The global nonlinear stability of Minkowski space.

Einstein equations, f(R)–modified gravity, and Klein-Gordon fields

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Abstract. We study the initial value problem for two fundamental theories of gravity, that is, Einstein's field equations of general relativity and the (fourth-order) field equations of $f(R)$-modified gravity. For both of these physical theories, we investigate the global dynamics of a self-gravitating massive matter field when an initial data set is prescribed on an asymptotically flat and spacelike hypersurface, provided these data are sufficiently close to data in Minkowski spacetime. Under such conditions, we thus establish the global nonlinear stability of Minkowski spacetime in presence of massive matter. In addition, we provide a rigorous mathematical validation of the $f(R)$-theory based on analyzing a singular limit problem, when the function $f(R)$ arising in the generalized Hilbert-Einstein functional approaches the scalar curvature function R of the standard Hilbert-Einstein functional. In this limit we prove that $f(R)$-Cauchy developments converge to Einstein's Cauchy developments in the regime close to Minkowski space.

Our proofs rely on a new strategy, introduced here and referred to as the *Euclidian-Hyperboloidal Foliation Method (EHFM).* This is a major extension of the Hyperboloidal Foliation Method (HFM) which we used earlier for the Einstein-massive field system but for a restricted class of initial data. Here, the data are solely assumed to satisfy an asymptotic flatness condition and be small in a weighted energy norm. These results for matter spacetimes provide a significant extension to the existing stability theory for vacuum spacetimes, developed by Christodoulou and Klainerman and revisited by Lindblad and Rodnianski.

The EHFM approach allows us to solve the global existence problem for a broad class of nonlinear systems of coupled wave-Klein-Gordon equations. As far as gravity theories are concerned, we cope with the nonlinear coupling between the matter and the geometry parts of the field equations. Our method does not use Minkowski's scaling field, which is the essential challenge we overcome here in order to encompass wave-Klein-Gordon equations.

We introduce a spacetime foliation based on glueing together asymptotically Euclidian hypersurfaces and asymptotically hyperboloidal hypersurfaces. Well-chosen frames of vector fields (null-semi-hyperboloidal frame, Euclidian-hyperboloidal frame) allow us to exhibit the structure of the equations under consideration as well as to analyze the decay of solutions in time and in space. New Sobolev inequalities valid in positive cones and in each domain of our Euclidian-hyperboloidal foliation are established and a bootstrap argument is formulated, which involves a hierarchy of (almost optimal) energy and pointwise bounds and distinguishes between low- and high-order derivatives of the solutions. Dealing with the (fourth-order) $f(R)$-theory requires a number of additional ideas: a conformal wave gauge, an augmented formulation of the field equations, the quasi-null hyperboloidal structure, and a singular analysis to cope with the limit $f(R) \to R$.

Contents

Part I
Formulation of the problem

1 Purpose of this Monograph

1.1 Main objectives

1.1.1 Evolution of self-gravitating massive matter

Two theories of gravity. We are interested in the *global evolution problem* for self-gravitating massive matter, which we formulate as an initial value problem associated with the Einstein equations of general relativity or with the field equations of the $f(R)$-modified theory of gravity. Einstein's theory goes back to 1915, while the $f(R)$-theory was formulated in 1970 by Buchdahl [21]. In this latter gravity theory, the integrand of the Hilbert-Einstein functional, i.e. the spacetime scalar curvature R (cf. (1.1), below), is replaced by a nonlinear function satisfying $f(R) \simeq R + \frac{\kappa}{2}R^2$ for some small constant $\kappa > 0$ (cf. (1.5), below).

Both theories play a central role in the foundations of astrophysics and cosmology and it is natural to investigate the global nonlinear stability of the 'ground state' of these theories, i.e. Minkowski spacetime. This spacetime represents the geometry of a vacuum Universe at equilibrium. While building on the pioneering work by Christodoulou and Klainerman [33] on the *vacuum stability problem* (cf. our review in Section 1.2.1, below), our aim in this Monograph is to solve the *global stability problem for massive matter*, i.e. to study the global evolution of a *self-gravitating massive field*.

Three main results. More precisely, by prescribing an initial data set on a spacelike hypersurface which is asymptotically flat (in a suitable sense) and is sufficiently close to a slice of (empty, flat) Minkowski spacetime, we solve the following three nonlinear stability problems:

(1) *The global existence problem for the Einstein-massive field system.*

(2) *The global existence problem for the field equations of $f(R)$-gravity.*

(3) *The global validation of the $f(R)$-theory in the limit*[1] $f(R) \to R$.

In particular, we prove that global Cauchy developments of $f(R)$-gravity are "close" (in a sense we specify) to global Cauchy developments of Einstein theory.

The problems under consideration are geometric in nature and require a choice of gauge. Our analysis relies on the wave gauge, i.e on coordinate functions satisfying the wave equation (in the unknown metric). We built here on Lindblad and Rodnianski's pioneering work [95] concerning the vacuum Einstein equations, while bringing in significantly new insights and techniques required in order to cope with the three new problems above. We emphasize that, even for vacuum spacetimes, our results are stronger than those established earlier in [95] since we include a broader class of initial data sets.

1.1.2 A new method for the global existence problem

The mathematical analysis of self-gravitating massive fields was initiated by the authors in [87, 88], where the first of the above three problems was solved for the restricted class of initial data sets coinciding with Schwarzschild data outside of a spatially compact domain. In the present Monograph, we remove this restriction entirely and, moreover, provide a resolution to all of the three problems above. Our stability theorems encompass initial data sets which are characterized in terms of weighted Sobolev-type norms and pointwise decay conditions at spacelike infinity.

Several novel techniques of analysis are introduced in this Monograph, which are expected to be useful also for other problems. Namely, the fourth contribution of the present Monograph is a vast generalization of the Hyperboloidal Foliation Method (HFM) discussed by the authors in the earlier work [87, 88]. We refer to our approach as:

(4) *The Euclidian-Hyperboloidal Foliation Method* (EHFM).

[1] We can also refer to this problem as the *infinite mass problem.*

While we develop this approach primarily for problems arising in gravity theory, we expect it to be useful to tackle the global existence problem for a broad class of nonlinear systems of coupled wave-Klein-Gordon equations.

1.1.3 Two conjectures made by physicists

Asymptotically flat spacetimes. Numerical investigations of the stability problem (1) above for small perturbations of massive fields had originally suggested a potential mechanism for *nonlinear instability*. Indeed, the family of oscillating soliton stars (defined in [48, 116]) provides a potential candidate for instabilities that may develop during the evolution of matter governed by the Einstein equations. After several controversies, only the most recent numerical developments(cf. [106]) have finally led to a definite conjecture: in asymptotically flat spacetimes, one expects massive fields to be globally *nonlinearly stable*.

Advanced numerical methods (including mesh refinement and high-order accuracy) were necessary and, in the long-time evolution of arbitrarily small perturbations of oscillating soliton stars, the following nonlinear mechanism takes place: in a first phase of the evolution, the matter tends to collapse and, potentially, could evolve to a black hole; however, in an intermediate phase of the evolution and below a certain threshold in the mass amplitude, the collapse significantly slows down, until the final phase of the evolution is reached and a strong dissipation mechanism eventually prevents the collapse from occurring.

It was thus conjectured that *dispersion effects* are dominant in the long-time evolution of self-gravitating matter. The present work provides the first rigorous proof of this conjecture.

Asymptotically AdS spacetimes. It is worth mentioning that, in asymptotically anti-de Sitter (AdS) spacetimes, it was observed numerically that the evolution of generic (and arbitrarily small) initial perturbations leads to the formation of black holes. In such spacetimes, matter is confined (i.e. timelike geodesics reach the AdS boundary in a finite proper time) and *cannot disperse;* the effect of gravity remains dominant during all of the evolution, unavoidably leading to the formation of black holes; cf. Bizon et al. [16, 17, 18] and Dafermos and Holzegel [35, 36].

It is only in a recent work by Moschidis [103, 104] that the instability phenomena in asymptotically AdS spacetimes was rigorously established when the spherically symmetric Einstein–massless Vlasov system is considered.

The theory of $f(R)$-gravity. Another important conjecture made by physicists is relevant to the present study and concerns the theory of modified gravity theory. By construction, this theory is expected to be a "correction" to Einstein's theory, so that solutions to the $f(R)$-field equations (discussed below) are conjectured to be "close" to those of Einstein's equations when the defining function $f(R)$ is close to the scalar curvature function R.

Of course, at a *formal level* it is straightforward to check that the field equations determined from the nonlinear function $f(R) \simeq R + \frac{\kappa}{2} R^2$ "converge" to Einstein's field equations when the defining parameter $\kappa \to 0$. The challenge is to establish this property rigorously for a class of solutions. Indeed, in this work we resolve this conjecture for the class of spacetimes under consideration.

We establish that $f(R)$-Cauchy developments converge to Einstein's Cauchy developments, and this result therefore provides a mathematical validation of the $f(R)$-modified theory of gravity.

1.2 Background and methodology

1.2.1 Massless vs. massive problems

Previous works. As far as the global evolution problem in $3+1$ dimension is concerned, only spacetimes satisfying the vacuum Einstein equations have received attention in the past twenty five years. The subject was initiated in 1993 when the global nonlinear stability of Minkowski spacetime was established by Christodoulou and Klainerman in a breakthrough and very influential work [33]. The method of proof introduced therein is fully geometric in nature and, among other novel ideas, relies on a clever use of the properties of the Killing fields of Minkowski spacetime in order to define suitably weighted Sobolev norms and on a decomposition of the Einstein equations within a (null) frame associated with the Lorentzian structure of the spacetimes.

Later on, Bieri succeeded to weaken the spacelike asymptotic decay assumptions required in [33]; cf. the monograph [15] written together with Zipser. Next, Lindblad and Rodnianski [95] discovered an alternative proof, which is technically simpler but provides less control on the asymptotic behavior of solutions. (For instance, Penrose's peeling estimates were not established in [95]. This method relies on a decomposition of the Einstein equations in wave coordinates [94] and takes its roots in earlier work by Klainerman [76, 77, 79, 80] on the global existence problem for nonlinear wave equations; see also [31] for a different approach and [67, 68].

Restriction to massless fields. Importantly, both methods [33, 95] only apply to *vacuum* spacetimes or, more generally, to *massless* fields, since the scaling vector field of Minkowski spacetime is used in a very essential manner therein. In contrast, massive fields considered in the present Monograph are governed by Klein-Gordon equations which are *not invariant under scaling*. Minkowski's scaling vector field does not commute with the Klein-Gordon operator and, therefore, cannot be used in defining weighted Sobolev norms and deriving energy estimates. Without this field, the weighted Sobolev norms in [33, 95] are too lax to provide a suitable control on solutions.

1.2.2 The Euclidian-Hyperboloidal Foliation Method (EHFM)

A new method. The main challenge we overcome in this Monograph is the resolution of the global evolution problem for *massive* matter fields. We introduce a method of mathematical analysis for nonlinear wave equations, which does not require Minkowski's scaling field. This new method, referred to here as the *Euclidian-Hyperboloidal Foliation Method* (EHFM), generalizes the method we introduced in [85, 87, 88] and referred to there as the Hyperboloidal Foliation Method (HFM). The latter already allowed us to tackle a large class of nonlinear systems of coupled wave-Klein-Gordon equations.

In [85, 87, 88], we relied on a foliation of the interior of the light cone in Minkowski spacetime by spacelike hyperboloids and we derive sharp pointwise and energy estimates in order to be able to study the nonlinear coupling taking place between wave and Klein-Gordon equations. Our method had the advantage of relying on the Lorentz boosts and the translations only, rather than on the full family of (conformal) Killing fields of Minkowski spacetime.

With the Hyperboloidal Foliation Method, we solved the global nonlinear stability problem when the Einstein equations are coupled to a massive scalar field and are expressed in wave gauge. Only the restricted class of initial data sets coinciding with Schwarzschild data outside a spatially compact domain was treated in [88].

The present Monograph provides a vast extension of our previous arguments and covers a broad class of initial data sets while, simultaneously, treating a significant generalization of Einstein's theory. The approach we propose here relies on a spacetime foliation based on glueing together asymptotically Euclidian hypersurfaces and asymptotically hyperboloidal hypersurfaces. For our main tools, we refer to Part III and IV below. A presentation of the method is also provided in [90].

Hyperboloidal foliations. In the case of wave-Klein-Gordon equations in $1 + 1$ dimensions to be discussed in [98], solutions enjoy much better properties [37, 123]. The use of hyperboloidal foliations for wave equations was suggested first by Klainerman [77, 78] and, later, investigated by Hormander [62]. Earlier on, in [49, 50, 51], Friedrich also studied hyperboloidal foliations of Einstein spacetimes and succeed to establish global existence results for the Cauchy problem for the conformal vacuum field equations.

Hyperboloidal foliations can also be constructed in fully geometric manner by generalizing Christodoulou-Klainerman's method. In this direction, Wang [137] independently also obtained a very different proof of the restricted theorem [88]. We also mention that Donninger and Zenginoglu [40] recently studied cubic wave equations which exhibit a non-dispersive decay property. Furthermore, such foliations for the Einstein equations were constructed numerically by Moncrief and Rinne [100], and is a very active domain of research for numerical relativity [5, 45, 46, 59, 141].

1.2.3 Kinetic equations and other generalizations

Kinetic equations. We expect the Euclidian-Hyperboloidal Foliation Method of this Monograph to be relevant also for the study of the coupling of the Einstein equations with *kinetic equations*, such as the Vlasov equation (cf. [4, 111] for an introduction).

In this direction, recall that Fajman, Joudioux, and Smulevici [41, 43] have analyzed the global existence problem for a class of relativistic transport equations and their coupling to wave equations, and relying on [88] together with a new vector field technique [121, 41], established the stability of Minkowski spacetime for the Einstein-Vlasov system [43] for initial data sets coinciding with vacuum Schwarzschild data outside a spatially compact domain. Such a stability result for the Einstein-Vlasov system was also independently proven by Lindblad and Taylor by a completely different method [96, 132].

The application to kinetic equations, therefore, needs to be be further investigated and it would be desirable to extend our new approach EHFM to the Einstein-Vlasov equations and, more generally, to the Einstein-Boltzmann equations.

Klein-Gordon equations. Concerning the global existence problem for the Einstein-Klein-Gordon system with non-compact data, we also mention an ongoing research project by Wang [138] based on the fully geometric approach [33, 137] and, while our project came under completion, we also learned that Ionescu and Pausader [66] were also studying this problem by using a completely different methodology including the notion of resonances from [118] and [14, 55, 56, 81, 109, 110]. In this direction, our wave-Klein-Gordon model in [87] is already successfully revisited by Ionescu and Pausader [65] for a class of non-compact matter fields.

We conclude this brief review by mentioning that nonlinear wave equations of Klein-Gordon type, especially when they are posed on curved spacetimes, have been the subject of extensive research in the past two decades. A vast literature is available in this topic and we refer the interested reader to our former review in the introduction of [88], as well as [8, 9, 62, 63, 69, 70, 78, 92, 117, 119, 125] and the references cited therein.

1.3 Gravity field equations

1.3.1 The Hilbert-Einstein functional

Einstein curvature. Throughout this Monograph, we are interested in four-dimensional spacetimes $(\mathcal{M}, g)$ where $\mathcal{M}$ is a topological manifold (taken to be $\mathcal{M} \simeq [1, +\infty) \times$

$\mathbb{R}^3$) and g is a Lorentzian metric with signature $(-,+,+,+)$. The Levi-Civita connection of the metric g is denoted by $\nabla_g = \nabla$ from which we can determine the (Riemann, Ricci, scalar) curvature of the spacetime.

The standard theory of gravity is based on the Hilbert-Einstein action defined from the scalar curvature $R_g = R$ of the metric g:

$$\int_{\mathcal{M}} \Big(R_g + 16\pi\, L[\phi, g]\Big)\, dV_g, \tag{1.1}$$

where dV_g denotes the canonical volume form on $(\mathcal{M}, g)$. It is well-known that critical points of this action satisfy *Einstein's field equations*

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta} \qquad \text{in } (\mathcal{M}, g), \tag{1.2}$$

in which

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{R_g}{2}\, g_{\alpha\beta} \tag{1.3}$$

is the *Einstein curvature tensor* and $R_{\alpha\beta}$ denotes the Ricci curvature of the metric g, which later we will express explicitly in coordinates in Proposition 13.1.

Matter tensor. The Lagrangian $16\pi\, L[\phi, g]$ describes the matter content of the spacetime and allows us to determine the right-hand side of (1.3), given by the *energy-momentum tensor*

$$T_{\alpha\beta} = T_{\alpha\beta}[\phi, g] := -2\, \frac{\delta L}{\delta g^{\alpha\beta}}[\phi, g] + g_{\alpha\beta}\, L[\phi, g]. \tag{1.4}$$

Throughout, Greek indices describe $0, 1, 2, 3$ and we use the standard convention of implicit summation over repeated indices, as well as raising and lowering indices with respect to the metric $g_{\alpha\beta}$ and its inverse denoted by $g^{\alpha\beta}$. For instance, we write $X_\alpha = g_{\alpha\beta} X^\beta$ for the duality between vectors and 1-forms.

1.3.2 The generalized gravity functional

A modified gravity action. The so-called $f(R)$-modified theory of gravity is a generalization of Einstein's theory and is based on the following modified gravity action

$$\int_{\mathcal{M}} \Big(f(R_g) + 16\pi\, L[\phi, g]\Big)\, dV_g, \tag{1.5}$$

in which $f : \mathbb{R} \to \mathbb{R}$ is a given (sufficiently smooth) function. In order for the modified theory to be a formal extension of Einstein's theory, we assume that $f(r) \simeq r$ in the zero curvature limit $r \to 0$. More precisely, we assume that

$$f(R) \simeq R + \frac{\kappa}{2} R^2 \qquad \text{as } R \to 0 \tag{1.6}$$

for some positive coefficient[1] $\kappa > 0$. For some historical background about this theory, we refer to Buchdahl [21] and Brans and Dicke [19], as well as [25, 26, 105, 114] and the references therein.

Critical points of (1.5) satisfy the following $f(R)$-*field equations*

$$N_{\alpha\beta} = 8\pi \, T_{\alpha\beta}[\phi, g] \qquad \text{in } (\mathcal{M}, g), \tag{1.7}$$

in which

$$N_{\alpha\beta} := f'(R_g) \, G_{\alpha\beta} - \frac{1}{2}\Big(f(R_g) - R_g f'(R_g)\Big) g_{\alpha\beta} + \Big(g_{\alpha\beta} \, \Box_g - \nabla_\alpha \nabla_\beta\Big) \big(f'(R_g)\big) \tag{1.8}$$

is referred to as the *modified gravity tensor*. Here, $\Box_g := \nabla_\alpha \nabla^\alpha$ is the wave operator associated with the metric g.

The limit problem $f(R) \to R$. We observe the following:

- Within the above theory, Einstein's equations are obviously recovered by choosing the function $f(R)$ to be the linear function R.
- In the limit $\kappa \to 0$ in (1.6), one has $f(R) \to R$ and the field equations (1.7) formally converge to the Einstein equations (1.2).

Clearly, taking the limit $f(R) \to R$ is *singular* in nature, since (1.2) involves up to *second-order* derivatives of the unknown metric g, while (1.7) contains *fourth-order* terms. One of our objectives will be to rigorously prove that solutions of (1.7) converge (in a sense to be specified) to solutions of (1.2).

[1] The positive sign of $\kappa := f''(0)$ is essential for the global-in-time stability of solutions, but is irrelevant for their local-in-time existence.

1.3.3 Evolution of the spacetime scalar curvature

Next, taking the trace of the field equations (1.7), we obtain

$$3\,\Box_g f'(R_g) + \big(-2f(R_g) + R_g f'(R_g)\big) = 8\pi\, g^{\alpha\beta} T_{\alpha\beta} \qquad \text{in } (\mathcal{M}, g), \tag{1.9}$$

which is a *second-order evolution equation* for the spacetime curvature R. It will play a central role in our analysis. Interestingly enough, from (1.6), we have

$$\big(-2f(R) + f'(R)R\big) \simeq -R \quad (\text{when } R \to 0),$$

so that the linear part of the differential operator in (1.9) reads (with $\kappa > 0$)

$$\Box R_g - \frac{1}{3\,\kappa} R_g. \tag{1.10}$$

Hence, the scalar curvature function $R_g : \mathcal{M} \to \mathbb{R}$ satisfies a nonlinear Klein-Gordon equation. In view of (1.10), we can refer to $1/\kappa$ as the *mass parameter*.

The study of the convergence problem "modified gravity toward Einstein gravity", or *infinite mass problem*, will require to understand the singular limit $\kappa \to 0$ in solutions to the differential operator (1.10).

1.3.4 Dynamics of massive matter

Massive scalar field. From the twice-contracted Bianchi identities $\nabla^\alpha R_{\alpha\beta} = \frac{1}{2}\nabla_\beta R$, the Einstein tensor as well as the modified gravity tensor are easily checked to be divergence free, that is,

$$\nabla^\alpha G_{\alpha\beta} = \nabla^\alpha N_{\alpha\beta} = 0 \tag{1.11}$$

and, consequently, the following *matter evolution equations* hold

$$\nabla^\alpha T_{\alpha\beta} = 0 \qquad \text{in } (\mathcal{M}, g). \tag{1.12}$$

We are interested here in *massive scalar fields* $\phi : \mathcal{M} \to \mathbb{R}$ corresponding to the following energy-momentum tensor:

$$T_{\alpha\beta} := \nabla_\alpha\phi\nabla_\beta\phi - \Big(\frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi + U(\phi)\Big) g_{\alpha\beta}, \tag{1.13}$$

in which the potential $U = U(\phi)$ is a prescribed function depending on the nature of the matter field under consideration.

Nonlinear Klein-Gordon equation. By combining (1.12) and (1.13), we see that the field ϕ satisfies a *nonlinear Klein-Gordon equation* associated with the unknown curved metric g:

$$\Box_g \phi - U'(\phi) = 0 \qquad \text{in } (\mathcal{M}, g), \tag{1.14}$$

which is expected to uniquely determine the evolution of the matter (after prescribing suitable initial data). Throughout, we assume that

$$U(\phi) = \frac{c^2}{2}\phi^2 + \mathcal{O}(\phi^3). \tag{1.15}$$

for some constant $c > 0$, referred to as the *mass* of the scalar field.

For instance, with the choice $U(\phi) = \frac{c^2}{2}\phi^2$, (1.14) is nothing but the Klein-Gordon equation $\Box_g \phi - c^2\phi = 0$, which would be *linear* for a known metric g. However, our challenge is precisely to understand the *nonlinear coupling problem* when the metric g itself is one of the unknowns and is given by solving (1.3) or (1.7) (with suitably prescribed initial data).

2 The initial value formulation

2.1 The Cauchy problem for the Einstein equations

2.1.1 Einstein's constraint equations

We begin with a discussion of the initial value problem for Einstein's field equations coupled with a massive scalar field and, next, present our generalization to the $f(R)$-field equations.

The formulation of the initial value problem for the Einstein equations is based on the prescription of an initial data set, denoted here by $\big(\mathcal{M}_0, g_0, k_0, \phi_0, \phi_1\big)$, which provides us with the intrinsic and extrinsic geometry of the initial hypersurface (i.e. the induced metric and the second fundamental form of the hypersurface), and with the matter variables on this hypersurface, denoted here by (ϕ_0, ϕ_1), which provide us with the initial value of the scalar field and its Lie derivative in the timelike direction (normal to the hypersurface).

Importantly, such an initial data cannot be chosen arbitrarily, but must satisfy certain constraint conditions of Gauss-Codazzi-type. We assume that the spacetime under consideration is foliated (at least locally, for the sake of this argument) by a time function $t : \mathcal{M} \to [1, +\infty)$, so that a local chart can be expressed as (t, x^a) with $a = 1, 2, 3$. Obviously, a solution to (1.2) may exist only if, in these coordinates, at least the equations[1]

$$G_{00} = 8\pi T_{00}, \qquad G_{0a} = 8\pi T_{0a}, \tag{2.1}$$

are satisfied. However, these four equations turn out to involve only the induced metric and second fundamental form to the initial hypersurface $\{t = 1\}$ as well as matter data. This motivates the introduction of the following definition.

Observe that the following definition does not require any topological or asymptotic assumptions on the data.

Definition 2.1. *An* initial data set *for the Einstein-massive field system* (1.2) *consists of data* $(\mathcal{M}_0, g_0, k_0, \phi_0, \phi_1)$ *satisfying the following conditions:*

- $\mathcal{M}_0$ *is a* $3-$*dimensional manifold, endowed with a Riemannian metric* g_0 *and a symmetric* $(0, 2)$*-tensor field* k_0*.*
- ϕ_0 *and* ϕ_1 *are scalar fields defined on* $\mathcal{M}_0$*.*
- *The so-called* Hamiltonian constraint *holds:*

$$R_0 - k_{0,ab} k_0^{ab} + (k_{0,a}^a)^2 = 8\pi \Big(\nabla_{0,a}\phi_0 \nabla_0^a \phi_1 + (\phi_1)^2 + 4\, U(\phi_0) \Big), \tag{2.2}$$

  in which $R_0 = R_{g_0}$ *denotes the scalar curvature of the Riemannian metric* g_0 *and* ∇_0 *denotes its Levi-Civita connection.*
- *The so-called* momentum constraints *hold:*

$$\nabla_{0,b} k_{0,a}^a - \nabla_{0,a} k_{0,b}^a = 8\pi\, \phi_1 \nabla_{0,b}\phi_0. \tag{2.3}$$

Of course, the indices in (2.2) and (2.3) are raised or lowered with the metric g_0, so that for instance $k_{0,ab} k_0^{ab} = g_0^{aa'} g_0^{bb'} k_{0,ab} k_{0,a'b''}$ and $\nabla_{0,a}\phi_0 \nabla_0^a \phi_1 = {g_0}^{ab} \nabla_{0,a}\phi_0 \nabla_{0,b}\phi_1$.

[1]The remaining equations $G_{ab} = 8\pi T_{ab}$ provide us with genuine evolution equations.

2.1.2 Einstein's Cauchy developments

A geometric notion of solution. Relying on Definition 2.1, we can now formulate the Cauchy problem, as follows. (In the following, we focus on the future evolution of the data so that $\mathcal{M}_0$ is taken to be the past boundary of the spacetime.)

