \mathcal{G}_B(\mathbf{x}) d\mathbf{x} = (2\delta)^N - (2\delta + 2)^N + (2\delta + 1)^N.$$

Since the right hand side remains positive if we divide by $(2\delta)^N$ it suffices to determine when

$$1 - \left(1 + \frac{1}{\delta}\right)^N + \left(1 + \frac{1}{2\delta}\right)^N > 0.$$

Setting $\delta = N/c$ for some $c > 0$ we find that for large N

$$1 - \left(1 + \frac{1}{\delta}\right)^N + \left(1 + \frac{1}{2\delta}\right)^N \approx 1 - e^c + e^{c/2}.$$

The equation $1 - e^c + e^{c/2} = 0$ has one real solution, namely $c = 2 \log(\phi)$. The function $c \mapsto 1 - e^c + e^{c/2}$ is a decreasing function at $c = 2 \log(\phi)$ so if $c < 2 \log(\phi)$ is a constant independent of N , then for N sufficiently large we have

$$1 - \left(1 + \frac{1}{\delta}\right)^N + \left(1 + \frac{1}{2\delta}\right)^N > 0.$$

On the other hand, if $c > 2 \log(\phi)$ then

$$1 - \left(1 + \frac{1}{\delta}\right)^N + \left(1 + \frac{1}{2\delta}\right)^N < 0.$$

The proof of the first statement is complete upon setting $\delta = ((2 \log(\phi))^{-1} \pm \epsilon) N$. \square

It follows from the above corollaries that Montgomery's minorants are better approximates when δ is very large compared to N , and Selberg's are better when N is large compared to δ .

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