Definition 2.2. *Given an initial data set* $(\mathcal{M}_0, g_0, k_0, \phi_0, \phi_1)$*, the* initial value problem *for the Einstein equations consists of finding a Lorentzian manifold* $(\mathcal{M}, g)$ *(endowed with a time-orientation) and a scalar field* $\phi : \mathcal{M} \to \mathbb{R}$ *such that:*

- $(\mathcal{M}, g)$ *satisfies Einstein's field equations* (1.2)-(1.3)-(1.13).
- *There exists an embedding* $i : (\mathcal{M}_0, g_0) \to (\mathcal{M}, g)$ *with pull-back metric* $i^*g = g_0$ *and second fundamental form* k_0*, so one can write* $\mathcal{M}_0 \subset \mathcal{M}$*.*
- *The field* ϕ_0 *coincides with the restriction* $\phi|_{\mathcal{M}_0}$*, while* ϕ_1 *coincides with the Lie derivative* $\mathcal{L}_n \phi|_{\mathcal{M}_0}$*, where* n *denotes the unit future-oriented normal along* $\mathcal{M}_0$*.*

A solution satisfying the conditions in Definition 2.8 is referred to as *Einstein's Cauchy development* of the initial data set $(\mathcal{M}_0, g_0, k_0, \phi_0, \phi_1)$. Moreover, by definition, a *globally hyperbolic development* is a solution such that every inextendible causal curve (whose tangent vector is either timelike or null) meets the initial hypersurface exactly once. Furthermore, by analogy with the theory of ordinary differential equations, one actually seek for the *maximal development* associated with a given initial data set. These notions were introduced in Choquet-Bruhat together with Geroch [29]; see [44] and the textbook [30] for further details.

The main challenge of the gravitation theory. In general relativity, the main mathematical chalenge is to determine, at least for certain classes of initial data sets, the global geometry of the Cauchy developments. It is known that solutions can become "singular" and the evolution may lead to the formation of trapped surfaces and eventually to the formation of black holes. (Cf. the textbook by Hawking and Ellis [58].)

It is natural to restrict this problem to initial data sets that are perturbations of particular solutions and, of course, the Minkowski geometry stands out as the *ground state* of the theory of gravity. By considering perturbations of Minkowski spacetime, possibly in presence of matter as we do in this Monograph, the challenge is to establish that the Cauchy developments in Definition 2.2 are *geodesically complete* in future timelike directions. This means that a timelike geodesic can be extended for arbitrary large values of its affine parameter. From a physical standpoint, in such a spacetime, an observer (represented by a timelike curve) can live forever and its proper time may approach infinity.

2.2 The formulation of the $f(R)$-gravity theory

2.2.1 Evolution of the scalar curvature

A new unknown metric. Next, we turn our attention to the $f(R)$-field equations (1.7) which, as we already pointed out, are *fourth-order* in the metric g. Remarkably, these equations can be conformally transformed to a *third-order* form, as follows.

Definition 2.3. *The (unphysical)* conformal metric $g^\dagger$ *associated with a solution* $(\mathcal{M}, g)$ *to the* $f(R)$*-field equations* (1.7) *is defined from the (physical) metric* g *by setting*

$$g^\dagger{}_{\alpha\beta} = e^{\kappa\rho} g_{\alpha\beta}, \qquad \rho = \frac{1}{\kappa} \ln(f'(R_g)), \tag{2.4}$$

in which the conformal factor $\rho : (\mathcal{M}, g) \to \mathbb{R}$ *is referred to as the* scalar curvature field.

In view of (1.6), we have $f'(R) \simeq 1 + \kappa R$, so that the field ρ approaches the scalar curvature R in the limit $\kappa \to 0$. Denoting by $\nabla^\dagger$ and $\Box_{g^\dagger}$ the Levi-Civita connection and wave operator associated with the conformal metric $g^\dagger$, respectively, the trace equation (1.9) yields us

$$3\,\Box_{g^\dagger} f'(R_g) = 8\pi e^{-\kappa\rho} g^{\alpha\beta} T_{\alpha\beta} + e^{-\kappa\rho}\Big(2f(R_g) - f'(R_g)R_g\Big) - 3\kappa^2 g^{\alpha\beta}\nabla_\alpha\rho\nabla_\beta\rho.$$

It is straightforward to check the identity

$$\Box_{g^\dagger} f'(R_g) = \Box_{g^\dagger} e^{\kappa\rho} = e^{\kappa\rho}\Big(\kappa\,\Box_{g^\dagger}\rho - \kappa^2\, g^{\dagger\alpha\beta}\nabla^\dagger{}_\alpha\rho\nabla^\dagger{}_\beta\rho\Big),$$

and we therefore obtain

$$3\kappa\,\Box_{g^\dagger}\rho = 8\pi e^{-2\kappa\rho} g^{\dagger\alpha\beta} T_{\alpha\beta} + e^{-2\kappa\rho}\Big(2f(R_g) - f'(R_g)R_g\Big). \tag{2.5}$$

A Klein-Gordon equation. To better extract the structure of the equation (2.5), it is then convenient to introduce the following notation:

- We set

$$
\begin{aligned}
& f_0(s) := \frac{f(R)}{f'(R)^2}, \qquad f_1(s) := \frac{f(R) - f'(R)R}{f'(R)^2}, \\
& e^{2\kappa s} := f'(R), \qquad R \in \mathbb{R}.
\end{aligned}
\tag{2.6}
$$

- The Legendre transform of the function f, by definition, reads

$$
\begin{aligned}
f^*(y) :=& \sup_{R \in \mathbb{R}} y\, R - f(R) \\
=& f'(R(y))R(y) - f(R(y)), \qquad f'(R(y)) = y = e^{\kappa s}.
\end{aligned}
\tag{2.7}
$$

Hence, we find

$$
f_1(s) = -e^{-2\kappa s} f^*(e^{\kappa s}), \qquad s \in \mathbb{R}
$$

and thus the last term in (2.5) reads

$$
e^{-4\kappa\rho}\Big(2f(R_g) - f'(R_g)R_g\Big) = f_0(\rho) + f_1(\rho) =: \rho + f_2(\rho). \tag{2.8}
$$

We can thus rewrite (2.5) as follows.

Proposition 2.4. *With the notation above, the $f(R)$-field equations* (1.7) *imply the following* Klein-Gordon equation for the curvature field

$$
3\kappa\, \Box_{g^\dagger}\rho - \rho = f_2(\rho) + 8\pi e^{-\kappa\rho} g^{\dagger\alpha\beta} T_{\alpha\beta}, \tag{2.9}
$$

in which $f_2(\rho) = \kappa \mathcal{O}(\rho^2)$ is of quadratic order at least, as follows from (1.6)*.*

2.2.2 Field equations for the conformal metric

Modified gravity tensor. Next, rewriting (1.8) as

$$
\begin{aligned}
N_{\alpha\beta} =& e^{\kappa\rho} R_{\alpha\beta} - \frac{1}{2} f(R) g_{\alpha\beta} + \kappa e^{\kappa\rho}\left(g_{\alpha\beta}\, \Box_g - \nabla_\alpha \nabla_\beta\right)\rho \\
& + \kappa^2 e^{\kappa\rho}\left(g_{\alpha\beta}\, g(\nabla\rho, \nabla\rho) - \nabla_\alpha\rho \nabla_\beta\rho\right)
\end{aligned}
$$

and, recalling the conformal transformation formula for the Ricci curvature

$$R_{\alpha\beta} = R^{\dagger}_{\alpha\beta} + \nabla_\alpha \nabla_\beta \rho - \frac{\kappa^2}{2} \nabla_\alpha \rho \nabla_\beta \rho + \frac{\kappa}{2} \Big(\Box_g \rho + \kappa g(\nabla \rho, \nabla \rho) \Big) g_{\alpha\beta},$$

we obtain

$$\begin{aligned} N_{\alpha\beta} = & e^{\kappa\rho} R^{\dagger}_{\alpha\beta} - \frac{3\kappa^2}{2} e^{\kappa\rho} \nabla_\alpha \rho \nabla_\beta \rho + \frac{3\kappa}{2} e^{\kappa\rho} g_{\alpha\beta} \Box_g \rho \\ & + \frac{3\kappa^2}{2} e^{2\kappa\rho} g_{\alpha\beta} g(\nabla \rho, \nabla \rho) - \frac{1}{2} f(R) g_{\alpha\beta}. \end{aligned} \tag{2.10}$$

The trace of this tensor reads

$$\begin{aligned} \text{tr } (N) = & f'(R) R - 2 f(R) + 3 \Box_g f'(R) \\ = & f'(R) R - 2 f(R) + 3\kappa e^{2\kappa\rho} \Box_{g^\dagger} \rho + 3\kappa^2 e^{\kappa\rho} g(\nabla \rho, \nabla \rho) \end{aligned}$$

and, therefore, from (2.10) we deduce that

$$3\kappa e^{2\kappa\rho} \Box_{g^\dagger} \rho = \frac{1}{2} \text{tr } (N_g) + f(R) - \frac{R}{2} f'(R) - 6\kappa^2 e^{2\kappa\rho} g(\nabla \rho, \nabla \rho).$$

Hence, the identity (2.10) is equivalent to saying

$$N_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \text{tr } (N) = e^{2\kappa\rho} R^{\dagger}_{\alpha\beta} + \frac{1}{2} \Big(f(R) - R f'(R) \Big) g_{\alpha\beta} - 6\kappa^2 e^{2\kappa\rho} \nabla_\alpha \rho \nabla_\beta \rho. \tag{2.11}$$

In view of this relation, we have reached the following.

Proposition 2.5. *With the notation above, the $f(R)$-field equations* (1.7) *take the equivalent form*

$$e^{2\kappa\rho} R^{\dagger}_{\alpha\beta} - 6\kappa^2 \, e^{2\kappa\rho} \nabla^{\dagger}{}_\alpha \rho \nabla^{\dagger}{}_\beta \rho - \frac{1}{2} f_1(\rho) \, g^{\dagger}{}_{\alpha\beta} = 8\pi \Big(T_{\alpha\beta} - \frac{1}{2} g^{\dagger}{}_{\alpha\beta} g^{\dagger \alpha'\beta'} T_{\alpha'\beta'} \Big), \tag{2.12}$$

which will be referred to as the conformal field equations *of the $f(R)$-gravity theory and in which $f_1(\rho) = \kappa \mathcal{O}(\rho^2)$ is of quadratic order.*

A third-order modification of Einstein equations. At this stage, the following observations are in order:

- Our reduced equations involve up to three derivatives of the new metric, only.

- The expression (2.12) contains
  - Ricci curvature terms,
  - only up to first-order derivatives of the scalar curvature,
  - but no other third-order derivatives,
  - nor fourth-order derivative of the new metric.
- Also, it is easy to compute the limit $\kappa \to 0$ in (2.12) and, since $f(r) \to r$ and $f_1(r) \to 0$, we obtain $g \to g^\dagger$,

$$\begin{aligned} \rho &\to R, \\ R^\dagger_{\alpha\beta} - 6\kappa^2\, \nabla^\dagger{}_\alpha\rho\nabla^\dagger{}_\beta\rho + \frac{1}{2}e^{-4\kappa\rho} f^*(e^{2\kappa\rho})\, g^\dagger{}_{\alpha\beta} &\to R_{\alpha\beta} \end{aligned} \tag{2.13}$$

  and in this limit (2.12) are nothing but Einstein's equations

$$R^\dagger_{\alpha\beta} = 8\pi T_{\alpha\beta} - 4\pi g_{\alpha\beta}\big(g^{\alpha'\beta'}T_{\alpha'\beta'}\big). \tag{2.14}$$

2.2.3 The conformal field formulation of $f(R)$-gravity

Coupling to matter fields. Although the vacuum $f(R)$-equations are interesting in their own sake, in our analysis we also allow for the coupling with a massive scalar field, denoted again by ϕ and described by the same energy-momentum tensor (1.13). In the physics literature, the $f(R)$-equations (1.7)-(1.13) are referred to as the Jordan coupling.

On the other hand, we point out that the alternative formulation of the matter model in which the physical metric is replaced by the conformal metric leads to an *ill-posed evolution problem*; see [89] for more details. We also point out that it would not be difficult to generalize the nonlinear stability theory established in this Monograph and encompass the coupling with a *massless* scalar field.

A summary. In summary, in terms of the conformal metric $g^\dagger$ defined in (2.4), the field equations of modified gravity (1.7) contain only up to *third-order* derivatives of the metric $g^\dagger$ and read

$$\begin{aligned} & R^\dagger_{\alpha\beta} - 6\kappa^2\, \partial_\alpha\rho\partial_\beta\rho + \frac{1}{2}e^{-4\kappa\rho} f^*(e^{2\kappa\rho})\, g^\dagger{}_{\alpha\beta} \\ & = 8\pi\left(e^{-2\kappa\rho}\nabla^\dagger{}_\alpha\phi\nabla^\dagger{}_\beta\phi + e^{-4\kappa\rho}U(\phi)\, g^\dagger_{\alpha\beta}\right), \end{aligned} \tag{2.15}$$

while the scalar curvature field $\rho : \mathcal{M} \to \mathbb{R}$ satisfies

$$3\kappa \,\Box_{g^\dagger} \rho - \rho = f_2(\rho) - 8\pi \Big({g^\dagger}^{\alpha\beta} {\nabla^\dagger}_\alpha \phi {\nabla^\dagger}_\beta \phi + 4 e^{-2\kappa\rho} U(\phi) \Big), \tag{2.16}$$

and the matter field $\phi : \mathcal{M} \to \mathbb{R}$ satisfies

$$\widetilde{\Box}_{g^\dagger} \phi - c^2 \phi = c^2 \big(e^{-\kappa\rho} - 1 \big) \phi + \kappa {g^\dagger}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \rho. \tag{2.17}$$

The following definition (cf. [89]) is motivated by the fact that the compatibility condition $\rho = \frac{1}{2\kappa} \ln(f'(R_g))$ does hold throughout the spacetime, *provided* it holds (together with a timelike Lie derivative of this condition) on a Cauchy hypersurface. This property is stated in Proposition 2.9, below, after introducing a suitable notion of Cauchy development.

Definition 2.6. *The* **conformal field formulation of $f(R)$-gravity** *consists of*

- *the equations* (2.15)–(2.17)*, which provide one with a second-order system in* $(g^\dagger, \rho, \phi)$*,*
- *together with the defining condition relating ρ to the spacetime scalar curvature R_g of the physical metric $g_{\alpha\beta} = e^{-2\kappa\rho} {g^\dagger}_{\alpha\beta}$:*

$$\rho = \frac{1}{2\kappa} \ln(f'(R_g)). \tag{2.18}$$

2.3 The Cauchy problem for the $f(R)$-field equations

2.3.1 The $f(R)$-constraint equations

Since ρ satisfies a second-order evolution equation, we may anticipate that, similarly to what we did for the matter field ϕ, two initial data for ρ are required for formulating the initial value problem. Indeed, the formulation of the initial value problem associated with the $f(R)$-equations requires an initial data set $\big(\mathcal{M}_0, g_0, k_0, R_0, R_1, \phi_0, \phi_1\big)$ which, in addition to the intrinsic and extrinsic geometry and matter content, also provides us with the *spacetime scalar curvature*, denoted by R_0, and its Lie derivative in time R_1. Observe again that the following definition is very general and does not require any topological or asymptotic assumptions.

Definition 2.7. *An* **initial data set for modified gravity** *consists of data*

$$(\mathcal{M}_0, g_0, k_0, R_0, R_1, \rho_0, \rho_1)$$

satisfying the following conditions:

- $\mathcal{M}_0$ *is a* $3-$*dimensional manifold, endowed with a Riemannian metric* g_0 *and a symmetric 2-covariant tensor field* k_0*.*

- R_0 *and* R_1 *denote two scalar fields defined on* $\mathcal{M}_0$*, which represent the scalar curvature field and a timelike Lie derivative of this field.*

- ϕ_0 *and* ϕ_1 *denote two scalar fields defined on* $\mathcal{M}_0$*, which represent the matter field and a timelike Lie derivative of this field.*

- *In terms of the conformal metric* ${g^\dagger}_0$ *and the conformal second fundamental form* ${k^\dagger}_0$ *defined (by analogy with* (2.4)*) by*

$$\begin{aligned} {g^\dagger}_{0,\alpha\beta} &:= f'(R_0) g_{0,\alpha\beta}, \\ {k^\dagger}_{0,\alpha\beta} &:= f'(R_0) k_{0,\alpha\beta} + f''(R_0) R_1 g_{0,\alpha\beta}, \end{aligned} \tag{2.19}$$

  the Hamiltonian constraint *of modified gravity holds:*

$$\begin{aligned} &{R^\dagger}_0 - {g^\dagger}_0^{aa'} {g^\dagger}_0^{bb'} {k^\dagger}_{0,ab} {k^\dagger}_{0,a'b'} + (k_{0,b}^b)^2 \\ &= 8\pi\, e^{-\kappa\rho_0} \Big({g_0^\dagger}^{ab} {\nabla^\dagger}_{0,a} \phi_0 {\nabla^\dagger}_{0,b} \phi_0 + (\phi_1)^2 + 4 e^{-\kappa\rho_0} U(\phi_0) \Big) \\ &\quad + 6\kappa^2 (\rho_1)^2 + 6\kappa^2 {g_0^\dagger}^{ab} {\nabla^\dagger}_{0,a} \rho_0 {\nabla^\dagger}_{0,b} \rho_0 - e^{-2\kappa\rho_0} f^*(e^{2\kappa\rho_0}), \end{aligned} \tag{2.20}$$

  in which ${R^\dagger}_0$ *denotes the scalar curvature of the Riemannian metric* $g_0^\dagger$*, and* ${\nabla^\dagger}_0$ *is the corresponding Levi-Civita connection.*

- *The* momentum constraints *of modified gravity hold:*

$${\nabla^\dagger}_{0b} {k^\dagger}_{0a}^{a} - {\nabla^\dagger}_{0a} {k^\dagger}_{0b}^{a} = 8\pi\, e^{-\kappa\rho_0}\, \phi_1 {\nabla^\dagger}_{0,b} \phi_0 + 6\kappa^2 \rho_1 {\nabla^\dagger}_{0,b} \rho_0. \tag{2.21}$$

2.3.2 The notion of $f(R)$-Cauchy developments

We are now able to generalize Definition 2.2 (which concerned the Einstein equations), as follows.

Definition 2.8. *Given an initial data set* $(\mathcal{M}_0, g_0, k_0, R_0, R_1, \phi_0, \phi_1)$ *as in Definition 2.7, the* initial value problem for modified gravity *consists of finding a Lorentzian manifold* $(\mathcal{M}, g)$ *(endowed with a time-orientation) and scalar fields* ρ, ϕ *defined on* $\mathcal{M}$ *such that:*

- *The field equations of modified gravity* (1.7)-(1.8)-(1.13) *are satisfied.*
- *There exists an embedding* $i : (\mathcal{M}_0, g_0) \to (\mathcal{M}, g)$ *with pull-back metric* $i^*g = g_0$ *and second fundamental form* k_0*, hence one can write* $\mathcal{M}_0 \subset \mathcal{M}$.
- *The field* ρ_0 *coincides with the restriction* $\rho|_{\mathcal{M}_0}$*, while* ρ_1 *coincides with the Lie derivative* $\mathcal{L}_n \rho|_{\mathcal{M}_0}$*, where* n *denotes the unit future-oriented normal to* $\mathcal{M}_0$.
- *The scalar fields* ϕ_0 *and* ϕ_1 *coincide with the restriction of* ϕ *and its Lie derivative* $\mathcal{L}_n \phi$ *on* $\mathcal{M}_0$*, respectively.*

A solution satisfying the conditions in Definition 2.8 is referred to as a *modified gravity Cauchy development* of the initial data set $(\mathcal{M}_0, g_0, k_0, R_0, R_1, \phi_0, \phi_1)$. The notion of *maximally hyperbolic development* then follows by straightforwardly extending the definition in Choquet-Bruhat [30]. In comparison with the classical gravity theory, the modified gravity theory has *two extra degrees of freedom* specified from the two additional data (R_0, R_1). Similarly as in classical gravity, these fields cannot be arbitrarily prescribed and suitable constraint equations must be assumed, as stated by (2.20)-(2.21).

The following observations are in order:

- From Definition 2.8, in the special case of vanishing geometric data $R_0 = R_1 = \phi_0 = \phi_1 \equiv 0$ we recover the classical formulation in Definition 2.8.
- On the other hand, by taking a vanishing matter field $\phi \equiv 0$ but non-vanishing geometric data (R_0, R_1), the spacetimes in Definition 2.8 generally do not satisfy Einstein's vacuum equations.

2.3.3 Equivalent formulation in the conformal metric

All of the above notions could be equivalently stated in terms of the conformal metric $g^\dagger$, while regarding the field ρ as an independent unknown. This standpoint is justified by the following property of the $f(R)$-field equations [89].

Proposition 2.9. *If* $(\mathcal{M}, g^\dagger, \rho, \phi)$ *is a Cauchy development to the augmented conformal system of modified gravity* (2.15)-(2.16)-(2.17)*, then by introducing the physical metric*

$$g_{\alpha\beta} = e^{-2\kappa\rho} g^\dagger_{\alpha\beta},$$

the condition

$$\rho = \frac{1}{2\kappa} \ln(f'(R_g)) \tag{2.22}$$

holds throughout the spacetime $\mathcal{M}$*, provided it holds on the initial Cauchy hypersurface* $\mathcal{M}_0$ *and the same identity also holds for the normal time Lie derivative on* $\mathcal{M}_0$*.*

While the conformal formulation will play an essential role in the forthcoming proofs, as far as the statements of our main results are concerned it is more natural to give statements in the physical metric g rather than in the unphysical metric $g^\dagger$., as we have done in this section. However, let us also observe that the scalar curvature data R_0, R_1 can be equivalently replaced by initial data ρ_0, ρ_1 with the correspondence:

$$\rho_0 = \frac{1}{\kappa} \ln(f'(R_0)), \qquad \rho_1 = \frac{1}{\kappa} \frac{f''(R_0)}{f'(R_0)} R_1. \tag{2.23}$$

Part II

Nonlinear stability statements

3 The global stability of Minkowski space

3.1 Initial data sets of interest

3.1.1 Minkowski and Schwarzschild spacetimes

Minkowski metric. We are interested in solving the initial value problem when the initial data sets represent perturbations of a flat hypersurface in Minkowski spacetime. The initial data sets are assumed to have the topology of $\mathbb{R}^3$ and be covered by a single coordinate chart denoted by $x = (x^a) = (x_a) \in \mathbb{R}^3$ with $a = 1, 2, 3$. Our spacetimes will also be endowed with a foliation determined by a global time function denoted by $t : M \mapsto [1, +\infty)$ in which $\big\{t = 1\big\}$ represents the initial hypersurface. It is convenient also to define the radial vector (ω_a) and the radius $r > 0$ by

$$\omega_a := \frac{x_a}{r}, \qquad r^2 := \sum_{a=1}^{3} (x^a)^2. \tag{3.1}$$

Recall first that, in standard Cartesian coordinates (which are also wave coordinates, in the sense (4.2) below), the Minkowski metric g_M reads

$$\begin{aligned} g_M = (g_{M,\alpha\beta}) &= -dt^2 + \sum_{a=1}^{3} (dx^a)^2 \\ &= -dt^2 + (g_{M,ab}) = -dt^2 + (\delta_{ab}). \end{aligned} \tag{3.2}$$

We also use the notation

$$g_{M,0} = (\delta_{ab}) \tag{3.3}$$

to denote the restriction of the Minkowski metric on $t = 1$.

Schwarzschild metric. By the positive mass theorem [115, 139], a metric coinciding with the flat metric outside a compact region necessarily coincides with the Minkowski metric. Hence, only mild decay in spacelike directions is compatible with the Einstein equations.

Given any parameter $m_S > 0$, consider next the so-called Schwarzschild metric defined for $r > m_S$ by its components

$$
\begin{aligned}
g_{S,00} &= -\frac{r - m_S}{r + m_S}, \\
g_{S,0a} &= 0, \\
g_{S,ab} &= \frac{r + m_S}{r - m_S}\omega_a\omega_b + \frac{(r + m_S)^2}{r^2}(\delta_{ab} - \omega_a\omega_b),
\end{aligned}
\tag{3.4}
$$

in which we use the notation (3.1). This metric represents the geometry of a spherically symmetric and static black hole with mass $m_S > 0$ and, more precisely, the domain of outer communication of this solution to the vacuum Einstein equations. As observed in [7], the formulas (3.4) provide the metric in wave coordinates (cf. (4.2) below).

Furthermore, the restriction of the Schwarzschild metric to a hypersurface of constant time $t =$ and the corresponding second fundamental form $k_{S,ab}$ of this hypersurface satisfy

$$
\begin{aligned}
g_{S,ab} &\simeq \Big(1 + \frac{2m_S}{r}\Big)\delta_{ab} + (m_S)^2\, \mathcal{O}(r^{-2}), \qquad r \to +\infty, \\
k_{S,ab} &= 0,
\end{aligned}
\tag{3.5}
$$

since the coefficients in (3.4) is independent of t.

For our purpose, the Schwarzschild metric only tells us about the asymptotic behavior at spacelike infinity which is compatible with the constraint equations and, therefore, physically admissible. We conclude that (3.5) is the typical asymptotic behavior that our global stability theory should cover. In fact, our framework will be more general and will include (3.5) as a special case.

3.1.2 The class of initial data sets of interest

Asymptotically tame data. We begin by presenting our sup-norm conditions which play a special role in the present work and have no analogue within the standard global existence theory for nonlinear wave equations. These conditions concern the geometry of the initial hypersurface but not the matter and curvature components of the initial data sets. In addition the conditions below, "small energy" conditions will be also required on the initial data sets (cf. (3.11) and (3.12), below).

Definition 3.1. *Let $\epsilon > 0$ and $\sigma \in [0, 1/2)$ be given parameters and N be an integer. An initial data set*

$$
\begin{aligned}
&(\mathcal{M}_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1) && \textit{(as in Definition 2.2)} \\
\textit{or } &(\mathcal{M}_0 \simeq \mathbb{R}^3, g_0, k_0, R_0, R_1, \phi_0, \phi_1) && \textit{(as in Definition 2.8)}
\end{aligned}
$$

*is said to be (σ, ϵ, N)–***asymptotically tame** *if in a coordinate chart $(x^a) = (r, \omega^A)$ (with $r = |x|$ and $A = 1, 2$) the components of the metric $h_0 = g_0 - \delta$ and the 2-tensor field k*

$$
h_0^{rr} := \frac{1}{r^2} h_{0,ab} x^a x^b, \qquad k_{0,A}^A := \frac{1}{r^2} k_{0,ab} \big(\delta^{ab} - x^a x^b\big), \tag{3.6}
$$

satisfy the decay conditions for all $r = |x| \lesssim R$:

$$
\begin{aligned}
|\partial^I h_0^{rr}(x)| &\leq \epsilon\, (1 + r)^{-|I| - 1 + \sigma}, \qquad |I| \leq N + 1, \\
|\partial^I k_{0,A}^A(x)| &\leq \epsilon\, (1 + r)^{-|I| - 2 + \sigma}, \qquad |I| \leq N,
\end{aligned} \tag{3.7}
$$

$$
\frac{1}{1 + R} \Big| \int_{S_R(x)} \Big(h_0^{rr}(y), (y - x) \nabla h_0^{rr}(y), R\, k_{0,A}^A(y) \Big) d\sigma(y) \Big| \leq \epsilon, \tag{3.8}
$$

over the sphere $S_R(x) := \big\{ y \in \mathbb{R}^3 \, / \, |x - y| = R \big\}$.

Examples of initial data sets. Other formulations of our asymptotic decay conditions will be provided and discussed below. At this stage, let us mention that it is straightforward to check that the Schwarzschild metric does satisfy both conditions (3.7) and (3.8) with $\sigma = 0$ while ϵ can be taken to be a multiple of the mass m_S (arising in (3.5)). Our theory for the stability of Minkowski spacetime for instance applies to this important class of metrics with Schwarzschild-like behavior. Our decay conditions in (3.7) allow for a decay at a slower rate in comparison to the Schwarzschild metric. For instance let us recall that the condition $\sigma < 1/2$ suffices in order for the ADM mass to be well-defined. Observe also that the decay (3.7) is guaranteed in [28] for a broad class of data sets with σ possibly arbitrarily close to 1.

3.2 Three main results of nonlinear stability

3.2.1 The Einstein theory

The global existence statement. The first mathematical result concerning the local-in-time existence for the Einstein equations (possibly coupled to matter fields) was established by Choquet-Bruhat in 1952 and in subsequent works; see the textbook [30] and the references therein. Solutions are defined according to Definition 2.2. Several theorems that are global-in-nature are established in this work.

We begin with our existence statements. The initial data are assumed to have small "energy" defined as a *weighted Sobolev-type norm* which involves the Killing fields of Minkowski spacetime. As far as the initial data sets are concerned, in addition to the **spatial translations** corresponding to the vector fields

$$\partial_a, \quad a = 1, 2, 3, \tag{3.9}$$

we also rely on the **spatial rotations** corresponding to the vector fields

$$\Omega_{ab} := x_b \partial_a - x_a \partial_b, \quad a, b = 1, 2, 3. \tag{3.10}$$

The notation $\partial^{I_1} \Omega^{I_2}$ is used in which I_1 and I_2 are multi-indices in $1, 2, 3$.

Theorem 3.2 (Nonlinear stability of Minkowski space for self-gravitating massive fields)**.** *For all sufficiently small $\epsilon > 0$ and all sufficiently large integer N, one can find $\sigma, \eta > 0$ depending upon (ϵ, N) so that the following property holds.*

Consider any (σ, ϵ, N)–asymptotically tame initial data set (cf. Definitions 2.2 and 3.1) for the Einstein-massive field system (1.2)*, i.e.*

$$(\mathcal{M}_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1),$$

which is assumed to be sufficiently close to a flat and vacuum spacelike slice of Minkowski spacetime $(\mathbb{R}^{3+1}, g_M)$ in the sense that, in a global coordinate chart $x = (x^a)$ with $r = |x|$,

$$\begin{aligned}
\|(1+r)^{\eta+|I|} Z^I \partial(g_0 - g_{M,0})\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|} Z^I k_0\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I \phi_0\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I \phi_1\|_{L^2(\mathbb{R}^3)} &\leq \epsilon,
\end{aligned} \tag{3.11}$$

for all $Z^I = \partial^{I_1}\Omega^{I_2}$ with $|I_1|+|I_2| \leq N$, in which $g_{M,0} = (\delta_{ab})$. Then, the corresponding initial value problem associated with (1.2) *admits a globally hyperbolic Cauchy development $(\mathcal{M}, g, \phi)$, which is future causally geodesically complete*[1] *and asymptotically approaches Minkowski spacetime.*

Properties of the solutions. The following observations should be made:

- In our proof we will establish energy and sup-norm estimates on the difference $g - g_M$, which show that the spacetime metric and the Minkowski metric are globally close to each other and the difference tends to zero in causal and spacelike directions.
- For instance, for the Schwarzschild metric (in the domain of outer communication away from the horizon), the energy (3.11) is finite and, therefore, our theorem applies to metrics that have a Schwarzschild-like decay (3.5) at spacelike infinity.
- In comparison with the geometric data (g_0, k_0), a stronger decay is assumed on the matter data (ϕ_0, ϕ_1). Analyzing a possible long-range effect[2] in these fields is out of the scope of the present work.

Remark 3.3. *Our proof will use the radial part of the spacetime boosts, defined as $L_r := t\partial_r + r\partial_t$. Our energy functional will be formulated in terms of the derivatives $\partial_\alpha h$ of the spacetime correction $h := g - g_M$, and the weight arising in the first two inequalities in* (3.11) *will be motivated from the following schematic calculation:*

$$\begin{aligned}\partial_\alpha^I L_r^J \partial(g - g_M)|_{\mathcal{M}_0} &\simeq \partial_\alpha^I \partial_r^J \partial(g - g_M)|_{\mathcal{M}_0} + (1+r)^{|J|}\partial_\alpha^I \partial_t^J \partial(g - g_M)|_{\mathcal{M}_0} \\ &\simeq (1+r)^{|J|}\partial^I\partial^J\partial(g_0 - g_{M,0}) + (1+r)^{|J|}\partial^I\partial^J k_0.\end{aligned}$$

3.2.2 The theory of $f(R)$-gravity

The method leading to Theorem 3.2 can be generalized in order to establish the following result in which κ is viewed as a fixed parameter.

[1] We recall that a future causally geodesically complete spacetime, by definition, has the property that every affinely parameterized geodesic (of null or timelike type) can be extended toward the future (for all values of its affine parameter).

[2] This question was recently studied in [24, 123] for $1+1$ scalar equations.

Theorem 3.4 (Nonlinear stability of Minkowski space in modified gravity)**.** *Let $f = f(R)$ be a function satisfying the condition* (1.6) *for some $\kappa \in (0,1)$. For all sufficiently small $\epsilon > 0$ and all sufficiently large integer N, one can find σ, η depending upon (ϵ, N) so that the following property holds.*

Consider any (σ, ϵ, N)–asymptotically tame initial data set (cf. Definitions 2.8 and 3.1) for the field equations of modified gravity (1.7)*, i.e.*

$$(\mathcal{M}_0 \simeq \mathbb{R}^3, g_0, k_0, R_0, R_1, \phi_0, \phi_1),$$

which is assumed to be sufficiently close to a flat spacelike slice of Minkowski spacetime $(\mathbb{R}^{3+1}, g_M)$ in the sense that, in a global coordinate chart $x = (x^a)$ with $r = |x|$,

$$\begin{aligned}
\|(1+r)^{\eta+|I|} Z^I \partial(g_0 - g_M)\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|} Z^I k_0\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I R_0\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I R_1\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I \phi_0\|_{L^2(\mathbb{R}^3)} &\leq \epsilon, \\
\|(1+r)^{\eta+|I|+1/2} Z^I \phi_1\|_{L^2(\mathbb{R}^3)} &\leq \epsilon
\end{aligned} \tag{3.12}$$

for all $Z^I = \partial^{I_1} \Omega^{I_2}$ with $|I_1| + |I_2| \leq N$. Then, the corresponding initial value problem associated with (1.7) *admits a globally hyperbolic Cauchy development $(\mathcal{M}, g, \phi)$, which is causally geodesically complete and asymptotically approaches Minkowski spacetime.*

More precise versions of our existence results will be stated in Section 5, after introducing a spacetime foliation and weighted Sobolev-type energy norms adapted to our problem. (Cf. Theorems 5.1 and 5.2.)

3.2.3 The infinite mass problem

The nonlinear function $f(R)$. In order to study the limit $f(R) \to R$, we will be able to specify the dependency of our estimates with respect to the parameter $\kappa \in (0,1)$:

- We are primarily interested in the quadratic action

$$\int_{\mathcal{M}} \Big(R_g + \frac{\kappa}{2} (R_g)^2 + 16\pi L[\phi, g] \Big) \, dV_g, \tag{3.13}$$

which is most often considered in the physics literature (and corresponds to a vanishing right-hand side in (3.14) below).

- More generally, expanding the function f at the origin we can assume that, in some interval $[-r_*, r_*]$ at least,

$$\Big|\frac{d^n}{dr^n}\Big(f(r) - r - \frac{\kappa}{2}r^2\Big)\Big| \lesssim \kappa^{n+2} r^3, \qquad r \in [-r_*, r_*], \tag{3.14}$$

  at any order $n = 0, \ldots, M$ up to a sufficiently large (but fixed) M.

Hence, the pointwise convergence property $f(R) \to R$ is equivalent to the condition that κ converges to zero.

Resolution of the singular limit problem. Our convergence statement below addresses the fundamental question of the mathematical consistency of the modified gravity theory: the field equations admit a well-defined limit $\kappa \to 0$ and we recover the Einstein theory. Our theorem applies to a sequence of initial data sets for the $f(R)$-theory which is assumed to satisfy suitable κ-independent uniform bounds (to be specified later).

Theorem 3.5 (Convergence of the modified gravity theory toward Einstein's gravity theory)**.** *Suppose that the condition* (3.14) *holds. Then, in the limit $\kappa \to 0$ when the defining function $f(R)$ arising in the generalized action* (1.5) *approaches the scalar curvature function R, the Cauchy developments* $(\mathcal{M}, g, \phi)$ *of $f(R)$-modified gravity given by Theorem 3.4 converge to Einstein's Cauchy developments given by Theorem 3.2.*

4 The wave-Klein-Gordon formulation in EH foliations

4.1 A class of wave-Klein-Gordon systems

4.1.1 Gauge freedom

Choosing the coordinate functions. Since the field equations under consideration are geometric in nature, it is essential to fix the degrees of gauge freedom before tackling any stability issue from the perspective of the initial value problem for partial differential equations. As already pointed out, our analysis relies on a single global coordinate chart $(x^\alpha) = (t, x^a)$ and, more specifically, on a choice of coordinate functions x^α satisfying the homogeneous linear wave equation in the unknown metric. This means that we impose that the functions $x^\alpha : \mathcal{M} \to \mathbb{R}$ satisfy the so-called *wave gauge conditions* ($\alpha = 0, 1, 2, 3$)

$$\Box_g x^\alpha = 0 \qquad \text{for the Einstein field equations,} \tag{4.1}$$

and, more generally, these functions are chosen to satisfy the *conformal wave gauge conditions* as we call them here ($\alpha = 0, 1, 2, 3$)

$$\Box_{g^\dagger} x^\alpha = 0 \qquad \text{for the } f(R)\text{-field equations.} \tag{4.2}$$

Second-order systems. In such a gauge, we obtain a nonlinear system of *second-order partial differential equations* with second-order constraints, as follows:

- For the Einstein theory, the unknowns are the metric coefficients $g_{\alpha\beta}$ in the chosen coordinates together with the scalar field ϕ. It is well-known that the constraints are preserved during the time evolution [30].
- For the $f(R)$-theory, the unknowns are also the conformal metric coefficients ${g^\dagger}_{\alpha\beta}$, the scalar curvature field ρ, and the matter field ϕ. It was established in [89] that the associated constraints are preserved during the time evolution.

In the present section, we write the equations in a schematic form, while the algebraic structure of this system of nonlinear and coupled equations will be analyzed in further details later in Part V. In the case of Einstein's gravity theory and a massless scalar field, we recover the nonlinear structure exhibited first by Lindblad and Rodnianski [94].

4.1.2 The formulation in coordinates

Christoffel symbols and Ricci curvature. Following [89], we thus choose coordinate functions x^α satisfying (4.2) or, equivalently,

$$\Gamma^{\dagger\alpha} = {g^\dagger}^{\alpha\beta} {\Gamma^\dagger}^\lambda_{\alpha\beta} = 0, \tag{4.3}$$

in terms of the Christoffel symbols of the conformal metric $g^\dagger$:

$$\Gamma^{\dagger}{}^{\lambda}_{\alpha\beta} = \frac{1}{2} g^{\dagger\lambda\lambda'} \big(\partial_\alpha g^\dagger{}_{\beta\lambda'} + \partial_\beta g^\dagger{}_{\alpha\lambda'} - \partial_{\lambda'} g^\dagger_{\alpha\beta}\big). \tag{4.4}$$

We also introduce the *modified wave operator*

$$\widehat{\Box}^\dagger = \widehat{\Box}_{g^\dagger} := g^{\dagger\alpha'\beta'} \partial_{\alpha'}\partial_{\beta'}, \tag{4.5}$$

which, in the coordinates under consideration, coincides with the geometric wave operator whenever the conditions (4.3) are enforced.

While we postpone the discussion of the expression of the Ricci curvature to Proposition 13.1, below, let us just recall here that the Ricci curvature reads

$$R^\dagger_{\alpha\beta} = \partial_\lambda \Gamma^{\dagger}{}^{\lambda}_{\alpha\beta} - \partial_\alpha \Gamma^{\dagger}{}^{\lambda}_{\beta\lambda} + \Gamma^{\dagger}{}^{\lambda}_{\alpha\beta} \Gamma^{\dagger}{}^{\delta}_{\lambda\delta} - \Gamma^{\dagger}{}^{\lambda}_{\alpha\delta} \Gamma^{\dagger}{}^{\delta}_{\beta\lambda}, \tag{4.6}$$

which is second-order in the unknown metric. The functions $F_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger)$ arising in (4.7) below represent the quadratic nonlinearities of the Ricci curvature.

The nonlinear wave system. The following formulation is explained further in Section 4.1.4, below.

Proposition 4.1 (The $f(R)$-gravity equations in conformal wave gauge)**.** *In the conformal wave gauge* (4.3)*, the field equations of modified gravity* (1.7) *for a massive field ϕ satisfying* (1.13) *take the following form of a nonlinear system of 11 coupled wave and Klein-Gordon equations for the conformal metric components $g^\dagger_{\alpha\beta}$, the scalar curvature field ρ, and the matter field ϕ:*

$$\begin{aligned}
\widehat{\Box}^\dagger g^\dagger_{\alpha\beta} &= F_{\alpha\beta}(g^\dagger, \partial g^\dagger) + A_{\alpha\beta} + B_{\alpha\beta}, \\
3\kappa\, \widehat{\Box}^\dagger \rho - \rho &= W_\kappa(\rho) - \sigma, \\
\widehat{\Box}^\dagger \phi - U'(\phi) &= 2\kappa\, g^{\dagger\alpha\beta} \partial_\alpha\rho\partial_\beta\phi,
\end{aligned} \tag{4.7}$$

in which

$$\begin{aligned}
A_{\alpha\beta} &:= -3\kappa^2 \partial_\alpha\rho\partial_\beta\rho - V_\kappa(\rho) g^\dagger{}_{\alpha\beta}, \\
B_{\alpha\beta} &:= 16\pi \Big(- e^{-\kappa\rho} \partial_\alpha\phi\partial_\beta\phi + U(\phi) e^{-2\kappa\rho}\, g^\dagger_{\alpha\beta} \Big), \\
\sigma &:= 8\pi e^{-\kappa\rho} \Big(g^{\dagger\alpha\beta} \partial_\alpha\phi\partial_\beta\phi + 4\, e^{-\kappa\rho} U(\phi) \Big),
\end{aligned} \tag{4.8}$$

for some nonlinear functions $V_\kappa = V_\kappa(\rho)$ and $W_\kappa = W_\kappa(\rho)$.

The algebraic and differential constraints. We observe the following:

- The functions V_κ and W_κ are quadratic, since (with the notation $\rho = \frac{1}{\kappa} \log f'(R)$)

$$\begin{aligned} V_\kappa(\rho) &:= \frac{f(R) - Rf'(R)}{f'(R)^2} = -\frac{\kappa}{2}\rho^2 + \kappa^2 \mathcal{O}(\rho^3), \\ W_\kappa(\rho) &:= 6\frac{V_\kappa(\rho)}{f'(R)} - \rho + \frac{f(R)}{f'(R)^2} = \kappa \mathcal{O}(\rho^2). \end{aligned} \tag{4.9}$$

- The system (4.7) is also supplemented with the algebraic constraint on the scalar curvature, that is,

$$\rho = \frac{1}{\kappa} \log f'(R_g), \qquad {g^\dagger}_{\alpha\beta} = e^{\kappa\rho} g_{\alpha\beta}. \tag{4.10}$$

- It must as well be supplemented with the gauge wave constraints (4.3)-(4.4), and the Hamiltonian and momentum constraints of modified gravity (2.20)-(2.21).

4.1.3 The formal limit toward Einstein's equations

In the limit $\kappa \to 0$ we can *formally* recover the Einstein equations since we find $f(R) \to R$ and $g^\dagger \to g$, so that

$$\begin{aligned} &A_{\alpha\beta} \to 0, \\ &B_{\alpha\beta} \to 16\pi \big(-\partial_\alpha\phi\partial_\beta\phi + U(\phi)\, g^\dagger_{\alpha\beta}\big). \end{aligned} \tag{4.11}$$

Moreover, the wave equation for ρ yields us (again *formally*)

$$\rho \to 8\pi\big(g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + 4\,U(\phi)\big) \quad \text{ when } \kappa \to 0, \tag{4.12}$$

which is the standard expression for the spacetime curvature of an Einstein-scalar field spacetime.

Proposition 4.2 (The Einstein equations in wave gauge)**.** *In wave gauge* (4.1)*, the Einstein equations* (1.2) *for a massive field ϕ satisfying* (1.13) *take the following form of a nonlinear system of 10 coupled wave-Klein-Gordon equations for the metric components $g_{\alpha\beta}$ and the scalar field ϕ:*

$$\begin{aligned} \widehat{\Box} g_{\alpha\beta} &= F_{\alpha\beta}(g, \partial g) + 16\pi \big(-\partial_\alpha\phi\partial_\beta\phi + U(\phi)\, g_{\alpha\beta}\big), \\ \widehat{\Box}\phi - U'(\phi) &= 0. \end{aligned} \tag{4.13}$$

Furthermore, the system (4.13) must be supplemented

- with the gauge wave constraints (4.3)-(4.4), that is,

$$
\begin{aligned}
\Gamma^\alpha &= g^{\alpha\beta}\Gamma^\lambda_{\alpha\beta} = 0, \\
\Gamma^\lambda_{\alpha\beta} &= \frac{1}{2} g^{\lambda\lambda'} \big(\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'} - \partial_{\lambda'} g_{\alpha\beta}\big),
\end{aligned}
\tag{4.14}
$$

- and with Einstein's Hamiltonian and momentum constraints (2.2)-(2.3).

4.1.4 Derivation of the wave-Klein-Gordon system

We give a proof of Proposition 4.1, which relies on the previous calculations made in [89]. We recall the following set of equations:

$$
\begin{aligned}
N_{g_{\alpha\beta}} - \frac{g_{\alpha\beta}}{2}\mathrm{tr}\,(N_g) &= e^{2\rho^\sharp} R^\dagger_{\alpha\beta} + \frac{1}{2} e^{2\rho^\sharp} g_{\alpha\beta} W_2^{\sharp\sharp}(\rho^{\sharp\sharp}) - 6 e^{2\rho^{\sharp\sharp}} \partial_\alpha \rho^{\sharp\sharp} \partial_\beta \rho^{\sharp\sharp}, \\
T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \mathrm{tr}\,(T) &= \partial_\alpha\phi\partial_\beta\phi - U(\phi) g_{\alpha\beta}, \\
\Box_{g^\dagger}\rho^{\sharp\sharp} &= \frac{W_2(\rho^{\sharp\sharp})}{6e^{2\rho^{\sharp\sharp}}} + \frac{W_3^{\sharp\sharp}(\rho^{\sharp\sharp})}{6e^{4\rho^{\sharp\sharp}}} + \frac{\mathrm{tr}\,(N_g)}{6e^{4\rho^{\sharp\sharp}}}, \\
\nabla^{\dagger\alpha} T_{\alpha\beta} &= e^{-2\rho^{\sharp\sharp}} \nabla^\alpha T_{\alpha\beta} + 4 g^{\dagger\alpha\gamma} \partial_\gamma \rho^{\sharp\sharp} T_{\alpha\beta}, \\
\Box_{g^\dagger}\phi - U'(\phi) &= 2\, g^\dagger(\nabla\rho^{\sharp\sharp}, \nabla\phi),
\end{aligned}
\tag{4.15}
$$

in which, in [89], we set ${g^\dagger}_{\alpha\beta} = e^{2\rho^{\sharp\sharp}} g_{\alpha\beta}$ for the conformal metric and $\rho^{\sharp\sharp} = \frac{1}{2}\log f'(R_g)$ is regarded as an additional unknown. The potential functions $W_2^{\sharp\sharp}$ and $W_3^{\sharp\sharp}$ are defined by

$$
\begin{aligned}
&W_2^{\sharp\sharp}(\rho^{\sharp\sharp}) := \frac{f(r) - f'(r) r}{f'(r)}, \\
&W_3^{\sharp\sharp}(\rho^{\sharp\sharp}) := f(r), \qquad\qquad e^{2\rho^{\sharp\sharp}} = f'(r), \qquad r \in \mathbb{R}.
\end{aligned}
\tag{4.16}
$$

Let us define the new functions (with $\rho^{\sharp\sharp} = \frac{1}{2}\log f'(r)$):

$$
V_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp}) := \frac{f(r) - r f'(r)}{f'(r)^2}, \qquad W_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp}) := \frac{W_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp})}{f'(r)} - \frac{\rho^{\sharp\sharp}}{3\kappa} + \frac{f(r)}{6 f'(r)^2},
\tag{4.17}
$$

With the notation above and according to [89], the field equations of modified gravity in conformal wave gauge read

$$
\begin{aligned}
\widehat{\Box}^\dagger {g^\dagger}_{\alpha\beta} =& \ F_{\alpha\beta}(g^\dagger, \partial g^\dagger) + A_{\alpha\beta} + B_{\alpha\beta},\\
\widehat{\Box}^\dagger \rho^{\sharp\sharp} - \frac{\rho^{\sharp\sharp}}{3\kappa} =& \ W_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp}) - \Sigma^{\sharp\sharp},\\
\widehat{\Box}^\dagger \phi - U'(\phi) =& \ 2\, {g^\dagger}^{\alpha\beta} \partial_\alpha \rho^{\sharp\sharp} \partial_\beta \phi,
\end{aligned}
\tag{4.18}
$$

where we have set

$$
\begin{aligned}
A_{\alpha\beta} =& - 12 \partial_\alpha \rho^{\sharp\sharp} \partial_\beta \rho^{\sharp\sharp} - V_\kappa^\sharp(\rho^{\sharp\sharp}) {g^\dagger}_{\alpha\beta},\\
B_{\alpha\beta} =& 8\pi \Big(- 2 e^{-2\rho^{\sharp\sharp}} \partial_\alpha \phi \partial_\beta \phi + 2 U(\phi) e^{-4\rho^{\sharp\sharp}} \, g^\dagger_{\alpha\beta} \Big),\\
\Sigma^{\sharp\sharp} =& \frac{4\pi}{3} e^{-2\rho^{\sharp\sharp}} \Big({g^\dagger}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 4\, e^{-2\rho^{\sharp\sharp}} U(\phi) \Big).
\end{aligned}
\tag{4.19}
$$

Expressing the above equations in our notation, we write $\rho = \frac{2}{\kappa}\rho^{\sharp\sharp}$ and we set $V_\kappa(\rho) := V_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp})$, $W_\kappa(\rho) := 6\, W_\kappa^{\sharp\sharp}(\rho^{\sharp\sharp})$, and $\sigma := 6\, \Sigma^{\sharp\sharp}$. This is equivalent to the system stated in Proposition 4.1.

4.2 The Euclidian-hyperboloidal foliation (EHF)

4.2.1 A decomposition of the spacetime

Three distinct regions. It remains to specify the time slicing of our spacetime. Our choice of a foliation is based on a decomposition of the future $\mathcal{M}$ of the initial hypersurface $\mathcal{M}_0$, which distinguishes between three regions, referred to as the *interior domain*, *transition domain,* and *exterior domain*, respectively, and denoted by

$$
\mathcal{M} = \mathcal{M}^{\text{int}} \cup \mathcal{M}^{\text{tran}} \cup \mathcal{M}^{\text{ext}}.
\tag{4.20}
$$

Without loss of generality, we label the initial hypersurface as $\big\{t = 1\big\}$ in our global coordinate chart (t, x^1, x^2, x^3). Recall also that $r^2 = \sum (x^a)^2$.

After identification, we thus write

$$
\mathcal{M} \simeq \big\{t \geq 1\big\} \subset \mathbb{R}^{3+1},
\tag{4.21}
$$

and we can use the symmetries of Minkowski spacetime and regard them as *approximate symmetries* for our spacetime $\mathcal{M}$.

The symmetries of Minkowski spacetime. The vector field method we propose in this Monograph relies on the following:

- The *translations* generated by the vector fields
$$\partial_\alpha, \quad \alpha = 0, 1, 2, 3, \tag{4.22}$$
which, for instance, will be tangent to the time slices in the exterior domain.

- The *Lorentz boosts* generated by the vector fields
$$L_a = x_a \partial_t + t\partial_a, \quad a = 1, 2, 3, \tag{4.23}$$
which, for instance, will be tangent to the time slices in the interior domain.

- The *spatial rotations* generated by the vector fields
$$\Omega_{ab} = x_a \partial_b - x_b \partial_a, \quad a = 1, 2, 3, \tag{4.24}$$
which will be tangent to the time slices in, both, the exterior and the interior domains.

We refer to (4.22)–(4.24) as the family of *admissible vector fieds.* Importantly, these fields *commute* with the wave and Klein-Gordon operators in Minkowski spacetime, namely
$$\big[Z, \Box_{g_M} - c^2\big]\phi = 0 \quad \text{for all admissible fields } Z, \tag{4.25}$$
so that, for any solution to the wave equation or Klein-Gordon equation,
$$\Box_{g_M}\phi - c^2\phi = f \quad \text{imply} \quad \Box_{g_M} Z\phi - c^2 Z\phi = Zf, \tag{4.26}$$
for any vector field Z in the list above.

Importantly, throughout our analysis we avoid to rely on *Minkowski's scaling field,* defined as
$$S = t\partial_t + r\partial_r, \tag{4.27}$$
or, at least, we never apply this vector field to our equations, since it does not commute with the Klein-Gordon operator in Minkowski spacetime.

Decay of solutions. More precisely, our basic strategy is as follows:

- Within an interior domain denoted by $\mathcal{M}^{\text{int}}$, we rely on the foliation based on the hyperboloidal slices
$$\left\{t^2 - r^2 = s^2\right\} \subset \mathbb{R}^{3+1} \tag{4.28}$$
with hyperbolic radius[1] $s \geq 2$. These slices are most convenient in order to analyze wave propagation issues and establish the decay of solutions in timelike directions.

- Within an exterior domain denoted by $\mathcal{M}^{\text{ext}}$, we rely on Euclidian slices of constant time $c \geq 1$
$$\left\{t = c\right\} \subset \mathbb{R}^{3+1}.$$
These slices are most relevant in order to analyze the asymptotic behavior of solutions in spacelike directions and the properties of asymptotically flat spacetimes.

4.2.2 Definition of the time foliation of interest

The time function. In order to take advantage of both foliations above, we propose to glue them together by introducing a transition region $\mathcal{M}^{\text{tran}}$, as follows. Consider a cut-off function $\chi = \chi(y)$ (see next paragraph) satisfying

$$\chi(y) = \begin{cases} 0, & y \leq 0, \\ 1, & y \geq 1, \end{cases} \tag{4.29}$$

which is globally smooth and is increasing within the interval $(0, 1)$. Then, we introduce the following *transition function*

$$\xi(s, r) := 1 - \chi(r + 1 - s^2/2) \in [0, 1], \tag{4.30}$$

which is globally smooth and is defined for all $s \geq 1$ and all $r \geq 0$. It also satisfies

$$\xi(s, r) = \begin{cases} 1, & r \leq -1 + s^2/2, \\ 0, & r \geq s^2/2. \end{cases} \tag{4.31}$$

[1]The region $s \in [1, 2]$ will require a specific treatment.

Hence, the function ξ is not constant precisely in a transition region around the light cone $2r \simeq s^2 = t^2 - r^2$.

Definition 4.3. *The* Euclidian-hyperboloidal time function *is the function* $T = T(s, r)$ *defined by the following ordinary differential problem:*

$$
\begin{aligned}
\partial_r T(s, r) &= \chi(s-1)\xi(s, r)\frac{r}{\sqrt{r^2 + s^2}}, \\
T(s, 0) &= s \geq 1.
\end{aligned}
\tag{4.32}
$$

Then, by definition, the *Euclidian-Hyperboloidal foliation* is determined from this time function and consists of the following family of spacelike hypersurfaces

$$
\mathcal{M}_s := \big\{(t, x) \,/\, t = T(s, |x|)\big\}. \tag{4.33}
$$

Analyzing the spacetime foliation. It is convenient to also define the following spacetime regions:

$$
\begin{aligned}
\mathcal{M}_{[s_0, s_1]} &:= \big\{(t, x) \,/\, T(s_0, |x|) \leq t \leq T(s_1, |x|)\big\} \subset \mathbb{R}^{3+1}, \\
\mathcal{M}_{[s_0, +\infty)} &:= \big\{(t, x) \,/\, T(s_0, |x|) \leq t\big\} \subset \mathbb{R}^{3+1},
\end{aligned}
\tag{4.34}
$$

and the interior, transition, and exterior domains (with $r = |x|$)

$$
\begin{aligned}
\mathcal{M}^{\text{int}}_s &:= \big\{t^2 = s^2 + r^2, \quad r \leq -1 + s^2/2\big\} && \text{hyperboloidal region}, \\
\mathcal{M}^{\text{tran}}_s &:= \big\{t = T(s, r), \quad -1 + s^2/2 \leq r \leq s^2/2\big\} && \text{transition region}, \\
\mathcal{M}^{\text{ext}}_s &:= \big\{t = T(s), \quad r \geq s^2/2\big\} && \text{Euclidian region}.
\end{aligned}
\tag{4.35}
$$

By construction, we thus have:

- In the interior, the relation $T^2 = s^2 + r^2$ holds and the slices consist of hyperboloids of Minkowski spacetime.
- In the exterior, one has $T = T(s) \simeq s^2$ which is independent of r and represents a "slow time", and the slices consists of flat hyperplanes of Minkowski spacetime.

One important task will be to analyze the geometric and algebraic properties of this foliation; see Part III.

4.2.3 Weighted Sobolev-type energy norms

In order to state our nonlinear stability theorems, we still need to define the Sobolev-type norm of interest. We use the energy norm associated to the wave and Klein-Gordon equations and induced on our Euclidian-hyperboloidal slices. In addition, in the exterior domain we introduce a weight function which provides us with the required control of the decay in spacelike directions.

Our weight depends upon the variable

$$q := r - t, \tag{4.36}$$

that is, the distance to the light cone from the origin and so, for some $\eta \in (0, 1]$ and using our cut-off function (4.29), we set

$$\omega_\eta(t, r) := \chi(q)(1 + q)^\eta = \chi(r - t)(1 + r - t)^\eta. \tag{4.37}$$

Clearly, ω_η is a smooth function vanishing in $\{r < t\}$.

To any function v defined in the domain $\mathcal{M}_{[s_0,s_1]}$ limited by two slices of our foliation (say $1 \le s_0 \le s \le s_1$), we define the (flat) *energy functional*

$$\begin{aligned} &E_{\eta,c}(s, v) \\ &:= \int_{\mathcal{M}_s} (1 + \omega_\eta)^2 \Big(\Big(1 - \chi(1 - s)^2 \xi^2 \frac{r^2}{t^2}\Big) \big(\partial_t v\big)^2 + \sum_a \Big(\xi \frac{x^a}{t} \partial_t v + \partial_a v\Big)^2 + c^2 v^2 \Big) dx, \end{aligned}$$

in which, by definition, one has $t = T(s, r)$ and $r = |x|$ on $\mathcal{M}_s$. With the notation (see Part III, below)

$$\begin{aligned} \overline{\partial}_a &:= \partial_a T(s, r) \partial_t + \partial_a \\ &= \chi(1 - s) \xi(s, r) \frac{x^a}{T(s, r)} \partial_t v + \partial_a, \end{aligned} \tag{4.38}$$

we obtain the alternative form

$$E_{\eta,c}(s, v) = \int_{\mathcal{M}_s} (1 + \omega_\eta)^2 \Big(|\zeta(s, r) \partial_t v|^2 + \sum_a |\overline{\partial}_a v|^2 + c^2 v^2 \Big)\, dx, \tag{4.39}$$

in which the coefficient $\zeta = \zeta(s, r) \in [0, 1]$ is defined as

$$\zeta := \chi(1 - s) \sqrt{\frac{s^2 + \chi^2(r - 1 + s^2/2) r^2}{s^2 + r^2}}. \tag{4.40}$$

This energy (together with its generalization to a curved metric) will lead us to a control of the weighted wave-Klein-Gordon energy associated with the operator $\Box v - c^2 v$ with $c \geq 0$.

The choice of the cut-off function will be presented later in Part 3.

5 The nonlinear stability in wave gauge

5.1 Main results

5.1.1 Einstein equations in wave gauge

We are now in a position to state our main results in coordinates. We treat first the Einstein equations (1.2) in which case the physical and the conformal metrics coincide. It is convenient to denote by

$$Z^I := \partial^{I_1} L^{I_2} \Omega^{I_3}, \quad I = (I_1, I_2, I_3),$$

an arbitrary combination of translations, boosts, and rotations, that is

$$\partial \in \big\{\partial_0, \partial_1, \partial_2, \partial_3\big\}, \quad L \in \big\{L_1, L_2, L_3\big\}, \quad \Omega \in \big\{\Omega_{12}, \Omega_{13}, \Omega_{23}\big\}. \tag{5.1}$$

Theorem 5.1 (Global nonlinear stability of self-gravitating massive fields)**.** *For all sufficiently small $\epsilon > 0$ and all sufficiently large integer N, one can find $\sigma \in [0, 1)$, $\eta, \delta \in (0, 1)$, and $C_0 > 0$ depending upon (ϵ, N) so that the following property holds for the Einstein-massive field equations in wave gauge* (4.13) *in the Euclidian-Hyperboloidal Foliation (EHF) (cf. Definition 4.3).*

Consider any (σ, ϵ, N)–asymptotically tame initial data $(h_{\alpha\beta} := g_{\alpha\beta} - g_{M,\alpha\beta}, \phi)$, prescribed on the hypersurface $\big\{t = 1\big\} = \big\{s = 1\big\}$ and satisfying the energy bounds

$$E_\eta(1, Z^I h_{\alpha\beta})^{1/2} + E_{c,\eta+1/2}(1, Z^I \phi)^{1/2} \leq \epsilon, \qquad |I| \leq N, \tag{5.2}$$

as well as the sup-norm estimates (3.7) *in Definition 3.1. Then, a global solution $(h_{\alpha\beta} := g_{\alpha\beta} - g_{M,\alpha\beta}, \phi)$ satisfying the Einstein equations* (4.13) *exists, which enjoys the following bounds (for all $s \geq 1$)*

$$\begin{aligned} E_\eta(s, Z^I h_{\alpha\beta})^{1/2} &\leq C_0 \epsilon s^\delta, & |I| &\leq N, \\ E_{c,\eta+1/2}(s, Z^I \phi)^{1/2} &\leq C_0 \epsilon s^{\delta+1/2}, & |I| &\leq N, \\ E_{c,\eta+1/2}(s, Z^I \phi)^{1/2} &\leq C_0 \epsilon s^\delta, & |I| &\leq N-4. \end{aligned} \tag{5.3}$$

Moreover, the spacetime $(\mathcal{M} \simeq \mathbb{R}^3, g, \phi)$ *determined by this solution is future timelike geodesically complete.*

5.1.2 The $f(R)$-equations in wave gauge

Next, our statement for the $f(R)$-theory reads as follows.

Theorem 5.2 (Global nonlinear stability for the theory of modified gravity)**.** *Let* $f = f(R)$ *be a function satisfying the condition* (1.6) *for some* $\kappa \in (0,1)$. *For all sufficiently small* $\epsilon > 0$ *and all sufficiently large integer* N, *one can find* $\sigma \in [0,1)$, $\eta, \delta \in (0,1)$, *and* $C_0 > 0$ *depending upon* (ϵ, N, κ) *so that the following property holds for the* $f(R)$*-field equations in wave gauge* (4.7) *in the Euclidian-Hyperboloidal Foliation (EHF) (cf. Definition 4.3).*

Consider any (σ, ϵ, N)*–asymptotically tame initial data consisting of a metric* $h_{\alpha\beta} := g_{\alpha\beta} - g_{M,\alpha\beta}$, *a scalar curvature field* ρ, *and a matter field* ϕ *prescribed on the hypersurface* $\{t=1\} = \{s=1\}$ *and satisfying the energy bounds*

$$E_\eta(1, Z^I h_{\alpha\beta})^{1/2} + \frac{1}{b} E_{b,\eta+1/2}(1, Z^I \rho)^{1/2} + E_{c,\eta+1/2}(1, Z^I \phi)^{1/2} \le \epsilon, \qquad |I| \le N, \quad (5.4)$$

in which $b := \kappa^{-1/2}$, *as well as the sup-norm estimates* (3.7) *in Definition 3.1. Then a global solution* $(h_{\alpha\beta} = g^\dagger{}_{\alpha\beta} - g^\dagger{}_{M,\alpha\beta}, \rho, \phi)$ *satisfying the* $f(R)$*-field equations in conformal wave gauge* (4.7) *exists, which enjoys the following bounds for all* $s \ge 1$

$$\begin{aligned}
E_\eta(s, Z^I h_{\alpha\beta})^{1/2} &\le C_0 \epsilon s^\delta, && |I| \le N, \\
E_{b,\eta+1/2}(s, Z^I \rho)^{1/2} &\le C_0 b \epsilon s^{\delta+1/2}, && |I| \le N, \\
E_{b,\eta+1/2}(s, Z^I \rho)^{1/2} &\le C_0 b \epsilon s^{\delta}, && |I| \le N-4. \\
E_{c,\eta+1/2}(s, Z^I \phi)^{1/2} &\le C_0 \epsilon s^{\delta+1/2}, && |I| \le N, \\
E_{c,\eta+1/2}(s, Z^I \phi)^{1/2} &\le C_0 \epsilon s^{\delta}, && |I| \le N-4.
\end{aligned} \quad (5.5)$$

Moreover, the spacetime $(\mathcal{M} \simeq \mathbb{R}^3, g^\dagger, \rho, \phi)$ *determined by this solution is future timelike geodesically complete.*

The mathematical validity of the theory of modified gravity will be justified by establishing that our estimates can be made to be essentially independent of $\kappa \in (0,1)$ and by next analyzing the convergence $\kappa \to 0$.

Remark 5.3. *1. Note that the energy may grow in time. We will actually show that the exponent $\delta = \delta(\epsilon)$ can be chosen to approach zero when the initial norms approach zero, that is,* $\limsup_{\epsilon \to 0} \delta(\epsilon) = 0$*. This type of behavior was first observed by Lindblad and Rodnianski [95] for asymptotically flat foliations of vacuum spacetimes.*

2. Sup-norm bounds will also be established for $g - g_M$ which shows that our sup-norm conditions (3.7) *in Definition 3.1 also hold within the spacetime.*

5.2 A new strategy of proof: the EHF method

5.2.1 Dealing simultaneously with the Einstein and $f(R)$-equations

While a more technical description of our method will be given in the next parts, at this stage we can already outline our strategy of proof.

Before we can proceed with the analysis of the Cauchy problem for the $f(R)$-system we perform a conformal formulation based on the unknown scalar curvature. Namely, as explained earlier, we propose to express the field equations and the initial value problem in terms of a conformally equivalent metric denoted by $g^\dagger$. Following [89], we supplement the field equations (1.7) with the equation (1.9) satisfied by the scalar curvature field.

We also emphasize that Theorems 5.1 and 5.2 are going to be established simultaneously. We write all of our estimates in the context of the $f(R)$-gravity equations. By taking $\kappa = 0$ and suppressing the equation for the scalar curvature field, we will establish that (essentially) all of our estimates carry over to the Einstein equations. In the last part of the analysis, we will give a proof of Theorem 3.5 by analyzing carefully the dependency of our estimates with respect to the mass parameter κ.

5.2.2 The notion of Euclidian-Hyperboloidal Foliation (EFH)

Our method is based on distinguishing between interior and exterior spacetime domains, in which different foliations are required, which we will glue together along a transition region concentrated near the light cone from the origin in our coordinate system.

- **Interior domain.** In our approach, this region $\mathcal{M}^{\text{int}} \subset \mathcal{M}$ is foliated by spacelike hypersurfaces which are (truncated) hyperboloids in Minkowski spacetime $\mathbb{R}^{3+1}$

and coincides with the future of a truncated asymptotically hyperboloidal initial hypersurface. We could solve the field equations globally from an initial data set prescribed on such a hypersurface.

- **Exterior domain.** In our approach, this region $\mathcal{M}^{\text{ext}} \subset \mathcal{M}$ is foliated by asymptotically flat hypersurfaces of $\mathcal{M}$, which are flat spacelike hypersurfaces in Minkowski spacetime $\mathbb{R}^{3+1}$. We can solve the field equations in this exterior domain (containing spacelike infinity) by prescribing suitable initial data.

- **Transition domain.** These two domains are glued together by introducing a transition region around the light cone in which the geometry of the foliation changes drastically from begin hyperboloidal to being Euclidian in nature.

We distinguish between several frames of vector fields. The Cartesian frame ∂_α, the semi-hyperboloidal frame $\underline{\partial}_\alpha$, as well as the null frame $\widetilde{\partial}_a$ will be used at various stages of our analysis:

- Vector fields tangent to the foliation: $\underline{\partial}_a = (x_a/t)\partial_t + \partial_a$ in the interior domain and ∂_a in the exterior domain, which for instance are used in expressing the energy estimates.

- Vector fields relevant for decomposing the metric and the nonlinearities: $\underline{\partial}_a$ in the interior domain and $\widetilde{\partial}_a = (x^a/r)\partial_t + \partial_a$ in the exterior domain.

5.2.3 Main challenge and difficulties

A major challenge is to cope with the nonlinear coupling taking place between the geometry and the matter terms of the gravity field equations, which potentially could lead to a blow-up phenomena and prevent global existence. To proceed, we need to handle the following difficulties:

- The Einstein equations, and more generally the $f(R)$–equations as we discuss them here, do not satisfy the standard null condition. Importantly, understanding the quasi-null structure of the (Einstein or $f(R)$) field equations is essential in the proof and requires understanding the interplay with the wave gauge condition and identifying certain cancellations.

- According to the positive mass theorem, physically admissible initial data (with the exception of Minkowski spacetime itself) must have a non-trivial tail at spatial infinity which typically behaves like Schwarzschild spacetime. We include a spatial weight ω_η in our energy norm which allows to encompass this behavior.

- We will formulate a bootstrap argument and establish (uniform in κ) energy bounds of sufficiently high order satisfied by the metric, the scalar curvature field, and the scalar matter field. Our bootstrap involves a suitable hierarchy of (almost optimal) energy and pointwise bounds, and distinguishes between low- and high-order derivatives of the solutions.

5.2.4 Main technical contributions in this Monograph

We rely on basic high-order energy estimates obtained by applying the translations, boosts, and spatial rotations of Minkowski spacetime.

- Considering a simpler model first and then analyzing the full problem of interest, we provide a *classification of the nonlinearities* arising in wave-Klein-Gordon equations and systematically we compare them with the terms we control with our energy functional.

- By proposing a general and synthetic proof, we establish that our frames of vector fields enjoy *favorable commutator estimates* in order for high-order energy estimates to be derived, and, especially, enjoy good commutation properties with the Killing fields of Minkowski spacetime

- We derive new *Sobolev inequalities for Euclidian-hyperboloidal foliations*, which are established by studying cone-like domains first.

- We establish *sharp sup-norm estimates* for solutions to wave and Klein-Gordon equations on curved spacetime, after decomposing the solution operators in suitable frames and building on the following three approaches: an ordinary differential equation argument along rays, a characteristic integration argument, and Kirchhoff's explicit formula.

The *wave gauge conditions* play a central role in the derivation of energy and pointwise bounds and especially provide a control on the components of the metric associated with a propagation equation that does not satisfy the null condition.

Importantly, the *dependency w.r.t. the mass parameter* κ must be carefully traced throughout, in order to encompass the regime $\kappa \to 0$. We introduce a scaling of the scalar curvature field which is most suitable for deriving uniform bounds and analyzing the singular convergence $\kappa \to 0$.

5.2.5 From geometric data to wave gauge data

Wave conditions on the initial hypersurface In order to complete the formulation in coordinates (4.7), we must connect the initial data required in wave coordinates to the prescribed geometric initial data introduced in Definitions 2.1 and 2.7. While the field equations are coordinate-independent, our analysis does require a choice of coordinates and all geometric degrees of freedom must be fixed.

This is a standard issue which is extensively treated in the literature on the local-in-time existence theory [30, 136] and we only briefly discuss this issue here. The starting point is the four wave gauge conditions for the conformal metric $g^\dagger$, that is,

$$g^{\dagger\alpha\beta}\big(2\partial_\alpha {g^\dagger}_{\beta\gamma} - \partial_\gamma {g^\dagger}_{\alpha\beta}\big) = 0, \qquad \gamma = 0, \ldots, 3. \tag{5.6}$$

By imposing the orthogonality condition

$${g^\dagger}_{0a} = 0 \quad \text{ on } t = 1, \tag{5.7}$$

on the *initial* slice (only), we obtain

$$\begin{aligned} g^{\dagger 00}\partial_t {g^\dagger}_{00} - g^{\dagger ab}\partial_t {g^\dagger}_{ab} = 0, \\ 2\, g^{\dagger 00}\partial_t {g^\dagger}_{0a} + 2\, g^{\dagger bc}\big(\partial_b {g^\dagger}_{ac} - \partial_a {g^\dagger}_{bc}\big) = 0. \end{aligned} \tag{5.8}$$

This suggests us how to define the initial data in coordinates on the initial hypersurface from the prescribed geometric data $g_0^\dagger$ and $k_0^\dagger$.

Lapse function The precise definition may depend on the choice of asymptotic conditions assumed on the initial data set. When the lapse function on the initial hypersurface is chosen to be 1, one defines the initial data for (4.7) on $\{t = 1\}$ as follows:

$$\begin{aligned}
&g^\dagger{}_{00}\big|_{t=1} = -1, \qquad g^\dagger{}_{0a}\big|_{t=1} = 0, \qquad g^\dagger{}_{ab}\big|_{t=1} = g^\dagger{}_{0ab},\\
&\partial_t g^\dagger{}_{00}\big|_{t=1} = 2 g_0^{\dagger ab} k^\dagger{}_{0ab},\\
&\partial_t g^\dagger{}_{0a}\big|_{t=1} = g_0^{\dagger bc}\partial_c g^\dagger_{0ba} - \frac{1}{2} g_0^{\dagger bc}\partial_a g^\dagger_{0bc},\\
&\partial_t g^\dagger{}_{ab}\big|_{t=1} = -2\, k^\dagger_{0ab},
\end{aligned}\tag{5.9}$$

while, for the matter and curvature fields,

$$\rho\big|_{t=1} = \rho_0, \quad \partial_t\rho\big|_{t=0} = \rho_1, \quad \phi\big|_{t=1} = \phi_0, \quad \partial_t\phi\big|_{t=0} = \phi_1. \tag{5.10}$$

On the other hand, for the class of metrics that are asymptotic to a given Schwarzschild metric with mass m_S, we can choose the lapse function $l > 0$ on the initial hypersurface from its expression for the Schwarzschild metric expressed in wave gauge. In this case, we select a smooth interpolating function l satisfying

$$l(r) = \begin{cases} 1, & r \le 1/2,\\ \frac{r-2m_S}{r+2m_S}, & r \ge 1. \end{cases} \tag{5.11}$$

We define the initial data for (4.7) on $\{t = 1\}$ as follows:

$$\begin{aligned}
&g^\dagger{}_{00}\big|_{t=1} = -l^2, \qquad g^\dagger{}_{0a}\big|_{t=1} = 0, \qquad g^\dagger{}_{ab}\big|_{t=1} = g^\dagger{}_{0ab},\\
&\partial_t g^\dagger{}_{00}\big|_{t=1} = 2 l^3 g_0^{\dagger ab} k^\dagger{}_{0ab},\\
&\partial_t g^\dagger{}_{0a}\big|_{t=1} = l^2 g_0^{\dagger bc}\partial_c g^\dagger_{0ba} - \frac{l^2}{2} g_0^{\dagger bc}\partial_a g^\dagger_{0bc} - l\partial_a l,\\
&\partial_t g^\dagger{}_{ab}\big|_{t=1} = -2l\, k^\dagger_{0ab},
\end{aligned}\tag{5.12}$$

while, for the matter and curvature fields, we have

$$\rho\big|_{t=1} = \rho_0, \quad \partial_t\rho\big|_{t=0} = l\rho_1, \quad \phi\big|_{t=1} = \phi_0, \quad \partial_t\phi\big|_{t=1} = l\phi_1. \tag{5.13}$$

Part III

The notion of Euclidian-Hyperboloidal foliations

6 Defining Euclidian-hyperboloidal foliations (EHF)

6.1 Defining the time function

The cut-off function This part together with the next one presents some of main tools required for the EHF method (with the exception of the sup-norm estimates that will be presented later), while the treatment of a class of equations is presented in the following. From now on we focus on the region $s \geq s_0 = 2$, so that the factor $\chi(s-1)$ introduced in the previous part is now identically 1. For this reason and for clarity in the presentation, we allow ourselves to repeat some of the notation that were introduce earlier. Hence, this Part together with the next one can be read mostly independently from the previous two Parts and therefore provide the reader an introduction to the new tools required in our method.

First of all, a cut-off function $\chi : \mathbb{R} \to [0,1]$ is defined by introducing first a function $\rho : \mathbb{R} \to [0,+\infty)$ by

$$\rho(y) := \begin{cases} e^{\frac{-2}{1-(2y-1)^2}}, & 0 < y < 1, \\ 0, & \text{otherwise}, \end{cases} \tag{6.1}$$

which is clearly globally smooth. Setting

$$\rho_0 := \int_{-\infty}^{+\infty} \rho(y)\, dy = \int_0^1 \rho(y)\, dy > 0, \tag{6.2}$$

our cut-off function is defined as

$$\chi(y) := \rho_0^{-1} \int_{-\infty}^{y} \rho(y')\, dy', \quad y \in \mathbb{R}. \tag{6.3}$$

It follows that χ is a smooth function satisfying

$$\chi(y) = \begin{cases} 0, & y \le 0, \\ 1, & y \ge 1, \end{cases} \tag{6.4}$$

which is also strictly increasing on the interval $(0, 1)$.

Within the spacetime, we define a smooth function[1] $\chi = \chi(s, r)$ by

$$\xi(s, r) := 1 - \chi(r - 1 + s^2/2), \quad s \ge 2,\, r > 0, \tag{6.5}$$

which satisfies

$$\xi(s, r) = \begin{cases} 1, & r \le -1 + s^2/2, \\ 0, & r \ge s^2/2. \end{cases} \tag{6.6}$$

This function is chosen to be non-constant precisely in a neighborhood of the light cone, as will become clear below.

The time function. We define the function $T = T(s, r) \ge 1$ by the following two conditions:

$$\partial_r T(s, r) = \xi(s, r) \frac{r}{\sqrt{r^2 + s^2}}, \qquad T(s, 0) = s. \tag{6.7}$$

In view of (6.6), it is clear that, for sufficiently small values of r,

$$T(s, r) = \sqrt{s^2 + r^2} \quad \text{for all } r \le -1 + s^2/2. \tag{6.8}$$

On the other hand, in the intermediate region, we find

$$\begin{aligned} T(s, r) =& \frac{1}{2}\sqrt{s^4 + 4} + \int_{-1+s^2/2}^{r} (s^2 + \rho^2)^{-1/2} \rho\, \xi(s, \rho) d\rho \\ & \text{for all } s^2/2 \ge r \ge -1 + s^2/2, \end{aligned} \tag{6.9}$$

and finally

$$\begin{aligned} T(s, r) &= \frac{1}{2}\sqrt{s^4 + 4} + \int_{-1+s^2/2}^{s^2/2} (s^2 + \rho^2)^{-1/2} \rho\, \xi(s, \rho) d\rho \\ &=: T(s) \quad \text{for all } r \ge s^2/2, \end{aligned} \tag{6.10}$$

which does not depend upon the radius r. Observe that this last coefficient satisfies

$$\frac{1}{2}\sqrt{s^4 + 4} < T(s) < \frac{1}{2}\sqrt{s^4 + 4s^2}. \tag{6.11}$$

[1] We now restrict attention to $s \ge 2$.

6.2 Defining the foliation

The slices. Consider the family of slices associated with the time function T:

$$\mathcal{M}_s := \{(t,x) \in \mathbb{R} \times \mathbb{R}^3 \,/\, t = T(s,r),\, r = |x|\}. \tag{6.12}$$

The following property is easily checked. The slices $\mathcal{M}_s$ form a family of smooth 3-dimensional spacelike hypersurfaces, whose future oriented normal vector n_s (with respect to the Euclidian metric) reads

$$n_s = \frac{1}{\sqrt{(1+\xi^2(s,r))r^2 + s^2}} \big(\sqrt{s^2+r^2}, -x^a \xi(s,r)\big) \tag{6.13}$$

and whose volume element σ_s (with respect to the Euclidian metric) reads

$$\sigma_s = \frac{\sqrt{s^2 + r^2(1+\xi(s,r)^2)}}{\sqrt{s^2+r^2}} dx. \tag{6.14}$$

We introduce the following notation:

$$\begin{aligned}
\mathcal{M}_s^{\text{int}} &:= \{(t,x) \in \mathbb{R}^{3+1} \,/\, t = \sqrt{s^2+r^2},\, r \leq -1 + s^2/2\}, \\
\mathcal{M}_s^{\text{tran}} &:= \{(t,x) \in \mathbb{R}^{3+1} \,/\, t = T(s,r),\, -1 + s^2/2 \leq r \leq s^2/2\} \\
\mathcal{M}_s^{\text{ext}} &:= \{(t,x) \in \mathbb{R}^{3+1} \,/\, t = t(s),\, r \geq s^2/2\}.
\end{aligned} \tag{6.15}$$

The decomposition of the spacetime. Our construction provides us with a decomposition of each slice in three parts

$$\mathcal{M}_s := \mathcal{M}_s^{\text{int}} \cup \mathcal{M}_s^{\text{tran}} \cup \mathcal{M}_s^{\text{ext}}. \tag{6.16}$$

Next, we set

$$\begin{aligned}
\mathcal{M}_{[s_0,+\infty)} &:= \{(t,x) \in \mathbb{R}^{3+1} \,/\, t \geq t(s_0,r)\}, \\
\mathcal{M}_{[s_0,s_1]} &:= \{(t,x) \in \mathbb{R}^{3+1} \,/\, t(s_0,r) \leq t \leq t(s_1,r)\},
\end{aligned} \tag{6.17}$$

and we thus have

$$\mathcal{M}_{[s_0,+\infty)} = \bigcup_{s \geq s_0} \mathcal{M}_s. \tag{6.18}$$

This foliation is referred to as the *Euclidian-Hyperboloidal Foliation* (EHF). We also write

$$
\begin{aligned}
\mathcal{M}^{\text{int}}_{[s_0,+\infty)} &= \bigcup_{s\geq s_0} \mathcal{M}^{\text{int}}_s,\\
\mathcal{M}^{\text{tran}}_{[s_0,+\infty)} &= \bigcup_{s\geq s_0} \mathcal{M}^{\text{tran}}_s,\\
\mathcal{M}^{\text{ext}}_{[s_0,+\infty)} &= \bigcup_{s\geq s_0} \mathcal{M}^{\text{ext}}_s,
\end{aligned}
\tag{6.19}
$$

and we refer to these regions as the *interior, transition, and exterior domains* of the spacetime, respectively.

Choice of coordinates. In $\mathcal{M}_{[s_0,+\infty)}$ we rely on the coordinate functions

$$
\overline{x}^0 = s, \qquad \overline{x}^a = x^a,
$$

and the natural frame associated with this parametrization reads

$$
\begin{aligned}
\partial_s &= \partial_s T(s,r)\partial_t,\\
\overline{\partial}_a &= \partial_a + \partial_a T(s,r)\partial_t = \partial_a + \frac{x^a \xi(s,r)}{\sqrt{s^2+r^2}}\partial_t.
\end{aligned}
\tag{6.20}
$$

Then we analyse the transition region $\mathcal{M}^{\text{tran}}_s$ we can write

$$
\begin{aligned}
t - r = T(s,r) - r &= \frac{1}{2}\sqrt{s^4+4} - 1 + s^2/2 + \int_{-1+s^2/2}^{r} \frac{\xi(s,r)\rho d\rho}{\sqrt{s^2+\rho^2}} - \big(r - 1 + s^2/2\big)\\
&= \frac{2}{\sqrt{1+4s^{-4}} + 1 - 2s^{-2}} + \int_{-1+s^2/2}^{r} \frac{\xi(s,r)\rho d\rho}{\sqrt{s^2+\rho^2}} - \big(r - 1 + s^2/2\big)\,,
\end{aligned}
\tag{6.21}
$$

thus we obtain the folllowing lower and upper bounds for the distance to the light cone within the transition region:

$$
1 < \frac{2}{\sqrt{1+4s^{-4}} + 1 - 2s^{-2}} - 1 < t - r \leq \frac{2}{\sqrt{1+4s^{-4}} + 1 - 2s^{-2}} < 2
\tag{6.22}
$$

in the transition region $\mathcal{M}^{\text{tran}}_s$.

6.3 Jacobian of the coordinate transformation

The following identity relates the volume elements in both coordinates (t, x) and (s, x):

$$dxdt = Jdxds, \tag{6.23}$$

in which J denotes the Jacobian of the corresponding change of variables, that is, $J = |\partial_s T(s, r)|$. The following technical observation will play a role later.

Lemma 6.1. *The Euclidian-hyperboloidal time-function $T = T(s, r)$ satisfies*

$$|\partial_s T(s,r)| \lesssim \begin{cases} \frac{s}{T(s,r)} = \frac{s}{(s^2+r^2)^{1/2}}, & r \leq -1 + s^2/2, \\ \frac{\xi(s,r)s}{(s^2+r^2)^{1/2}} + 2(1-\xi(s,r))s, & -1 + s^2/2 \leq r \leq s^2/2, \\ 2s, & r \geq s^2/2, \end{cases} \tag{6.24}$$

in which the implied constant is a universal constant.

Thanks for this lemma, we thus obtain

$$J \lesssim \begin{cases} s/t = \frac{s}{(s^2+r^2)^{1/2}}, & r \leq -1 + s^2/2, \\ \frac{\xi(s,r)s}{(s^2+r^2)^{1/2}} + 2(1-\xi(s,r))s, & -1 + s^2/2 \leq r \leq s^2/2, \\ 2s, & r \geq s^2/2. \end{cases} \tag{6.25}$$

Proof. We have

$$\partial_r \partial_s T(s,r) = \partial_s \partial_r T(s,r) = \frac{r \partial_s \xi(s,r)}{\sqrt{s^2+r^2}} - \frac{sr\xi(s,r)}{(s^2+r^2)^{3/2}}$$

and observe that $\partial_s \xi(s,r) = -s\partial_r \xi(s,r)$, so that

$$\partial_r \partial_s T(s,r) = \frac{-s\partial_r \xi(s,r) r}{\sqrt{s^2+r^2}} - \frac{sr\xi(s,r)}{(s^2+r^2)^{3/2}}. \tag{6.26}$$

Recall that, at the center $r = 0$, one has $t(s, 0) = s$ and thus $\partial_s T(s, 0) = 1$. Then we see that for $0 \leq r \leq -1 + s^2/2$,

$$\partial_s T(s,r) = 1 - \int_0^r \frac{s\rho}{(s^2+\rho^2)^{3/2}} d\rho = \frac{s}{\sqrt{s^2+r^2}} = s/t.$$

For $-1+s^2/2 \le r \le \frac{s^2}{2}$, we have

$$\begin{aligned}\partial_s T(s,r) &= 1 - \int_0^{-1+s^2/2} \frac{s\rho}{(s^2+\rho^2)^{3/2}} d\rho - \int_{-1+s^2/2}^{r} \frac{s\rho\xi(s,\rho)d\rho}{(s^2+\rho^2)^{3/2}} \\ &\quad - s\int_{-1+s^2/2}^{r} \frac{\partial_r \xi(s,\rho)\rho d\rho}{\sqrt{s^2+\rho^2}} \\ &= \frac{2s}{\sqrt{s^4+4}} - \int_{-1+s^2/2}^{r} \frac{s\rho\xi(s,\rho)d\rho}{(s^2+\rho^2)^{3/2}} - s\int_{-1+s^2/2}^{r} \frac{\partial_r \xi(s,\rho)\rho d\rho}{\sqrt{s^2+\rho^2}} \\ &= \frac{s\xi(s,r)}{(s^2+r^2)^{1/2}} - 2s\int_{-1+s^2/2}^{r} \frac{\rho\partial_r\xi(s,r)d\rho}{(s^2+\rho^2)^{1/2}} \\ &= \frac{s\xi(s,r)}{(s^2+r^2)^{1/2}} + 2sT_1(r).\end{aligned}$$

We write $T_s(\rho) := \frac{\rho}{s^2+\rho^2}$ and, by integration by parts, we find

$$\begin{aligned}T_1(r) &= \xi(s,\rho)T_s(\rho)\Big|_r^{-1+s^2/2} + \int_{-1+s^2/2}^{r} \xi(s,\rho)T_s'(\rho)d\rho \\ &= (\xi(s,-1+s^2/2) - \xi(s,r))T_s(r) + \xi(s,-1+s^2/2)\left(T_s(-1+s^2/2) - T_s(r)\right) \\ &\quad + \int_{-1+s^2/2}^{r} \xi(s,\rho)T_s'(\rho)d\rho \\ &= (1-\xi(s,r))T_s(r) + \left(T_s(-1+s^2/2) - T_s(r)\right) + \int_{-1+s^2/2}^{r} \xi(s,\rho)T_s'(\rho)d\rho.\end{aligned}$$

We observe that $T_s'(\rho) \ge 0$ and $\xi(s,r) \le 1$, hence

$$\int_{-1+s^2/2}^{r} \xi(s,\rho)T_s'(\rho)d\rho \le \int_{-1+s^2/2}^{r} T_s'(\rho)d\rho \le T_s(r) - T_s(-1+s^2/2).$$

Thus we see that

$$T_1(r) \le (1-\xi(s,r))T_s(r) \le 1 - \xi(s,r)$$

and this conclude the case $-1+s^2/2 \le r \le s^2/2$.

For $r \ge s^2/2$, we see that

$$\begin{aligned}\partial_s T(s,r) =& 1 - \int_0^{-1+s^2/2} \frac{s\rho d\rho}{(s^2+\rho^2)^{3/2}} - \int_{-1+s^2/2}^{s^2/2} \frac{s\rho\xi(s,\rho)d\rho}{(s^2+\rho^2)^{3/2}} \\ & - s\int_{-1+s^2/2}^{s^2/2} \frac{\partial_r\xi(s,\rho)\rho d\rho}{\sqrt{s^2+\rho^2}},\end{aligned}$$

thus we keep the bound given in the last case with $r = s^2/2$. $\square$

7 Frames of interest and the null condition

7.1 The semi-hyperboloidal frame (SHF)

We now introduce several frames of vector fields, which will play a key role on our analysis. From [85], we recall the definition of the *semi-hyperboloidal frame* defined globally in $\mathcal{M}_{[2,+\infty)}$ by

$$\underline{\partial}_0 := \partial_t, \qquad \underline{\partial}_a := \frac{x^a}{t}\partial_t + \partial_a. \tag{7.1}$$

The transition matrix between this frame and the natural frame ∂_α is given by the relations

$$\underline{\partial}_\alpha = \underline{\Phi}_\alpha^{\alpha'}\partial_{\alpha'}, \qquad \underline{\Phi}_\alpha^\alpha = 1, \quad \underline{\Phi}_a^0 = x^a/t, \tag{7.2}$$

that is

$$\left(\underline{\Phi}_\alpha^\beta\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/t & 1 & 0 & 0 \\ x^2/t & 0 & 1 & 0 \\ x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{7.3}$$

The inverse of $\underline{\Phi}$ is denoted by $\underline{\Psi}$ and satisfies

$$\partial_\alpha = \underline{\Psi}_\alpha^{\alpha'}\underline{\partial}_{\alpha'}, \qquad \underline{\Psi}_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/t & 1 & 0 & 0 \\ -x^2/t & 0 & 1 & 0 \\ -x^3/t & 0 & 0 & 1 \end{pmatrix}. \tag{7.4}$$

We also introduce the corresponding dual frame associated with $\underline{\partial}_\alpha$, that is,

$$\underline{\theta}^0 = dt - (x^a/t)\partial_a, \quad \underline{\theta}^a = dx^a. \tag{7.5}$$

Finally, observe that, in the semi-hyperboloidal frame, the Minkowski metric $g_M = m$reads as follows:

$$\left(\underline{g}_M^{\alpha\beta}\right) = \left(\underline{m}^{\alpha\beta}\right) = \begin{pmatrix} 1-(r/t)^2 & x^1/t & x^2/t & x^3/t \\ x^1/t & -1 & 0 & 0 \\ x^2/t & 0 & -1 & 0 \\ x^3/t & 0 & 0 & -1 \end{pmatrix}. \tag{7.6}$$

The semi-hyperboloidal frame will be most suitable in the interior domain in order to exhibit the (quasi-)null form structure of the nonlinearities of the field equations.

7.2 The null frame (NF)

Within the region $\{r > t/2\} \cap \mathcal{M}_{[2,+\infty)}$, we have $r > \frac{2}{\sqrt{3}}$ so that the radial variable is bounded below and we can introduce the *null frame* (NF):

$$\widetilde{\partial}_0 = \partial_t, \qquad \widetilde{\partial}_a := \frac{x^a}{r}\partial_t + \partial_a. \tag{7.7}$$

The transition matrix between this frame and the canonical frame is given by

$$\widetilde{\partial}_\alpha = \widetilde{\Phi}_\alpha^\beta \, \partial_\beta, \qquad \widetilde{\Phi}_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ x^1/r & 1 & 0 & 0 \\ x^2/r & 0 & 1 & 0 \\ x^3/r & 0 & 0 & 1 \end{pmatrix}. \tag{7.8}$$

The inverse of this transition matrix reads

$$\widetilde{\Psi}_\alpha^\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -x^1/r & 1 & 0 & 0 \\ -x^2/r & 0 & 1 & 0 \\ -x^3/r & 0 & 0 & 1 \end{pmatrix}. \tag{7.9}$$

We also note that the dual frame associated with this null frame is

$$\widetilde{\theta}^0 := dt - (x^a/r)dx^a, \qquad \widetilde{\theta}^a = dx^a. \tag{7.10}$$

Finally, in the null frame, the Minkowski metric takes the following simple form:

$$\left(\widetilde{g}_M^{\alpha\beta}\right) = \left(\widetilde{m}^{\alpha\beta}\right) = \begin{pmatrix} 0 & x^1/r & x^2/r & x^3/r \\ x^1/r & -1 & 0 & 0 \\ x^2/r & 0 & -1 & 0 \\ x^3/r & 0 & 0 & -1 \end{pmatrix}. \tag{7.11}$$

The null frame will be most suitable in the exterior domain in order to exhibit the (quasi-)null form structure of the nonlinearities of the field equations.

7.3 The Euclidian-hyperboloidal frame (EHF)

Based on the parametrization of $\mathcal{M}_{[2,+\infty)}$ by (s, x^a), we introduce the following *Euclidian-hyperboloidal frame* (EHF)

$$
\begin{aligned}
\breve{\partial}_0 &= \partial_t, \\
\breve{\partial}_a &= \partial_a t\, \partial_t + \partial_a = (x^a/r)\partial_r t\, \partial_t + \partial_a = \frac{\xi(s,r)x^a}{\sqrt{r^2+s^2}}\partial_t + \partial_a,
\end{aligned}
\tag{7.12}
$$

(which we sometimes call the Euclidian-semi-hyperboloidal frame (ESHF)). The transition matrix between this frame and the canonical natural frame is given by

$$
\breve{\partial}_\alpha = \breve{\Phi}_\alpha^{\beta}\partial_\beta, \qquad \breve{\Phi}_\alpha^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{\xi(s,r)x^1}{\sqrt{s^2+r^2}} & 1 & 0 & 0 \\ \frac{\xi(s,r)x^2}{\sqrt{s^2+r^2}} & 0 & 1 & 0 \\ \frac{\xi(s,r)x^3}{\sqrt{s^2+r^2}} & 0 & 0 & 1 \end{pmatrix}, \tag{7.13}
$$

while its inverse reads

$$
\breve{\Psi}_\alpha^{\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{-\xi(s,r)x^1}{\sqrt{s^2+r^2}} & 1 & 0 & 0 \\ \frac{-\xi(s,r)x^2}{\sqrt{s^2+r^2}} & 0 & 1 & 0 \\ \frac{-\xi(s,r)x^3}{\sqrt{s^2+r^2}} & 0 & 0 & 1 \end{pmatrix}. \tag{7.14}
$$

This frame coincides with the semi-hyperboloidal frame in the interior domain, but with the canonical frame in the exterior domain. In the region of transition, $\breve{\partial}_a$ are also tangent to the slices $\mathcal{M}_s^{\text{tran}}$.

The corresponding dual frame is given by

$$
\breve{\theta}^0 = dt - \frac{\xi(s,r)x^a}{\sqrt{s^2+r^2}}dx^a, \qquad \breve{\theta}^a = dx^a. \tag{7.15}
$$

In this frame the Minkowski metric g_M takes the form

$$
\left(\breve{g}_M^{\alpha\beta}\right) = \left(\breve{m}^{\alpha\beta}\right) = \begin{pmatrix} 1 - \frac{\xi^2(s,r)r^2}{s^2+r^2} & \frac{\xi(s,r)x^1}{\sqrt{s^2+r^2}} & \frac{\xi(s,r)x^2}{\sqrt{s^2+r^2}} & \frac{\xi(s,r)x^3}{\sqrt{s^2+r^2}} \\ \frac{\xi(s,r)x^1}{\sqrt{s^2+r^2}} & -1 & 0 & 0 \\ \frac{\xi(s,r)x^2}{\sqrt{s^2+r^2}} & 0 & -1 & 0 \\ \frac{\xi(s,r)x^3}{\sqrt{s^2+r^2}} & 0 & 0 & -1 \end{pmatrix}. \tag{7.16}
$$

The Euclidian-hyperboloidal frame is appropriate in order to express our (high-order) energy functionals associated with solutions to the field equations.

Later, it will be convenient to use the coefficient $\zeta = \zeta(s, r)$ defined by

$$\zeta(s, r) := \sqrt{1 - \frac{\xi^2(s, r) r^2}{s^2 + r^2}}. \tag{7.17}$$

Finally, we also point out the relation

$$\partial_t = (\partial_s t)^{-1} \partial_s = J^{-1} \partial_s. \tag{7.18}$$

Of course, a tensor can be expressed in different frames and we adopt the following notation for any two-tensor T:

$$\begin{aligned} T = T^{\alpha\beta} \partial_\alpha \otimes \partial_\beta &= \underline{T}^{\alpha\beta} \underline{\partial}_\alpha \otimes \underline{\partial}_\beta \\ &= \breve{T}^{\alpha\beta} \breve{\partial}_\alpha \otimes \breve{\partial}_\beta = \widetilde{T}^{\alpha\beta} \widetilde{\partial}_\alpha \otimes \widetilde{\partial}_\beta. \end{aligned} \tag{7.19}$$

8 The weighted energy estimate

8.1 Statement of the energy estimate

In the domain $\mathcal{M}_{[s_0, +\infty)}$, we introduce the distance to the light cone

$$q = r - t$$

and for any $\eta > 0$ we define

$$\omega_\eta(q) := \begin{cases} 0, & q \leq 0 \Leftrightarrow t \geq r, \\ \chi(q)(1+q)^\eta, & q \geq 0 \Leftrightarrow r \geq t. \end{cases} \tag{8.1}$$

Clearly, ω_η is smooth within $\mathcal{M}_{[s_0, +\infty)}$ and

$$\begin{aligned} \partial_\alpha \omega_\eta &= \omega_\eta'(q) \partial_\alpha q \\ &= \begin{cases} 0, & q \leq 0 \Leftrightarrow t \geq r, \\ \left(\chi'(q)(1+q)^\eta + \eta \chi(q)(1+q)^{\eta-1}\right) \partial_\alpha q, & q \geq 0 \Leftrightarrow r \geq t, \end{cases} \end{aligned} \tag{8.2}$$

while $\omega_\eta'(q) \geq 0$.

We are ready to establish a basic energy estimate. Recall that the wave operator on Minkowski spacetime is denoted by

$$\Box = \Box_{g_M} = -\partial_t \partial_t + \sum_{a=1}^{3} \partial_a \partial_a. \tag{8.3}$$

A weighted energy estimate valid in a general curved spacetime will also be established later on.

Proposition 8.1 (Weighted energy estimate in flat spacetime)**.** *Let u be a smooth function defined in $\mathcal{M}_{[s_0,s_1]}$ and decaying sufficiently fast at infinity (with respect to $r \to \infty$). If*

$$\Box u - c^2 u = f, \tag{8.4}$$

then the following estimate holds

$$\begin{aligned} E_{\eta,c}(s,u)^{1/2} \leq E_{\eta,c}(s_0,u)^{1/2} &+ C\|f\|_{L^2(\mathcal{M}_s^{int})} \\ &+ C\left\|(1+(1-\xi(s,r))^{1/2}s)f\right\|_{L^2(\mathcal{M}_s^{tran})} \\ &+ C\left\|s(1+\omega_\eta)f\right\|_{L^2(\mathcal{M}_s^{ext})} \end{aligned} \tag{8.5}$$

for some uniform constant $C > 0$, in which the energy on the slice $\mathcal{M}_s$ is defined by

$$\begin{aligned} E_{\eta,c}(s,u) := \int_{\mathcal{M}_s} (1+\omega_\eta)\Bigg(&\frac{s^2+(1-\xi^2(s,r))r^2}{s^2+r^2}\left(\partial_t u\right)^2 \\ &+ \sum_a \left(\frac{\xi(s,r)x^a}{\sqrt{s^2+r^2}}\partial_t u + \partial_a u\right)^2 + c^2u^2\Bigg)\, dx. \end{aligned} \tag{8.6}$$

8.2 Derivation of the energy estimate

We provide here a proof of Proposition 8.1 by applying the classical multiplier $\partial_t u$ to the equation (8.4). We obtain

$$\begin{aligned} 2(1+\omega_\eta)^2\partial_t u \Box u = \partial_t &\Big((1+\omega_\eta)^2|\partial_t u|^2 + \sum_a \left(\partial_a u\right)^2\Big) \\ &- \partial_a\Big(2(1+\omega_\eta)^2\partial_t u \partial_a u\Big) \\ &+ 2(1+\omega_\eta)\omega_\eta'(q)\sum_a \left((x^a/r)\partial_t u - \partial_a u\right)^2. \end{aligned} \tag{8.7}$$

Let

$$V_\eta = (1+\omega_\eta)^2\big(|\partial_t u|^2 + \sum_a \big(\partial_a u\big)^2, -2\partial_t u \partial_a u\big) \tag{8.8}$$

be the associated energy flux. Integrating the identity (8.7) in the region $\mathcal{M}_{[s_0,s_1]}$ and assuming that u tends to zero sufficiently fast, Stokes' theorem applies and gives us

$$\begin{aligned}
&2\int_{\mathcal{M}_{[s_0,s_1]}} (1+\omega_\eta)^2 \partial_t u \,\Box u \, dxdt \\
&= \int_{\mathcal{F}_{s_1}} V_\eta \cdot n_{s_1} d\sigma_{s_1} - \int_{\mathcal{F}_{s_0}} V_\eta \cdot n_{s_0} d\sigma_{s_0} \\
&\quad + 2\sum_a \int_{\mathcal{M}_{[s_0,s_1]}} (1+\omega_\eta)\omega_\eta'(q) \left((x^a/r)\partial_t u - \partial_a u\right)^2 dxdt.
\end{aligned}$$

This leads us to the energy identity

$$\begin{aligned}
&2\int_{\mathcal{M}_{[s_0,s_1]}} (1+\omega_\eta)^2 \partial_t u \,\Box u \; Jdxds \\
&= \int_{\mathcal{F}_{s_1}} V_\eta \cdot n_{s_1} d\sigma_{s_1} - \int_{\mathcal{F}_{s_0}} V_\eta \cdot n_{s_0} d\sigma_{s_0} \\
&\quad + 2\sum_a \int_{\mathcal{M}_{[s_0,s_1]}} (1+\omega_\eta)\omega'(q) \left((x^a/r)\partial_t u - \partial_a u\right)^2 Jdxds.
\end{aligned}$$

We then differentiate the above equation with respect to s and obtain

$$\begin{aligned}
\int_{\mathcal{M}_s} (1+\omega_\eta)\partial_t u \,\Box u \; Jdx &= \frac{d}{ds}\int_{\mathcal{F}_{s_1}} V_\eta \cdot n_s d\sigma_s \\
&\quad + \sum_a \int_{\mathcal{M}_s} (1+\omega_\eta(q))\omega'(q^\eta) \left((x^a/r)\partial_t u - \partial_a u\right)^2 Jdx \\
&\geq \frac{d}{ds}\int_{\mathcal{F}_{s_1}} V_\eta \cdot n_s d\sigma_s.
\end{aligned} \tag{8.9}$$

Introducing as in (8.6)

$$
\begin{aligned}
E_{\eta,c}(s,u) &= \int_{\mathcal{M}_s} V_\eta \cdot n_s d\sigma_s \\
&= \int_{\mathcal{M}_s} (1+\omega_\eta)^2 \left(|\partial_t u|^2 + \sum_a \big(\partial_a u\big)^2 + \frac{2\xi(s,r)x^a}{\sqrt{s^2+r^2}} \partial_t u \partial_a u \right) \, dx \\
&= \int_{\mathcal{M}_s} (1+\omega_\eta)^2 \left(\frac{s^2+(1-\xi^2(s,r))r^2}{s^2+r^2} |\partial_t u|^2 + \sum_a \left(\frac{\xi(s,r)x^a}{\sqrt{s^2+r^2}} \partial_t u + \partial_a u \right)^2 \right) \, dx
\end{aligned}
$$

or

$$
\begin{aligned}
E_{\eta,c}(s,u) = & \int_{\mathcal{M}_s^{\text{int}}} \left((s/t)^2 |\partial_t u|^2 + \sum_a \left(\frac{x^a}{t} \partial_t u + \partial_a u \right)^2 \right) dx \\
& + \int_{\mathcal{M}_s^{\text{tran}}} \left(\frac{s^2+(1-\xi^2(s,r))r^2}{s^2+r^2} |\partial_t u|^2 + \sum_a \left(\frac{\xi(s,r)x^a}{\sqrt{s^2+r^2}} \partial_t u + \partial_a u \right)^2 \right) \, dx \\
& + \int_{\mathcal{M}_s^{\text{ext}}} (1+\omega_\eta)^2 \left(|\partial_t u|^2 + \sum_a \big(\partial_a u\big)^2 \right) \, dx,
\end{aligned}
$$

we see that

$$
\begin{aligned}
& \int_{\mathcal{M}_s} (1+\omega_\eta)^2 \partial_t u \, \Box u \, J dx \\
& = \left(\int_{\mathcal{M}_s^{\text{int}}} + \int_{\mathcal{M}_s^{\text{tran}} \cup \mathcal{M}_s^{\text{ext}}} \right) \\
& \quad \frac{J\sqrt{s^2+r^2}}{\sqrt{s^2+(1-\xi^2(s,r))r^2}} (1+\omega_\eta)^2 \frac{\sqrt{s^2+(1-\xi^2(s,r))r^2}}{\sqrt{s^2+r^2}} \partial_t u \, \Box u \, J dx.
\end{aligned}
\tag{8.10}
$$

Finally, we use the following bound deduced from (6.25)

$$
\frac{J\sqrt{s^2+r^2}}{\sqrt{s^2+(1-\xi^2(s,r))r^2}} \leq \begin{cases} 1, & \mathcal{M}_s^{\text{int}}, \\ C\big(1+s\sqrt{1-\xi(s,r)}\big), & \mathcal{M}_s^{\text{tran}}, \\ 2s, & \mathcal{M}_s^{\text{ext}}. \end{cases}
\tag{8.11}
$$

Now (8.10) leads to (observing also that $|\omega_\eta(q)| = 1$ on $\mathcal{M}_s^{\rm int}$)

$$
\begin{aligned}
&\int_{\mathcal{M}_s} (1+\omega_\eta)^2 \partial_t u \, \Box u \, J dx \\
&\leq C\, E_{\eta,c}(s,u)^{1/2} \Big(\|\Box u\|_{L^2(\mathcal{M}_s^{\rm int})} + \big\|(1+s(1-\xi_s(r))^{1/2})(1+\omega_\eta)\Box u\big\|_{L^2(\mathcal{M}_s^{\rm tran})} \\
&\qquad\qquad + \|s(1+\omega_\eta)\Box u\|_{L^2(\mathcal{M}_s^{\rm ext})}\Big).
\end{aligned}
\tag{8.12}
$$

Combining the above estimate with (8.9), we arrive at

$$
\begin{aligned}
&\frac{d}{ds} E_{\eta,c}(s,v) \\
&\leq C\, E_{\eta,c}(s,v)^{1/2} \Big(\|\Box u\|_{L^2(\mathcal{M}_s^{\rm int})} \quad + \big\|(1+s(1-\xi_s(r))^{1/2})(1+\omega_\eta)\Box u\big\|_{L^2(\mathcal{M}_s^{\rm tran})} \\
&+ \|s(1+\omega_\eta)\Box u\|_{L^2(\mathcal{M}_s^{\rm ext})}\Big)
\end{aligned}
$$

which leads us to

$$
\begin{aligned}
\frac{d}{ds} E_{\eta,c}(s,v)^{1/2} \leq C \Big(&\|\Box u\|_{L^2(\mathcal{M}_s^{\rm int})} + \big\|(1+s(1-\xi_s(r))^{1/2})(1+\omega_\eta)\Box u\big\|_{L^2(\mathcal{M}_s^{\rm tran})} \\
&+ \|s(1+\omega_\eta)\Box u\|_{L^2(\mathcal{M}_s^{\rm ext})}\Big).
\end{aligned}
$$

It remains to integrate this inequality in s over an interval $[s_0, s_1]$.

Part IV

Sobolev and commutator estimates for EHF

9 Revisiting standard Sobolev's inequalities

9.1 A Sobolev inequality on positive cones. I

With the notation $x = (x^a) \in \mathbb{R}^n$ and $a = 1, \ldots, n$ we introduce the region

$$\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \,/\, x^a \geq 0\}$$

and throughout this Part we restrict attention to sufficiently regular functions defined in $\mathbb{R}^n_+$ and smoothly extendible outside $\mathbb{R}^n_+$. As usual, a function $u : \mathbb{R}^n_+ \to \mathbb{R}$ is said to be *compactly supported* if there exists a real $R > 0$ such that $u(x)$ vanishes for all $|x| \geq R$.

We begin with the following result.

Proposition 9.1. *For all sufficiently regular and compactly supported functions u defined on $\mathbb{R}^3_+$, one has the inequality*

$$\|u\|_{L^6(\mathbb{R}^3_+)} \lesssim \sum_{a=1}^{3} \|\partial_a u\|_{L^2(\mathbb{R}^3_+)}, \tag{9.1}$$

where the implied constant is a universal constant.

Proof. Writing $\partial_a u^6 = 6u^5 \partial_a u$ and integrating this identity with respect to the variable x^a, we obtain

$$u^4 \leq 6 \int_0^{+\infty} |u^3 \, |\partial_a u| \, dx^a. \tag{9.2}$$

Then, introducing the functions

$$\begin{aligned} w^1(x) = w^1(x^2, x^3) &:= \sup_{x^1 \geq 0} |u(x)|^2, \\ w^2(x) = w^2(x^1, x^3) &:= \sup_{x^2 \geq 0} |u(x)|^2, \\ w^3(x) = w^3(x^1, x^2) &:= \sup_{x^3 \geq 0} |u(x)|^2. \end{aligned} \tag{9.3}$$

We thus see that

$$|w^a(x)|^2 \leq 6 \int_0^{+\infty} |u^3| \, |\partial_a u| \, dx^a. \tag{9.4}$$

Next, writing $d\hat{x}^1 = dx^2 dx^3$, $d\hat{x}^2 = dx^1 dx^3$, and $d\hat{x}^3 = dx^1 dx^2$, we can write for each $a = 1, 2, 3$

$$\begin{aligned} \int_{x^b \geq 0, b \neq a} (w^a)^2 d\hat{x}^a &\leq 6 \int_{x^b \geq 0, b \neq a} \int_0^{+\infty} |u^3| \, |\partial_a u| dx^a \\ &= 6 \int_{\mathbb{R}^3_+} |u^3| \, |\partial_a u| dx \\ &\leq 6 \|u^3\|_{L^2(\mathbb{R}^3_+)} \, \|\partial_a u\|_{L^2(\mathbb{R}^3_+)}. \end{aligned} \tag{9.5}$$

On the other hand, we can write

$$\int_{\mathbb{R}^3_+} u^6 dx \leq \int_{\mathbb{R}^3_+} w^1(x) \, w^2(x) \, w^3(x) \, dx. \tag{9.6}$$

We see that

$$\Big| \int_{x_1 \geq 0} w^2(x^1, x^3) \, w^3(x^1, x^2) dx^1 \Big| \leq \|w^2(x^1, x^2)\|_{L^2(\mathbb{R}^+_{x_1})} \, \|w^3(x^1, x^2)\|_{L^2(\mathbb{R}^+_{x_1})} \tag{9.7}$$

and

$$\begin{aligned} &\Big| \int_{x_1, x_2 \geq 0} w^1(x^2, x^3) \, w^2(x^1, x^3) \, w^3(x^1, x^2) dx^1 dx^2 \Big| \\ &\leq \|w^2(x^1, x^3)\|_{L^2(\mathbb{R}^+_{x_1})} \int_{x_2 \geq 0} |w^1(x^2, x^3)| \, \|w^3(x^1, x^2)\|_{L^2(\mathbb{R}^+_{x_1})} dx^2 \\ &\leq \|w^2(x^1, x^3)\|_{L^2(\mathbb{R}^+_{x_1})} \|w^1(x^2, x^3)\|_{L^2(\mathbb{R}^+_{x_2})} \Big\| \|w^3(x^1, x^2)\|^2_{L^2(\mathbb{R}^+_{x_1})} \Big\|_{L^2(\mathbb{R}^+_{x_2})} \\ &= \|w^2(x^1, x^3)\|_{L^2(\mathbb{R}^+_{x_1})} \|w^1(x^2, x^3)\|_{L^2(\mathbb{R}^+_{x_2})} \|w^3(x_1, x_2)\|_{L^2(\mathbb{R}^+_{x^1, x^2})}. \end{aligned} \tag{9.8}$$

Then we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3_+} w^1(x^2,x^3)\, w^2(x^1,x^3)\, w^3(x^1,x^2) dx^1 dx^2 dx^3\Big| \\
&\le \|w^3(x_1,x_2)\|_{L^2(\mathbb{R}^+_{x^1,x^2})} \int_{x^3\ge 0} \|w^2(x^1,x^3)\|_{L^2(\mathbb{R}^+_{x_1})} \|w^1(x^2,x^3)\|_{L^2(\mathbb{R}^+_{x_2})} dx^3 \\
&\le \|w^3(x_1,x_2)\|_{L^2(\mathbb{R}^+_{x^1,x^2})} \Big\| \|w^1(x^2,x^3)\|^2_{L^2(\mathbb{R}^+_{x_2})}\Big\|_{L^2(\mathbb{R}^+_{x^3})} \Big\| \|w^2(x^1,x^3)\|^2_{L^2(\mathbb{R}^+_{x_1})}\Big\|_{L^2(\mathbb{R}^+_{x^3})} \\
&= \|w^1(x_2,x_3)\|_{L^2(\mathbb{R}^+_{x^2,x^3})} \|w^2(x_1,x_3)\|_{L^2(\mathbb{R}^+_{x^1,x^3})} \|w^3(x_1,x_2)\|_{L^2(\mathbb{R}^+_{x^1,x^2})}.
\end{aligned}
$$

Then applying (9.5), we find

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^3_+} w^1(x^2,x^3)\, w^2(x^1,x^3)\, w^3(x^1,x^2) dx^1 dx^2 dx^3\Big| \\
&\le C\|u^3\|_{L^2(\mathbb{R}^3_+)} \|\partial_1 u\|_{L^2(\mathbb{R}^3_+)} \|\partial_2 u\|_{L^2(\mathbb{R}^3_+)} \|\partial_3 u\|_{L^2(\mathbb{R}^3_+)}
\end{aligned}
$$

and combined with (9.6), we obtain

$$
\begin{aligned}
\|u\|_{L^6(\mathbb{R}^3_+)} &\le C \Big(\|\partial_1 u\|_{L^2(\mathbb{R}^3_+)} \|\partial_2 u\|_{L^2(\mathbb{R}^3_+)}\, \|\partial_3 u\|_{L^2(\mathbb{R}^3_+)} \Big)^{1/3} \\
&\le C \sum_a \|\partial_a u\|_{L^2(\mathbb{R}^3_+)}.
\end{aligned}
$$

□

9.2 A Sobolev inequality on positive cones. II

Next, to any point $x \in \mathbb{R}^3_+$ and any scale $\rho > 0$, we associate the cube

$$
C_{\rho,x} := \{y \in \mathbb{R}^3_+ | x^a \le y^a \le x^a + \rho\}. \tag{9.9}
$$

We then establish the following result.

Proposition 9.2. *For all sufficiently regular functions u defined on $\mathbb{R}^3_+$, one has*

$$
|u(x)| \le C(\rho) \sum_{|I|\le 2} \|\partial^I u\|_{L^2(C_{\rho,x})}, \qquad x \in \mathbb{R}^3_+,\ \rho > 0, \tag{9.10}
$$

where $C(\rho) > 0$ is a constant depending upon ρ only.

Proof. Fix a point $x_0 \in \mathbb{R}^3_+$ and consider the cube C_{ρ,x_0}. For all $x \in C_{\rho,x_0}$, we have

$$|u(x_0) - u(x)| \le \int_0^1 |(x - x_0) \cdot \nabla u(x_0 + (x - x_0)t)|\, dt,$$

which leads us to

$$\begin{aligned} \left| u(x_0) - \rho^{-3} \int_{C_{\rho,x_0}} u(x) dx \right| &\le \rho^{-3} \int_{C_{\rho,x_0}} |u(x) - u(x_0)| dx \\ &\le \rho^{-3} \int_{C_{\rho,x_0}} \int_0^1 |(x - x_0) \cdot \nabla u(x_0 + (x - x_0)t)|\, dt dx \\ &= \rho^{-3} \int_0^1 \int_{C_{\rho,x_0}} |(x - x_0) \cdot \nabla u(x_0 + (x - x_0)t)|\, dt dx \end{aligned}$$

and, therefore,

$$\begin{aligned} \left| u(x_0) - \rho^{-3} \int_{C_{\rho,x_0}} u(x) dx \right| &\le C\rho^{-2} \int_0^1 \int_{C_{\rho,x_0}} |\nabla u(x_0 + (x - x_0)t)|\, dx\ dt \\ &= C\rho^{-2} \int_0^1 t^{-3} \int_{C_{t\rho,0}} |\nabla u(x_0 + y)| dy\ dt. \end{aligned} \tag{9.11}$$

We then observe that

$$\int_{C_{t\rho,0}} |\nabla u(x_0 + y)| dy \le \|\nabla u(x_0 + \cdot)\|_{L^6(C_{t\rho,0})} (t\rho)^{5/2} \le \|\nabla u\|_{L^6(C_{\rho,x_0})} (t\rho)^{5/2}$$

and we write

$$\left| u(x_0) - \rho^{-3} \int_{C_{\rho,x_0}} u(x) dx \right| \le C\rho^{1/2} \|\nabla u\|_{L^6(C_{\rho,x_0})}. \tag{9.12}$$

In the same manner we obtain

$$\left| u(x_1) - \rho^{-3} \int_{C_{\rho,x_0}} u(x) dx \right| \le C\rho^{1/2} \|\nabla u\|_{L^6(C_{\rho,x_0})}. \tag{9.13}$$

Combining (9.12) and (9.12), we thus arrive at the inequality:

$$|u(x_0) - u(x_1)| \le C\rho^{1/2} \|\nabla u\|_{L^6(C_{\rho,x_0})}. \tag{9.14}$$

Next, we introduce a cut-off function χ defined on $\mathbb{R}$, which is smooth and satisfy $\chi(x) = 0$ for $x \leq 0$ and $\chi(x) = 1$ for $x \geq 1$. We use this cut-off function in order to construct, from the given function u, an auxiliary function v defined by

$$v_{x_0}(x) := \big(1 - \chi(d^{-2}|x - x_0|^2)\big)\, u(x), \qquad x \in C_{d,x_0}. \tag{9.15}$$

We also set $x_1 := (x_0^1 + d, x_0^2 + d, x_0^3 + d)$ and then we see that

$$v_{x_0}(x_0) = u(x_0), \qquad v_{x_0}(x_1) = 0 \tag{9.16}$$

as well as

$$|v_{x_0}(x_0) - v_{x_0}(x_1)| \leq C(d) \|\nabla v_{x_0}\|_{L^6(C_{d,x_0})}. \tag{9.17}$$

The function $\partial_\alpha v_{x_0}$ is sufficiently regular and by construction is compactly supported in $\mathbb{R}^3_+$, thus we conclude that

$$\begin{aligned}|v_{x_0}(x_0) - v_{x_0}(x_1)| &\leq C(d) \sum_{1 \leq |I| \leq 2} \|\partial^I v_{x_0}\|_{L^2(\mathbb{R}^3_+)} \\ &= C(d) \sum_{1 \leq |I| \leq 2} \|\partial^I v_{x_0}\|_{L^2(C_{d,x_0})}.\end{aligned}$$

This establishes the desired result since, in view of (9.15), the norm of v can be bounded by the norm of u, namely

$$\sum_{1 \leq |I| \leq 2} \|\partial^I v_{x_0}\|_{L^2(C_{d,x_0})} \leq C \sum_{|I| \leq 2} \|\partial^I u\|_{L^2(C_{d,x_0})}.$$

□

10 Sobolev inequalities for EHF

10.1 Notation

We introduce the following regions in $\mathbb{R}^3$:

$$\begin{aligned}D_{\mathrm{trs}}(s) &:= \big\{x \in \mathbb{R}^3 \,/\, |x| \geq -1 + s^2/2\big\}, \\ D_{\mathrm{ext}}(s) &:= \big\{x \in \mathbb{R}^3 \,/\, |x| \geq s^2/2\big\},\end{aligned} \tag{10.1}$$

consider functions defined on $D_{\mathrm{trs}}(s)$ or on $D_{\mathrm{ext}}(s)$, which are assumed to be sufficiently regular and extendible outside their domain of definition. As usual, such a function u is said to be compactly supported, if there exists a real $R \geq -1 + s^2/2$ or a real $R \geq s^2/2$ such that u vanishes outside the ball $\{|x| > R\}$.

Recall our expressions of the boosts and rotations generated by the vector fields:

$$L_a = x^a \partial_t + t\partial_a, \qquad \Omega_{ab} = x^a \partial_b - x^b \partial_a, \qquad a, b = 1, 2, 3. \tag{10.2}$$

In the parametrization (s, x^a) of $\mathcal{M}_{[s_0,+\infty)}$ introduced earlier, the associated canonical frame is

$$\partial_s = \frac{\partial t}{\partial s}\,\partial_t, \qquad \overline{\partial}_a = \partial_a + \frac{\partial t}{\partial x^a}\partial_t.$$

Observe that, in the interior domain $\{r \leq -1 + s^2/2\} \cap \mathcal{M}_{[s_0\infty)}$, one has

$$\begin{aligned}\overline{\partial}_s &= (s/t)\partial_t,\\ \overline{\partial}_a &= \partial_r t(x^a/t)\partial_t + \partial_a = \frac{x^a}{t}\partial_t + \partial_a.\end{aligned} \tag{10.3}$$

First of all, by combining Propositions 9.1 and 9.2 together, we deduce the following result.

Proposition 10.1. *For any sufficiently regular and compactly supported function u defined on $D_{trs}(s)$ or $D_{ext}(s)$, respectively, one has*

$$\begin{aligned}\sup_{D_{trs}(s)} |u| &\leq C \sum_{|I|\leq 2} \|\partial^I u\|_{L^2(D_{trs}(s))},\\ \sup_{D_{ext}(s)} |u| &\leq C \sum_{|I|\leq 2} \|\partial^I u\|_{L^2(D_{ext}(s))},\end{aligned} \tag{10.4}$$

respectively

10.2 A sup-norm Sobolev estimate

We begin with the following preliminary inequality.

Proposition 10.2. *For any sufficiently regular function u defined on $D_{trs}(s)$ or $D_{ext}(s)$, respectively, the following inequalities hold:*

$$
\begin{aligned}
|u(x)| &\leq C(1+|x|^{-1}) \sum_{|I|+|J|\leq 2} \|\partial^I \Omega^J u\|_{L^2(D_{trs}(s))}, \quad |x| \geq -1 + s^2/2, \\
|u(x)| &\leq C(1+|x|^{-1}) \sum_{|I|+|J|\leq 2} \|\partial^I \Omega^J u\|_{L^2(D_{ext}(s))}, \qquad |x| \geq s^2/2,
\end{aligned}
\tag{10.5}
$$

respectively.

Proof. We only treat the case of $D_{\text{ext}}(s)$, since the case of $D_{\text{trs}}(s)$ is completely analogous. For a point $x_0 \in D_{\text{ext}}(s)$, without loss of generality we can suppose that $x_0 = (r_0, 0, 0)$ (by a rotation). Then we consider the standard spherical coordinates, and we consider the following region:

$$
R_{x_0} := \big\{ r_0 \leq r \leq r_0 + 1,\ 0 \leq \theta \leq \pi/6,\ \pi/3 \leq \varphi \leq \pi/2 \big\}. \tag{10.6}
$$

Case $r_0 \geq 1$. We focus first on the difficut case $r_0 \geq 1$ and we consider the function

$$
v_{x_0}(r, \theta, \varphi) := u(x^1, x^2, x^3)
$$

with

$$
x^1 = r \sin\varphi \, \cos\theta, \qquad x^2 = r \sin\varphi \, \sin\theta, \qquad x^3 = r\cos\varphi.
$$

Observe that

$$
\partial_\theta = \sin\varphi \, \Omega_{12}, \qquad \partial_\varphi = -\sin\varphi \, \Omega_{23} - \cos\theta \, \Omega_{13},
$$

$$
\partial_\theta \partial_\theta = \sin^2\varphi \, \Omega_{12}\Omega_{12},
$$

$$
\partial_\theta \partial_\varphi = -\sin^2\varphi \, \Omega_{12}\Omega_{23} + \sin\theta \cdot \Omega_{13} - \sin\varphi\cos\theta \, \Omega_{12}\Omega_{13},
$$

$$
\begin{aligned}
\partial_\varphi \partial_\varphi = {} & \sin^2\varphi \, \Omega_{23}\Omega_{23} + \sin\varphi\cos\theta \, \Omega_{12}\Omega 23 \\
& + \cos\theta \sin\varphi \, \Omega_{23}\Omega_{13} + \cos^2\theta \, \Omega_{13}\Omega_{13} - \cos\varphi \, \Omega_{23}.
\end{aligned}
$$

Now we observe the function v_{x_0} taking (r, θ, ϕ) as its variables with $\phi = \pi/2 - \varphi$, is a function defined in the truncated positive cone

$$
\big\{ r_0 \leq r \leq r_0 + 1,\ 0 \leq \theta \leq \pi/6,\ 0 \leq \phi \leq \pi/6 \big\}.
$$

Thus by Proposition 9.2 we have

$$|u(x_0)| = |v_{x_0}(0)| \le C \sum_{|I|\le 2} \|\partial^I v_{x_0}\|_{L^2(C_{1/2,0})}. \tag{10.7}$$

And we observe that

$$|\nabla v_{x_0}| \le C \sum_{\alpha} |\partial_\alpha u| + C \sum_{a\ne b} |\Omega_{ab} u|$$

and

$$|\partial^{I'} v_{x_0}| \le C \sum_{|I|+|J|\le 2} |\partial^I \Omega^J u|, \qquad |I'| \le 2.$$

On the other hand, we see that for a

$$\begin{aligned}
\|\partial^{I'} v_{x_0}\|^2_{L^2(\mathbb{R}^3_{+,1/2})} &= \int_{\mathbb{R}^3_{+,1/2}} |\partial^{I'} v_{x_0}(r,\theta,\phi)|^2 \, drd\theta d\phi \\
&\le \sum_{|I|+|J|\le 2} \int_{\mathbb{R}^3_{+,1/2}} |\partial^I \Omega^J u|^2 \, drd\theta d\varphi \\
&\le C r_0^{-2} \sum_{|I|+|J|\le 2} \int_{R_{x_0}} |\partial^I \Omega^J u|^2 \, r^2 \sin\varphi drd\theta d\varphi,
\end{aligned}$$

where we have used that, in R_{x_0}, $1 \le r/r_0 \le 2$ and that $\sqrt{3}/2 \le \sin\varphi \le 1$. Then we see that

$$\begin{aligned}
\|\partial^{I'} v_{x_0}\|^2_{L^2(\mathbb{R}^3_{+,1/2})} &\le C r_0^{-2} \sum_{|I|+|J|\le 2} \|\partial^I \Omega^J u\|^2_{L^2(R_{x_0})} \\
&\le C(1+r_0)^{-2} \sum_{|I|+|J|\le 2} \|\partial^I \Omega^J u\|^2_{L^2(D_{\text{ext}}(s))}.
\end{aligned}$$

In combination with (10.7), the desired result is established for $r_0 \ge 1$.

Case $r_0 \le 1$. This region is treated by a standard Sobolev inequality and details are omitted. □

10.3 A Sobolev inequality for the transition and exterior domains

Proposition 10.3 (Global Sobolev inequality for the transition and exterior domains)**.** *For all sufficiently regular functions defined on $\mathcal{M}_{[s_0,s_1]}$ with $2 \le s_0 \le s \le s_1$, one has* *for all* $x \in \mathcal{M}_s^{tran} \cup \mathcal{M}_s^{ext}$

$$|u(x)| \leq C(1+r)^{-1} \sum_{|I|+|J| \leq 2} \|\overline{\partial}_x^I \Omega^J\|_{L^2(\mathcal{M}_s^{tran} \cup \mathcal{M}_s^{ext})}, \tag{10.8}$$

and for all $x \in \mathcal{M}_s^{ext}$

$$|u(x)| \leq C(1+r)^{-1} \sum_{|I|+|J| \leq 2} \|\partial_x^I \Omega^J\|_{L^2(\mathcal{M}_s^{ext})}. \tag{10.9}$$

Here, $\overline{\partial}_x^I$ *denotes any* $|I|$*-order operator determined from the fields* $\{\overline{\partial}_a\}_{a=1,2,3}$*, while* ∂_x^I *denotes any a* $|I|$*-order operator determined from the fields* $\{\partial_a\}_{a=1,2,3}$*.*

Proof. We consider the parametrization (s,r) of $\mathcal{M}_{[s_0,+\infty)}$, and recall that on $\mathcal{M}_s^{\text{tran}} \cup \mathcal{M}_s^{\text{ext}}$, s is constant and $t = T(s,r)$, $-1 + s^2/2 \leq r$. We consider the restriction of u on $\mathcal{M}_s^{\text{tran}} \cup \mathcal{M}_s^{\text{ext}}$, that is, the function

$$v_s(x) = u(T(s,r), x) \tag{10.10}$$

and we remark the relations

$$\begin{aligned} \partial_a v_s &= \overline{\partial}_a u = (x^a/r)\frac{\partial t}{\partial r}\partial_t u + \partial_a u = \frac{\xi_s(r) x^a}{t}\partial_t u + \partial_a u, \\ \partial_b \partial_a v_s &= \overline{\partial}_b \overline{\partial}_a u, \\ \Omega_{ab} v_s &= (x^a \partial_b - x^b \partial_a) u = \Omega_{ab} u. \end{aligned} \tag{10.11}$$

Now we apply Proposition 10.2 on v_s, and we see that (10.8) is established, while (10.9) is established in the same manner. □

10.4 A Sobolev inequality for the interior domain

Proposition 10.4 (Global Sobolev inequality for interior domain). *For all sufficiently regular functions defined in a neighborhood of the hypersurface* $\mathcal{M}_s^{int}$*, the following estimate holds*

$$t^{3/2}\,|u(x)| \leq \sum_{|J| \leq 2} \|L^J u\|_{L^2(\mathcal{M}_s^{int})}, \qquad x \in \mathcal{M}_s^{int} \tag{10.12}$$

for some uniform constant $C > 0$*.*

Proof. We consider the restriction of the function u on the hyperboloid $\mathcal{H}_s$ with $|x| \leq -1 + s^2/2$:

$$v_s(x) := u(\sqrt{s^2 + r^2}, x).$$

Then we see that

$$\partial_a v_s = \overline{\partial}_a u = t^{-1} L_a u = (s^2 + r^2)^{-1/2} L_a u.$$

Take a $x_0 \in \mathcal{H}_s$, with out loss of generality, we can suppose that $x_0 = -3^{-1/2}(r_0, r_0, r_0)$. We consider the positive cone $C_{s/2,x_0} \subset \{|x| \leq -1 + s^2/2\}$. (Recall that $s \geq 2$ is assumed throughout.)

In this cone we introduce the following change of variable:

$$y^a := s^{-1}(x^a - x_0^a).$$

and we define

$$w_{s,x_0}(y) := v_s(sy + x_0), \quad y \in C_{1/2,0}. \tag{10.13}$$

Therefore, we obtain

$$\begin{aligned} \partial_a w_{s,x_0} &= s\partial_a v_s = \frac{s}{\sqrt{s^2 + r^2}} L_a u, \\ \partial_b \partial_a w_{s,x_0} &= \frac{s^2}{s^2 + r^2} L_b L_a u - s^2 x^b (s^2 + r^2)^{-3/2} L_a u. \end{aligned} \tag{10.14}$$

Thus, for $|I| \leq 2$, we obtain

$$|\partial^I w_{s,x_0}| \leq C \sum_{|J| \leq 2} |L^J u|.$$

Then by Proposition 9.2, we see that

$$|w_{s,x_0}(0)|^2 \leq C \sum_{|I| \leq 2} \int_{C_{1/2,0}} |\partial^I w_{s,x_0}|^2 dy = C s^{-3} \sum_{|J| \leq 2} \int_{C_{1/2,0}} |L^J u|^2 dx,$$

which leads to

$$|u(x_0)| \leq C s^{-3/2} \|L^J u\|_{L^2(\mathcal{H}_s)}. \tag{10.15}$$

On the other hand, when $r_0 \geq 1$, we consider the cone $C_{r_0/2,x_0}$ and we introduce the function

$$w_{x_0}(y) := v_s(r_0 y + x_0), \quad y \in C_{1/2,0}.$$

It is clear that

$$\partial_a w_{x_0} = \frac{r_0}{\sqrt{r^2+s^2}} L_a u \tag{10.16}$$

and

$$\partial_b \partial_a w_{x_0} = \frac{r_0^2}{r^2+s^2} L_b L_a u - r_0^2 x^b (s^2+r^2)^{-3/2} L_a u.$$

In the cube $C_{r_0/2,x_0}$ one has $r \geq \frac{\sqrt{3}}{2} r_0$. Thus for $|I| \leq 2$ we find

$$|\partial^I w_{x_0}| \leq \sum_{|J|\leq 2} |L^J u|.$$

Then by Proposition 9.2, we have

$$\begin{aligned} u(x_0)|^2 = |w_{x_0}(0)|^2 &\leq C \sum_{|J|\leq 2} \int_{C_{1/2,0}} |\partial^I w_{x_0}|^2 dy \\ &= C r_0^{-3} \sum_{|J|\leq 2} \int_{C_{1/2,0}} |L^J u|^2 dx, \end{aligned}$$

which leads us to

$$|u(x_0)| \leq C r_0^{-3/2} \sum_{|J|\leq 2} \|L^J u\|_{L^2(\mathcal{H}_s)}. \tag{10.17}$$

When $r_0 \leq 1$, we have $\sqrt{s^2+r_0^2} \leq 2s$, thus it remains to combine (10.16) with (10.17) and the desired result is proved. □

11 Commutators in the interior domain. A summary

11.1 Objective

An essential role will be played by the commutators $[X, Y]u := X(Yu) - Y(Xu)$ of certain operators X, Y, applied to function u and associated with the Euclidian-hyperboloidal foliation and the wave operators. We therefore present here various results concerning such commutators, by distinguishing between the exterior and interior domains.

Considering commutators involving the translations, Lorentz boosts, and hyperboloidal frame vectors, we will begin by deriving the estimates in $\mathcal{M}^{\text{int}}$, since this is comparatively easier and, in order to derive suitably uniform bounds, we will rely on homogeneity arguments and on the observation that all the coefficients in the following decomposition are smooth in $\mathcal{M}^{\text{int}}$. On the other hand, dealing with $\mathcal{M}^{\text{ext}}$ is more involved and require to introduce suitable weights, as we discuss below.

First of all, all admissible vector fields ∂_α, L_a under consideration are Killing fields for the flat wave operator $\Box$, so that the following commutation relations hold:

$$[\partial_\alpha, \Box] = 0, \qquad [L_a, \Box] = 0. \tag{11.1}$$

11.2 Commutators for vector fields in the interior domain

We find it convenient to summarize our results in this section first for $\mathcal{M}^{\text{int}}$, before providing a detailed proof covering both the interior and the exterior domains in the next section. All the estimates this section are restricted to $\mathcal{M}^{\text{int}}$.

We now analyze the relevant commutators of the vector fields under consideration (in the future of the hypersurface $\mathcal{H}_{s_0}$). On one hand, we need to decompose the differential operators of interest in the frame ∂_t, ∂_a and, on the other hand, we will be using the translations ∂^I and the boosts L_a^J. In any given equation or inequality, the contribution due to the commutators have better decay (compared to the vector fields under consideration): they either involve good derivatives (tangential to the hyperboloids) or contain a weight providing better decay in time.

- **Commutators of the semi-hyperboloidal frame:**

$$[\partial_t, \underline{\partial}_a] = -\frac{x_a}{t^2}\partial_t, \qquad [\underline{\partial}_a, \underline{\partial}_b] = 0. \tag{11.2}$$

- **Commutators for the boosts and tangent fields:**

$$[L^I, \underline{\partial}_b] = \sum_{|J,K|<|I|} \sigma_{bJ}^{Ia} \underline{\partial}_a L^J, \qquad \left|\partial^{I_1} L^{J_1} \sigma_{bJ}^{Ia}\right| \lesssim t^{-|I_1|}. \tag{11.3}$$

- **Commutators for the translations and tangent fields:**

$$[\partial^I, \underline{\partial}_a] = t^{-1} \sum_{|J,K|\leq|I|} \rho_{aJ}^I \partial^J, \qquad \left|\partial^{I_1} L^{J_1} \rho_{aJ}^I\right| \lesssim t^{-|I_1|}. \tag{11.4}$$

- **Commutators for the translations and boosts:**

$$
\begin{aligned}
&[\partial^I, L_a] = \sum_{|J,K|\le|I|} \xi_{aJ}^I \partial^J, && \xi_{aJ}^I \text{ being constant,}\\
&[L^I, \partial_\alpha] = \sum_{\substack{|J,K|<|I|\\ \gamma=0,\dots,3}} \theta_{\alpha J}^{I\gamma} \partial_\gamma L^J, && \theta_{\alpha J}^{I\gamma} \text{ being constant.}
\end{aligned}
\tag{11.5}
$$

- **Commutators of the semi-hyperboloidal frame and the admissible fields:**

$$
[\partial^I L^J, \underline{\partial}_\alpha] = \sum_{\substack{|J'|\le|J|\\ |I'|\le|I|\\ \beta=0,\dots,3}} \underline{\theta}_{\alpha I'J'}^{IJ\beta} \partial_\beta \partial^{I'} L^{J'}
\tag{11.6}
$$

and

$$
\left|\partial^{I_1} L^{J_1} \underline{\theta}_{\alpha I'J'}^{IJ\beta}\right| \lesssim \begin{cases} t^{-|I_1|}, & |J'|<|J|,\\ t^{-|I_1|-1}, & |I'|<|I|,\\ 0, & \text{otherwise.} \end{cases}
\tag{11.7}
$$

Next from the decompositions above, for the operators $\partial^I L^J$ we deduce that

$$
\left|[\partial^I L^J, \underline{\partial}_a]u\right| \lesssim \sum_{\substack{|J'|<|J|\\ |I'|\le|I|}} \sum_{b=1,2,3} |\underline{\partial}_b \partial^{I'} L^{J'} u| + t^{-1} \sum_{\substack{|I|\le|I'|\\ |J,K|\le|J'|}} |\partial^{I'} L^{J'} u|,
\tag{11.8}
$$

$$
\left|[\partial^I L^J, \partial_\alpha]u\right| \lesssim \sum_{\substack{|J'|<|J|\\ \beta=0,\dots,3}} |\partial_\beta \partial^I L^{J'} u|,
\tag{11.9}
$$

and

$$
\left|[\partial^I L^J, \underline{\partial}_\alpha]u\right| \lesssim t^{-1} \sum_{\substack{|I'|<|I|\\ |J'|\le|J|}} \sum_{\beta=0,\dots,3} \left|\partial_\beta \partial^{I'} L^{J'} u\right| + \sum_{\substack{|I'|\le|I|\\ |J'|<|J|}} \sum_{\beta=0,\dots,3} \left|\partial_\beta \partial^{I'} L^{J'} u\right|.
\tag{11.10}
$$

Observe again that the terms in the right-hand sides contain either fewer derivatives in L or a favorable factor $1/t$.

11.3 Commutators and estimates for second-order operators

Similarly, still in the interior region, for the second-order operators $\partial_\alpha \partial_\beta$ and $\underline{\partial}_\alpha \underline{\partial}_\beta$, we find

$$\big|[\partial^I L^J, \partial_\alpha \partial_\beta] u\big| \lesssim \sum_{\substack{|I| \le |I'| \\ |J'| < |I|}} \sum_{\gamma, \gamma' = 0, \ldots, 3} \big|\partial_\gamma \partial_{\gamma'} \partial^{I'} L^{J'} u\big| \tag{11.11}$$

and

$$\begin{aligned} & \big|[\partial^I L^J, \underline{\partial}_a \underline{\partial}_\beta] u\big| + \big|[\partial^I L^J, \underline{\partial}_\alpha \underline{\partial}_b] u\big| \\ & \lesssim \sum_{\substack{|I'| \le |I| \\ |J'| < |J|}} \sum_{\substack{c=1,2,3 \\ \gamma = 0, \ldots, 3}} \big|\underline{\partial}_c \underline{\partial}_\gamma \partial^{I'} L^{J'} u\big| + \sum_{\substack{|I'| < |I| \\ |J'| \le |J|}} \sum_{\substack{c=1,2,3 \\ \gamma = 0, \ldots, 3}} t^{-1} \big|\underline{\partial}_c \underline{\partial}_\gamma \partial^{I'} L^{J'} u\big| \\ & \quad + \sum_{\substack{|I'| \le |I| \\ |J'| \le |J|}} \sum_{\gamma = 0, \ldots, 3} t^{-1} \big|\partial_\gamma \partial^{I'} L^{J'} u\big|. \end{aligned} \tag{11.12}$$

We now derive further identities involving the semi-hyperboloidal frame and the boosts. These inequalities allow us to replace tangential derivatives (arising in our decomposition of the wave and Klein-Gordon operators) by Lorentz boosts (arising in our definition of the high-order energy). First of all, since $\underline{\partial}_a = t^{-1} L_a$, we write

$$\partial^I L^J \underline{\partial}_a u = \partial^I L^J \big(t^{-1} L_a u\big) = \sum_{\substack{I_1 + I_2 = I \\ J_1 + J_2 = J}} \partial^{I_1} L^{J_1} \big(t^{-1}\big) \partial^{I_2} L^{J_2} L_a u,$$

therefore we have proven the following.

Lemma 11.1. *An expression involving the semi-hyperbloidal derivatives $\underline{\partial}_a$ can be controlled from the boosts and translations, as follows:*

$$\big|\partial^I L^J \underline{\partial}_a u\big| \lesssim t^{-1} \sum_{\substack{|I'| \le |I| \\ |J'| \le |J|}} \big|\partial^{I'} L^{J'} L_a u\big|. \tag{11.13}$$

It follows that, with now one tangential derivative and one arbitrary derivative,

$$\big|\partial^I L^J \underline{\partial}_a \underline{\partial}_\nu u\big| \lesssim t^{-1} \sum_{\substack{|I'| \le |I| \\ |J'| \le |J|}} \big|\partial^{I'} L^{J'} L_a \underline{\partial}_\nu u\big| = t^{-1} \sum_{\substack{|I'| \le |I| \\ |J'| \le |J|}} \big|\partial^{I'} L^{J'} L_a \big(\underline{\Phi}_\nu^{\nu'} \partial_{\nu'} u\big)\big|,$$

in which the coefficients $\underline{\Phi}_\nu^{\nu'}$ are homogeneous of degree 0, implying the following result.

Lemma 11.2. *An expression involving a tangential derivative and a semi-hyperboloidal derivative can be controlled as follows:*

$$\big|\partial^I L^J \underline{\partial}_a \underline{\partial} u\big| + \big|\partial^I L^J \underline{\partial}\underline{\partial}_a u\big| \lesssim t^{-1} \sum_{\substack{|I'|\leq|I| \\ |J'|\leq|J|}} |\partial^{I'} L^{J'} L_a \partial u| \tag{11.14}$$

Finally, with two tangential derivatives, we find

$$\begin{aligned}
&\partial^I L^J \big(\underline{\partial}_a \underline{\partial}_b u\big) = \partial^I L^J \big(t^{-1} L_a (t^{-1} L_b) u\big) \\
&= \partial^I L^J \big(t^{-2} L_a L_b u\big) + \partial^I L^J \big(t^{-1} L_a (t^{-1}) u\big) \\
&= \sum_{\substack{I_1+I_2=I \\ J_1+J_2=J}} \partial^{I_1} L^{J_1} \big(t^{-2}\big) \partial^{I_2} L^{J_2} L_a L_b u + \sum_{\substack{I_1+I_2=I \\ J_1+J_2=J}} \partial^{I_1} L^{J_1} \big(t^{-1} L_a (t^{-1})\big) \partial^{I_2} L^{J_2} L_b u,
\end{aligned}$$

so that (improving upon (11.14), since we now consider only tangential derivatives) we reach the following conclusion.

Lemma 11.3. *An expression involving two tangential derivative can be controlled as follows:*

$$\big|\partial^I L^J \big(\underline{\partial}_a \underline{\partial}_b u\big)\big| \lesssim t^{-2} \sum_{\substack{|I'|\leq|I| \\ |J'|\leq|J|}} \big|\partial^{I'} L^{J'} L_a L_b u\big| + t^{-2} \sum_{\substack{|I'|\leq|I| \\ |J'|\leq|J|}} \big|\partial^{I'} L^{J'} L_b u\big|. \tag{11.15}$$

12 A general framework for the commutator estimates on the EHF

12.1 Basic commutation relations and homogeneous functions

We now present a general framework for establishing commutator estimates valid for the Euclidian-hyperboloidal foliation. We point out the following relations:

$$\begin{aligned}
&[\partial_t, L_b] = \partial_b, \qquad && [\partial_a, L_b] = \delta_{ab} \partial_t, \\
&[\partial_t, \Omega_{bc}] = 0, && [\partial_a, \Omega_{bc}] = \delta_{ab} \partial_c - \delta_{ac} \partial_b, \\
&[L_a, \Omega_{bc}] = \delta_{ab} L_c - \delta_{ac} L_b.
\end{aligned} \tag{12.1}$$

In $\{r < t-1\} \cap \{t > 1\}$ one has

$$
\begin{aligned}
&[(s/t)\partial_t, L_b] = (s/t)\underline{\partial}_b, \\
&[\underline{\partial}_a, L_b] = (x^a/t)\underline{\partial}_b, \\
&[(s/t)\partial_t, \Omega_{bc}] = 0, \\
&[\underline{\partial}_a, \Omega_{bc}] = \delta_{ab}\frac{t-x^b}{t}\underline{\partial}_c - \delta_{ac}\frac{t-x^c}{t}\underline{\partial}_b + \delta_{ac}\frac{(t-x^c)x^b}{t^2} - \delta_{ab}\frac{(t-x^b)x^c}{t^2}\partial_t.
\end{aligned}
\tag{12.2}
$$

In $\{r > t/2\} \cap \{t > 1\}$ one has

$$
\begin{aligned}
[\frac{r-t}{r}\partial_t, L_b] &= \frac{r-t}{r}\widetilde{\partial}_b + \frac{t}{r}\frac{r-t}{r}\partial_t, \\
[\widetilde{\partial}_a, L_b] &= \frac{x^a}{r}\widetilde{\partial}_b + \left(\delta_{ab} - \frac{x^a x^b}{r^2}\right)\frac{r-t}{r}\partial_t, \\
[\frac{r-t}{r}\partial_t, \Omega_{bc}] &= 0, \\
[\widetilde{\partial}_a, \Omega_{bc}] &= \delta_{ca}\widetilde{\partial}_b - \delta_{cb}\widetilde{\partial}_a.
\end{aligned}
\tag{12.3}
$$

For convenience, we introduce the following families of vector fields:

$$
\begin{aligned}
&\mathscr{T} := \{\partial_\alpha\}, \qquad \underline{\mathscr{T}} := \{(s/t)\partial_t, \underline{\partial}_a\}, \qquad \widetilde{\mathscr{T}} := \{(r-t)r^{-1}\partial_t, \widetilde{\partial}_a\} \\
&\mathscr{T}_x := \{\partial_a\}, \qquad \underline{\mathscr{T}}_x := \{\underline{\partial}_a\}, \qquad \widetilde{\mathscr{T}_x} := \{\widetilde{\partial}_a\}, \\
&\mathscr{L} := \{L_a\}, \qquad \mathscr{R} := \{\Omega_{ab}\}.
\end{aligned}
$$

We also introduce a special class of smooth functions defined in the interior domain $\{t-1 > r\}$ or in the exterior domain $\{r > t/2\}$.

Definition 12.1 (Homogeneous functions in the exterior domain)**.** *A smooth function u defined in the region $\{r > t/2\} \cap \{t > 0\}$ is called homogeneous of degree k in the exterior domain (EH of degree k for short) if u satisfies the following two properties:*

$$
\begin{aligned}
u(\lambda t, \lambda x) &= \lambda^k u(t,x), \qquad \lambda > 0, \\
|\partial^I u(t,\omega)| &\le C(I), \qquad \omega \in \mathbb{S}^2,\ 0 < t < 2.
\end{aligned}
$$

The set of EH functions of degree k is denoted by EH_k. We also denote by

$$
\mathcal{EH}_k := \bigcup_{j \le k} EH_j.
$$

An example of EH functions is given by the functions x^a/r, which are of degree zero.

Definition 12.2 (Homogeneous functions in the interior domain)**.** *A smooth function u defined in the region $\{r < t\} \cap \{t > 1\}$ is called homogeneous of degree k in the interior domain (IH of degree k for short) if u satisfies the following two properties:*

$$\begin{aligned} u(\lambda t, \lambda x) &= \lambda^k u(t,x), \qquad \lambda > 0, \\ |\partial^I u(2, 2x)| &\leq C(I), \qquad |x| < 1. \end{aligned}$$

The set of IH functions of degree k is denoted by IH_k and one also write

$$\mathcal{IH}_k := \bigcup_{j \leq k} IH_j.$$

An example of IH functions is the functions x^a/t, which are of degree zero. Then we discuss the relation between homogeneous functions and vector fields.

Lemma 12.3. *One has*

$$f \in EH_k, \quad A \in \mathscr{L} \cup \mathscr{R}, Af \in EH_k,$$

and

$$f \in IH_k, \quad A \in \mathscr{L} \cup \mathscr{R}, Af \in IH_k.$$

Furthermore, one has

$$f \in EH_k, \quad A \in \mathscr{T} \cup \underline{\mathscr{T}}_x \cup \widetilde{\mathscr{T}_x}, Af \in EH_{k-1},$$

and

$$f \in IH_k, \quad A \in \mathscr{T} \cup \underline{\mathscr{T}}_x \cup \widetilde{\mathscr{T}_x}, Af \in IH_{k-1},$$

Lemma 12.4. *Let u be EH of degree k. Then there exists a positive constant C, determined by I, J, K and u such that the following estimate holds in $\{r > t/2\} \cap \{t > 1\}$:*

$$|\partial^I \Omega^K L^J u| \leq C r^{-|I|}. \tag{12.4}$$

Furthermore, $\partial^I \Omega^K L^J u$ is EH of degree $k - |I|$.

Proof. Differentiate the following identity

$$u(\lambda t, \lambda x) = \lambda^k u(t,x),$$

we obtain

$$\partial_\alpha u(\lambda t, \lambda x) = \lambda^{k-1} u(t,x) \tag{12.5}$$

that is, $\partial_\alpha u$ is EH of degree $k-1$. This fact leads to

$$L_a u(\lambda t, \lambda x) = \lambda^k L_a u(t,x), \quad \Omega_{ab} u(\lambda t, \lambda x) = \lambda^k \Omega u(t,x). \tag{12.6}$$

That is, $L_a u$ and Ωu are EH of degree k. Thus it is direct by recurrence that $\partial^I \Omega^K L^J u$ is EH of degree $k - |I|$.

Now for a EH function u of degree k, we observe that for $r > t/2 > 0$,

$$u(t,x) = r^k u(t/r, x^a/r)$$

and $0 < t/r < 2$, $(x^a/r)_{a=1,2,3} \in \mathbb{S}^2$, thus bounded. So (12.4) is established. □

12.2 Decompositions of commutators. I

Definition 12.5. *Let D be a region of $\mathbb{R}^4$. Suppose that $\mathscr{A} = \{A_\alpha\}_{\alpha\in\mathfrak{I}}$ and $\mathscr{B} = \{B_\beta\}_{\beta\in\mathfrak{J}}$ be two families of vector field (with $\mathfrak{I}, \mathfrak{J}$ finite sets for index) defined on D:*

$$A_\alpha \in \mathscr{A}, \qquad B_\beta \in \mathscr{B}, \qquad [A_\alpha, B_\beta] = \Gamma_{\alpha\beta}^\gamma A_\gamma,$$

in which the coefficients $\Gamma_{\alpha\beta}^\gamma \in$ are constant, then one says that $\mathscr{B} \prec \mathscr{A}$.

It is clear that in $D = \mathbb{R}^4$, $\mathscr{R} \prec \mathscr{L}$, $\mathscr{R} \prec \mathscr{T}_x$, $\mathscr{R} \prec \mathscr{T}$ and $\mathscr{L} \prec \mathscr{T}$. Then we have the following result.

Lemma 12.6. *Let $A_\alpha \in \mathscr{A}$, B^J be a $J-th$ order derivative composed with operators in $\mathscr{B}$. Suppose that $\mathscr{B} \prec \mathscr{A}$, then*

$$[A_\alpha, B^J] = \sum_{\substack{\beta\in\mathfrak{I}\\ |J'|<|J|}} \Gamma_{\alpha J'}^{J\beta} A_\beta B^{J'},$$

where the coefficients $\Gamma_{\alpha J'}^{J\beta}$ are constants.

Proof. This result is proven by induction as follows on $|J|$. Observe that for $|J| = 1$ it is guaranteed by the definition (12.9). Now we consider

$$\begin{aligned}
[A_\alpha, B_\beta B^J] &= [A_\alpha, B_\beta]B^J + B_\beta\big([A_\alpha, B^J]\big) \\
&= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I} \\ |J'|<|J|}} B_\beta \left(\Gamma^{J\gamma}_{\alpha J'} A_\gamma B^{J'}\right) \\
&= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \cdot B_\beta A_\gamma B^{J'} \\
&= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \cdot A_\gamma B_\beta B^{J'} + \sum_{\substack{\gamma\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \cdot [B_\beta, A_\gamma] B^{J'},
\end{aligned}$$

so

$$[A_\alpha, B_\beta B^J] = \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \cdot A_\gamma B_\beta B^{J'} - \sum_{\substack{\gamma,\delta\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \Gamma^\delta_{\gamma\beta} A_\delta B^{J'}.$$

This concludes our proof by induction. □

Lemma 12.7. *Let A^I, B^J be $I-$th and $J-$th order derivatives composed with operators in $\mathscr{A}$ and $\mathscr{B}$ respectively. Suppose that $\mathscr{B} \prec \mathscr{A}$. Then*

$$[A^I, B^J] = \sum_{\substack{|I'|=|I| \\ |J'|<|J|}} \Gamma^{IJ}_{I'J'} A^{I'} B^{J'}, \tag{12.7}$$

where the coefficients $\Gamma^{IJ}_{I'J'}$ are constants.

Proof. This result is proven by induction as follows on I. The case $|I| = 1$ is guaranteed by lemma 12.6. Then we consider the following calculation:

$$[A^I A_\alpha, B^J] = A^I[A_\alpha, B^J] + [A^I, B^J]A_\alpha = \sum_{\substack{\beta\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{\beta J}_{\alpha J'} A^I A_\beta B^{J'} + \sum_{\substack{|I'|=|I| \\ |J'|<|J|}} \Gamma^{IJ}_{I'J'} A^{I'} B^{J'} A_\alpha.$$

For the second term, we find

$$A^{I'} B^{J'} A_\alpha = A^{I'} A_\alpha B^{J'} - \sum_{\substack{\beta\in\mathfrak{I} \\ |J''|<|J'|}} \Gamma^{J'\beta}_{\alpha J''} A^{I'} A_\beta B^{J''}.$$

Thus the desired result is established. □

Now we recall the relation (12.1) and see that in $D = \mathbb{R}^4$, $\mathscr{R} \prec \mathscr{L}$, $\mathscr{R} \prec \mathscr{T}_x$, $\mathscr{R} \prec \mathscr{T}$ and $\mathscr{L} \prec \mathscr{T}$. Then we establish the following relation:

$$\begin{aligned}
[\partial^I, L^J] &= \sum_{\substack{|I'|=|I| \\ |J'|<|J|}} \Gamma^{IJ}_{12,I'J'} \partial^{I'} L^{J'}, \\
[\partial^I, \Omega^K] &= \sum_{\substack{|I'|=|I| \\ |K'|<|K|}} \Gamma^{IK}_{13,I'K'} \partial^{I'} \Omega^{K'}, \\
[L^J, \Omega^K] &= \sum_{\substack{|J'|=|J| \\ |K'|<|K|}} \Gamma^{JK}_{23,J'K'} L^{J'} \Omega^{K'}.
\end{aligned} \tag{12.8}$$

Proposition 12.8. *For all multi-indices I, I', J, J' and K, K' and all sufficiently regular functions u defined on $\mathbb{R}^4$, one has*

$$\partial^{I'} L^{J'} \Omega^{K'} \partial^I L^J \Omega^K u = \sum_{\substack{|I''|=|I'|+|I|, |J''|=|J'|+|J| \\ |K''|\le|K'|+|K|}} \Gamma^{I'J'K'IJK}_{I''J''K''} \partial^{I''} L^{J''} \Omega^{K''} u, \tag{12.9}$$

where the coefficients $\Gamma^{I'J'K'IJK}_{I''J''K''}$ are constants.

Proof. This is established by induction from the previous identity. ☐

12.3 Decomposition of commutators. II

Definition 12.9. *Let D be a region of $\mathbb{R}^4$. suppose that $\mathscr{A} = \{A_\alpha\}_{\alpha\in\mathfrak{I}}$ and $\mathscr{B} = \{B_\beta\}_{\beta\in\mathfrak{J}}$ be two families of vector field (with $\mathfrak{I}, \mathfrak{J}$ finite sets for index) defined on D. Let Hom be a class of smooth functions defined on D. If $f \in Hom$, $B_\beta \in \mathscr{B}$, $B_\beta f \in Hom$ and*

$$A_\alpha \in \mathscr{A}, \qquad B_\beta \in \mathscr{B}, \quad [A_\alpha, B_\beta] = \Gamma^\gamma_{\alpha\beta} A_\gamma.$$

with $\Gamma^\gamma_{\alpha\beta} \in Hom$, then we say $\mathscr{B} \prec_{Hom} \mathscr{A}$.

Then we derive a general decomposition formula for the commutators.

Lemma 12.10. *Let $A_\alpha \in \mathscr{A}$, B^J be a $J-th$ order derivative composed with operators in $\mathscr{B}$. Suppose that $\mathscr{B} \prec_{Hom} \mathscr{A}$, then one has*

$$[A_\alpha, B^J] = \sum_{\substack{\beta\in\mathfrak{I} \\ |J'|<|J|}} \Gamma^{J\beta}_{\alpha J'} A_\beta B^{J'}, \tag{12.10}$$

where $\Gamma^{J\beta}_{\alpha J'} \in Hom$.

Proof. This result is proven by induction as follows on $|J|$. Observe that for $|J| = 1$ it is guaranteed by the definition (12.9). Now we consider

$$\begin{aligned}[A_\alpha, B_\beta B^J] &= [A_\alpha, B_\beta]B^J + B_\beta\big([A_\alpha, B^J]\big) \\ &= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} B_\beta \left(\Gamma^{J\gamma}_{\alpha J'} A_\gamma B^{J'}\right) \\ &= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} B_\beta \Gamma^{J\gamma}_{\alpha J'} A_\gamma B^{J'} + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} B_\beta A_\gamma B^{J'} \\ &= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} B_\beta \Gamma^{J\gamma}_{\alpha J'} A_\gamma B^{J'} + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} A_\gamma B_\beta B^{J'} \\ &\quad + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} [B_\beta, A_\gamma] B^{J'},\end{aligned}$$

so

$$\begin{aligned}[A_\alpha, B_\beta B^J] = &= \Gamma^\gamma_{\alpha\beta} A_\gamma B^J + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} B_\beta \Gamma^{J\gamma}_{\alpha J'} A_\gamma B^{J'} + \sum_{\substack{\gamma\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} A_\gamma B_\beta B^{J'} \\ &\quad - \sum_{\substack{\gamma,\delta\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\gamma}_{\alpha J'} \Gamma^\delta_{\gamma\beta} A_\delta B^{J'}.\end{aligned}$$

Recall that in the above expression, $B_\beta \Gamma^{J\gamma}_{\alpha J'} \in \mathrm{Hom}$, and the proof by induction is completed. □

Proposition 12.11. *Let $A_\alpha \in \mathscr{A}$. Suppose that B^J C^K are $|J|-$th and $|K|-$th order derivatives composed with operators in $\mathscr{B}$ and $\mathscr{C}$ respectively. Suppose that $\mathscr{B}, \mathscr{C} \prec_{Hom} \mathscr{A}$, then*

$$[A_\alpha, B^J C^K] = \sum_{\substack{|J'|\le|J|, |K'|\le|K|\\ |J'|+|K'|<|J|+|K|, \beta\in\mathfrak{I}}} \Gamma^{J\beta}_{\alpha J'} A_\beta B^{J'} C^{K'},$$

where $\Gamma^{J\beta}_{\alpha J'} \in$ Hom.

Proof. For convenience, we denote by (cf. Lemma 12.10)

$$[A_\alpha, B^J] = \sum_{\substack{\beta\in\mathfrak{I}\\ |J'|<|J|}} \Gamma^{J\beta}_{\alpha J'} A_\beta B^{J'}, \quad [A_\alpha, C^K] = \sum_{\substack{\beta\in\mathfrak{I}\\ |K'|<|K|}} \Delta^{K\beta}_{\alpha K'} A_\beta B^{C'}$$

with

$$\Gamma_{\alpha J'}^{J\beta},\ \Delta_{\alpha K'}^{K\beta} \in \text{Hom}.$$

Then we see that

$$\begin{aligned}
[A_\alpha, B^J C^K] &= [A_\alpha, B^J]C^K + B^J[A_\alpha, C^K] \\
&= \sum_{\substack{\gamma\in\mathfrak{I}\\|J'|<|J|}} \Gamma_{\alpha J'}^{J\gamma} A_\gamma B^{J'} C^K + \sum_{\substack{\delta\in\mathfrak{I}\\|K'|<|K|}} B^J\left(\Delta_{\alpha K'}^{K\delta} A_\delta C^{K'}\right) \\
&= \sum_{\substack{\gamma\in\mathfrak{I}\\|J'|<|J|}} \Gamma_{\alpha J'}^{J\gamma} A_\gamma B^{J'} C^K + \sum_{\substack{\delta\in\mathfrak{I},J_1+J_2=J\\|K'|<|K|}} B^{J_1}\Delta_{\alpha K'}^{K\delta}\cdot B^{J_2} A_\delta C^{K'}.
\end{aligned}$$

We observe that

$$B^{J_2}A_\delta C^{K'} = A_\delta B^{J_2} C^{K'} - [A_\delta, B^{J_2}]C^{K'} = A_\delta B^{J_2}C^{K'} - \sum_{\beta\in\mathfrak{I},|J_2'|<|J_2|} \Gamma_{\delta J_2'}^{J_2\beta} A_\beta B^{J_2} C^{K'},$$

and recall that $B^{J_1}\Delta_{\alpha K'}^{K\delta} \in$ Hom. This establishes the desired result. □

We take Hom $=$ IH_0,and observe that $f \in$ Hom, $L_a f, \Omega_{ab} f \in$ Hom. Thus we see that $\mathscr{R}, \mathscr{L} \prec_{\text{Hom}} \underline{\mathscr{T}}_x$. Then apply Proposition 12.11 and obtain the following decomposition:

$$\left[\underline{\partial}_a, L^J\Omega^K\right] = \underline{\Gamma}_{aJ'K'}^{JKb}\underline{\partial}_b L^{J'}\Omega^{K'}, \tag{12.11}$$

where $\underline{\Gamma}_{aJ'K'}^{JKb}\underline{\partial}_b L^{J'}$ are IH functions of degree zero.

We take Hom $=$ EH_0, and observe that $f \in$ Hom, $L_a f, \Omega_{ab} f \in$ Hom. Thus we see that $\mathscr{R}, \mathscr{L} \prec_{\text{Hom}} \widetilde{\mathscr{T}}$. Then apply Proposition 12.11 and obtain the following key decompositions:

$$\begin{aligned}
\left[\widetilde{\partial}_a, L^J\Omega^K\right] &= \sum_{\substack{|J'|\le|J|,|K'|\le|K|\\|J'|+|K'|<|J|+|K|}} \widetilde{\Gamma}_{aJ'K'}^{JK0}\frac{r-t}{r}\partial_t\partial^{J'}\Omega^{K'} + \sum_{\substack{b,|J'|\le|J|,|K'|\le|K|\\|J'|+|K'|<|J|+|K|}} \widetilde{\Gamma}_{aJ'K'}^{JKb}\widetilde{\partial}_b L^{J'}\Omega^{K'}, \\
\left[\frac{r-t}{r}\partial_t, L^J\Omega^K\right] &= \sum_{\substack{|J'|\le|J|,|K'|\le|K|\\|J'|+|K'|<|J|+|K|}} \widetilde{\Gamma}_{0J'K'}^{JK0}\frac{r-t}{r}\partial_t\partial^{J'}\Omega^{K'} + \sum_{\substack{b,|J'|\le|J|,|K'|\le|K|\\|J'|+|K'|<|J|+|K|}} \widetilde{\Gamma}_{0J'K'}^{JKb}\widetilde{\partial}_b L^{J'}\Omega^{K'}
\end{aligned} \tag{12.12}$$

the coefficients $\widetilde{\Gamma}_{aJ'K'}^{JKb}\underline{\partial}_b L^{J'}$ begin EH functions of degree zero.

12.4 Decomposition of commutators. III

We first establish the following properties.

Lemma 12.12. *For all multi-indices I, J one has*

$$\partial^I L^J (s/t) = \begin{cases} \underline{\Gamma}^J (s/t), & |I| = |K| = 0, \\ \sum_{I_1 + I_2 \cdots I_k = I} (s/t)^{1-2k} \underline{\Gamma}_k^I, & \end{cases} \tag{12.13}$$

where $\underline{\Gamma}^J \in IH_0$ and $\Gamma_k^I \in IH_{-|I|}$.

Lemma 12.13. *For all multi-indices I, K one has*

$$\partial^I \Omega^K \frac{r-t}{r} = \begin{cases} 0, & |K| \geq 1, \\ \widetilde{\Gamma}^I, & |I| \geq 1, \quad |K| = 0, \\ \dfrac{r-t}{r}, & |I| = |K| = 0, \end{cases} \tag{12.14}$$

where $\widetilde{\Gamma}^I \in EH_{-|I|}$.

Lemma 12.14. *For all multi-indices I, L, K one has*

$$\partial_x^I \Omega^K (1 + \omega_\gamma) = \begin{cases} 0, & |K| \geq 1, \\ \widetilde{\Gamma}_{\omega,\gamma}^I (1+q)^{\gamma - |I|}, & \end{cases} \tag{12.15}$$

where $\widetilde{\Gamma}_{\omega,\gamma}^I$ are smooth functions defined on $\{r \geq t/2\} \cap \{t \geq 1\}$, with all their derivatives bounded.

12.5 Decomposition of commutators. IV

We now focus on the tangent derivatives $\overline{\partial}_a$ on the region $\mathcal{M}_s$. We recall the parametrization of $\mathcal{M}_{[2,+\infty)}$ by coordinates $(\overline{x}^0, \overline{x}^a)$. We see that

$$[\overline{\partial}_a, \overline{\partial}_b] = 0 \tag{12.16}$$

and

$$[\overline{\partial}_a, \Omega_{bc}] = \delta_{ab} \overline{\partial}_c - \delta_{ac} \overline{\partial}_b. \tag{12.17}$$

We denote by $\overline{\mathscr{T}}_x = \{\overline{\partial}_a\}_{a=1,2,3}$. The above calculation shows that $\mathscr{R} \prec \overline{\mathscr{T}}_x$. Then by (12.6), in the region $\mathcal{M}_{[s_0,s_1]}$:

$$[\overline{\partial}_a, \Omega^K] = \sum_{b,|K'|<|K|} \overline{\Gamma}_{aK'}^{Kb} \overline{\partial}_b \Omega^{K'}, \tag{12.18}$$

where the coefficients $\overline{\Gamma}_{aK'}^{Kb}$ are constants. We thus see that for a function u defined in $\mathcal{M}_{[s_0,s_1]}$, sufficiently regular,

$$\overline{\partial}_x^I \Omega^{K'} \overline{\partial}_a \partial^I L^J \Omega^K u = \overline{\partial}_x^I \overline{\partial}_a \Omega^{K'} \partial^I L^J \Omega^K u - \sum_{b,|K''|<|K'|} \overline{\Gamma}_{aK''}^{K'b} \overline{\partial}_b \overline{\partial}_x^I \Omega^{K''} \partial^I L^J \Omega^K u.$$

Now recall Proposition 12.8, we see that both terms in right-hand-side of the above identity are linear combinations of the following terms with constant coefficients:

$$\overline{\partial}_b \overline{\partial}_x^I \partial^{I''} L^{J''} \Omega^{K''} u, \qquad |I''| = |I|, \quad |J''| = |J|, \quad |K''| \leq |K| + 2.$$

Finally, we establish the following estimate.

Proposition 12.15. *For all multi-index I with $|I| \leq 2$, the following estimate holds in $\mathcal{M}_{[s_0,s_1]}$*

$$|\overline{\partial}_a \overline{\partial}_x^I u| \leq C \sum_{\substack{b \\ |I'| \leq 2}} |\overline{\partial}_b \partial^{I'} u|. \tag{12.19}$$

Part V

Field equations for self-gravitating matter

13 Nonlinear structure of the field equations in conformal wave gauge

13.1 The wave Klein-Gordon formulation in symbolic notation

Wave-Klein-Gordon system. We will need to establish energy and sup-norm estimates for the Einstein system and its $f(R)$-generalization. For the latter, we seek for estimates that should be uniform and encompass the whole range $\kappa \in (0, 1]$. Later, we will distinguish between bounds satisfied by the metric, the scalar curvature, and the matter fields and between various orders of differentiation. In order to express such bounds, it is essential to *suitably rescale the scalar curvature* variable and this motivates us to introduce the notation:

$$\begin{aligned} \psi^\sharp &:= (\rho^\sharp, \phi), \qquad \rho^\sharp := \kappa\rho, \\ \psi^\flat &:= (\rho^\flat, \phi), \qquad \rho^\flat := \kappa^{1/2}\rho, \end{aligned} \tag{13.1}$$

Observe that no rescaling is required on ϕ. Moreover, in view of $\kappa\rho^2 = \rho^{\flat 2}$ and from (1.15) and (4.9), we can write

$$\begin{aligned} V_\kappa(\rho) &= \mathcal{O}(\rho^{\flat 2}), \qquad W_\kappa(\rho) = \mathcal{O}(\rho^{\flat 2}), \\ U(\phi) &= \mathcal{O}(\phi^2), \end{aligned} \tag{13.2}$$

together with analogous statements for the derivatives of these functions.

We can motivate our choice of scaling above by observing that the energy for the curvature field will take the schematic form

$$(\partial\rho^\sharp)^2 + (\rho^\flat)^2, \tag{13.3}$$

that is, does not explicitly involve the parameter κ. Hence, both $\rho^\sharp$ and $\rho^\flat$ will be uniformly controlled, although at different order of differentiability. However, the function ρ which formally approaches the scalar curvature R as $\kappa \to 0$ *will not be controlled* at a first stage of our analysis (e.g. in the bootstrap argument presented below) and a specific investigation will be necessary to establish that ρ converges (again with an unavoidable loss of derivatives).

We therefore consider the wave-Klein-Gordon formulation of the $f(R)$-gravity system in conformal wave coordinates, which was derived in Proposition 4.1. With our notation (13.1), this system reads

$$\begin{aligned}
\widehat{\Box}^\dagger {g^\dagger}_{\alpha\beta} =&\ F_{\alpha\beta}(g^\dagger, \partial g^\dagger) + A_{\alpha\beta} + B_{\alpha\beta},\\
3\,\widehat{\Box}^\dagger \rho^\sharp - \kappa^{-1/2}\rho^\flat =&\ -\sigma + W_\kappa(\rho) \simeq -\sigma + \mathcal{O}(\rho^{\flat 2}),\\
\widehat{\Box}^\dagger \phi - U'(\phi) =&\ 2\, {g^\dagger}^{\alpha\beta}\partial_\alpha\rho^\sharp\partial_\beta\phi.
\end{aligned} \tag{13.4}$$

We recall the notation $\widehat{\Box}^\dagger = {g^\dagger}^{\mu\nu}\partial_\mu\partial_\nu$ for the modified wave operator, while the Ricci nonlinearity $F_{\alpha\beta} = P_{\alpha\beta} + Q_{\alpha\beta}$ will be computed explicitly in Proposition 13.1 below. Since constant coefficients within the nonlinearities are irrelevant for the purpose of establishing global existence, it is convenient to use a symbolic notation and, with obvious convention, in view of (4.8) we express the nonlinearities $A_{\alpha\beta}, B_{\alpha\beta}, \sigma$ arising in (13.4) in the form

$$\begin{aligned}
A_{\alpha\beta} &\simeq \partial_\alpha\rho^\sharp\partial_\beta\rho^\sharp + V_\kappa(\rho){g^\dagger}^{\alpha\beta} \simeq \partial_\alpha\rho^\sharp\partial_\beta\rho^\sharp + \mathcal{O}(\rho^{\flat 2}){g^\dagger}_{\alpha\beta},\\
B_{\alpha\beta} &\simeq e^{-\rho^\sharp}\partial_\alpha\phi\partial_\beta\phi + U(\phi)e^{-2\rho^\sharp}\, g^\dagger_{\alpha\beta} \simeq e^{-\rho^\sharp}\partial_\alpha\phi\partial_\beta\phi + \mathcal{O}(\phi)^2 e^{-2\rho^\sharp}\, g^\dagger_{\alpha\beta},\\
\sigma &\simeq e^{-\rho^\sharp}{g^\dagger}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + e^{-2\rho^\sharp}U(\phi) \simeq e^{-\rho^\sharp}{g^\dagger}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + e^{-2\rho^\sharp}\mathcal{O}(\phi^2).
\end{aligned} \tag{13.5}$$

Expression of the Ricci curvature. Before we can proceed with the analysis of the system (13.4)-(13.5), we need to compute the expression of the Ricci curvature nonlinearity $F_{\alpha\beta}$ arising in the metric equation in (13.4). In general coordinates, the contracted Christoffel symbols read

$$\begin{aligned}\Gamma^{\dagger\gamma} = g^{\dagger\alpha\beta}\Gamma^{\dagger\gamma}_{\alpha\beta} &= \frac{1}{2}g^{\dagger\alpha\beta}g^{\dagger\gamma\delta}\big(\partial_\alpha g^\dagger{}_{\beta\delta} + \partial_\beta g^\dagger_{\alpha\delta} - \partial_\delta g^\dagger{}_{\alpha\beta}\big)\\ &= g^{\dagger\gamma\delta}g^{\dagger\alpha\beta}\partial_\alpha g^\dagger{}_{\beta\delta} - \frac{1}{2}g^{\dagger\alpha\beta}g^{\dagger\gamma\delta}\partial_\delta g^\dagger{}_{\alpha\beta}\end{aligned} \tag{13.6}$$

and, therefore,

$$\Gamma^\dagger{}_\lambda = g^\dagger{}_{\lambda\gamma}\Gamma^\gamma = g^{\dagger\alpha\beta}\partial_\alpha g^\dagger{}_{\beta\lambda} - \frac{1}{2}g^{\dagger\alpha\beta}\partial_\lambda g^\dagger{}_{\alpha\beta}.$$

The Ricci curvature in arbitrary coordinates, that is,

$$R^\dagger_{\alpha\beta} = \partial_\lambda\Gamma^{\dagger\lambda}_{\alpha\beta} - \partial_\alpha\Gamma^{\dagger\lambda}_{\beta\lambda} + \Gamma^{\dagger\lambda}_{\alpha\beta}\Gamma^{\dagger\delta}_{\lambda\delta} - \Gamma^{\dagger\lambda}_{\alpha\delta}\Gamma^{\dagger\delta}_{\beta\lambda}, \tag{13.7}$$

is nox decomposed into several very different contributions. The desired structure will be achieved by working in wave coordinates, which by definition satisfy $\Gamma^{\dagger\alpha} = 0$ but we not assume this in the following statement.

Proposition 13.1 (Expression of the curvature in coordinates)**.** *The Ricci curvature of the metric $g^\dagger$ admits the following decomposition*[1]

$$R^\dagger_{\alpha\beta} = -\frac{1}{2}g^{\dagger\lambda\delta}\partial_\lambda\partial_\delta g^\dagger_{\alpha\beta} + \frac{1}{2}\big(\partial_\alpha\Gamma^\dagger{}_\beta + \partial_\beta\Gamma^\dagger{}_\alpha\big) + \frac{1}{2}F_{\alpha\beta}, \tag{13.8}$$

where $F_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger) := P_{\alpha\beta} + Q_{\alpha\beta} + W_{\alpha\beta}$ is a sum of null terms, that is,

$$\begin{aligned}Q_{\alpha\beta} :&= g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\partial_\delta g^\dagger{}_{\alpha\lambda'}\partial_{\delta'}g^\dagger{}_{\beta\lambda} - g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\big(\partial_\delta g^\dagger{}_{\alpha\lambda'}\partial_\lambda g^\dagger{}_{\beta\delta'} - \partial_\delta g^\dagger{}_{\beta\delta'}\partial_\lambda g^\dagger{}_{\alpha\lambda'}\big)\\ &+ g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\big(\partial_\alpha g^\dagger{}_{\lambda'\delta'}\partial_\delta g^\dagger{}_{\lambda\beta} - \partial_\alpha g^\dagger{}_{\lambda\beta}\partial_\delta g^\dagger{}_{\lambda'\delta'}\big)\\ &+ \frac{1}{2}g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\big(\partial_\alpha g^\dagger{}_{\lambda\beta}\partial_{\lambda'}g^\dagger{}_{\delta\delta'} - \partial_\alpha g^\dagger{}_{\delta\delta'}\partial_{\lambda'}g^\dagger{}_{\lambda\beta}\big)\\ &+ g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\big(\partial_\beta g_{\lambda'\delta'}\partial_\delta g^\dagger{}_{\lambda\alpha} - \partial_\beta g^\dagger{}_{\lambda\alpha}\partial_\delta g^\dagger{}_{\lambda'\delta'}\big)\\ &+ \frac{1}{2}g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\big(\partial_\beta g^\dagger{}_{\lambda\alpha}\partial_{\lambda'}g^\dagger{}_{\delta\delta'} - \partial_\beta g^\dagger{}_{\delta\delta'}\partial_{\lambda'}g^\dagger{}_{\lambda\alpha}\big),\end{aligned}$$

quasi-null term

$$P_{\alpha\beta} := -\frac{1}{2}g^{\dagger\lambda\lambda'}g^{\dagger\delta\delta'}\partial_\alpha g^\dagger{}_{\delta\lambda'}\partial_\beta g^\dagger{}_{\lambda\delta'} + \frac{1}{4}g^{\dagger\delta\delta'}g^{\dagger\lambda\lambda'}\partial_\beta g^\dagger{}_{\delta\delta'}\partial_\alpha g^\dagger{}_{\lambda\lambda'}$$

and a remainder

$$W_{\alpha\beta} := g^{\dagger\delta\delta'}\partial_\delta g^\dagger_{\alpha\beta}\Gamma^\dagger{}_{\delta'} - \Gamma^\dagger{}_\alpha\Gamma^\dagger{}_\beta.$$

[1] This decomposition was derived first in [94], while the terminology "quasi-null term" was proposed in [88].

The full derivation of this decomposition is postponed to Section 15.

13.2 Homogeneous functions and nonlinearities of the field equations

Nonlinearities of the field equations. Recall that the notion of "homogeneous function" was introduced in Section 12. The notion above is now applied in order to classify the nonlinearities arising in our system (13.4)-(13.5). Recall our notation

$$h_{\alpha\beta} = g^{\dagger}{}_{\alpha\beta} - g_{M,\alpha\beta}, \tag{13.9}$$

The expressions of interest will be expressed from a limited list of nonlinear terms in $h, \rho^\sharp$, $\rho^\flat$, and ϕ. In the interior domain $\mathcal{M}^{\text{int}}$, the coefficient r/t is bounded above (by 1) and t is bounded away from zero.

Definition 13.2. *An expression (of derivatives of the metric, curvature, and matter field) is said to be a* **bounded linear combination** *if it can be expressed as a sum with homogeneous coefficients of degree* ≤ 0*.*

We are thus able to treat simultaneously the interior and exterior domains by using our global notation. We distinguish between the following classes of linear combinations with homogeneous coefficients of degree ≤ 0. Except for the last type of terms in our list, all terms are quadratic expressions in h, ρ, ϕ.

It is important to unify our notation and use

$$\widetilde{\underline{\partial}}_a := \begin{cases} \underline{\partial}_a & \text{in the interior } \mathcal{M}^{\text{int}}, \\ \widetilde{\partial}_a & \text{in the transition and exterior } \mathcal{M}^{\text{tran}} \cup \mathcal{M}^{\text{ext}}. \end{cases} \tag{13.10}$$

Another useul notation is obtain by writing an arbitrary combination of Killing fields in the form:

$$Z^{IJK} := \partial^I L^J \Omega^K. \tag{13.11}$$

We also use the notation (for instance)

$$|J, K| := |J| + |K|. \tag{13.12}$$

Quasi-linear terms.

- **Metric-metric interactions**[1] $\mathbb{Q}_{hh}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\partial_\mu\partial_\nu h_{\alpha\beta}, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}\partial_\mu\partial_\nu\partial^I L^J h_{\alpha\beta}, \quad |I,J,K|\le p,\ |J,K|<k.$$

- **Metric-curvature interactions** $\mathbb{Q}_{h\rho^\sharp}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\partial_\mu\partial_\nu \rho^\sharp, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}\partial_\mu\partial_\nu Z^{IJK}\rho^\sharp, \quad |I,J,K|\le p,\ |J,K|<k.$$

- **Metric-matter interactions** $\mathbb{Q}_{h\phi}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\partial_\mu\partial_\nu \phi, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}\partial_\mu\partial_\nu Z^{IJK}\phi, \quad |I,J,K|\le p,\ |J,K|<k.$$

- In view of the notation (13.1), we define $\mathbb{Q}_{h\psi^\sharp}(p,k) := \mathbb{Q}_{h\rho^\sharp}(p,k)\cup\mathbb{Q}_{h\phi}(p,k)$.

Good quasi-linear terms.

- **Metric-metric interactions** $\mathbb{GQ}_{hh}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu h_{\alpha\beta}, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b h_{\alpha\beta}, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu h_{\alpha\beta}, \qquad h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b h_{\alpha\beta}, \quad |I,J,K|\le p,\ |J,K|<k.$$

- **Metric-curvature interactions** $\mathbb{GQ}_{h\rho^\sharp}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu \rho^\sharp, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b \rho^\sharp, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu \rho^\sharp, \qquad h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b \rho^\sharp, \quad |I,J,K|\le p,\ |J,K|<k.$$

- **Metric-matter interactions** $\mathbb{GQ}_{h\phi}(p,k)$:

$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu \phi, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$Z^{I_1J_1K_1}h_{\alpha'\beta'}Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b \phi, \quad |I_1,I_2|\le p-k,\ |J_1,K_1,J_2,K_2|\le k,\ |I_2,J_2,K_2|\le p-1,$$
$$h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}_\mu \phi, \qquad h_{\alpha'\beta'}Z^{IJK}\underline{\widetilde{\partial}}_\mu\underline{\widetilde{\partial}}_b \phi, \quad |I,J,K|\le p,\ |J,K|<k.$$

- In view of (13.1), we analogously define $\mathbb{GQ}_{h\psi^\sharp}(p,k)$.

[1] p is the total number of vector fields while k is the total number of boosts and rotations.

Semi-linear terms.

- **Metric-metric interactions** $\mathbb{S}_{hh}(p,k)$:

$$Z^{I_1J_1K_1}\partial_\mu h_{\alpha\beta} Z^{I_2J_2K_2}\partial_\nu h_{\alpha'\beta'}, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k.$$

- **Curvature-curvature interactions** of first order $\mathbb{S}_{\rho^\sharp\rho^\sharp}(p,k)$:

$$Z^{I_1J_1K_1}\partial_\mu \rho^\sharp Z^{I_2J_2K_2}\partial_\nu \rho^\sharp, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k.$$

- **Curvature-curvature interactions** of zero order $\mathbb{S}_{\rho^\flat\rho^\flat}(p,k)$:

$$Z^{I_1J_1K_1}\rho^\flat Z^{I_2J_2K_2}\rho^\flat, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k.$$

- **Curvature-matter interactions** $\mathbb{S}_{\rho^\sharp\phi}(p,k)$:

$$Z^{I_1J_1K_1}\partial_\mu \rho^\sharp Z^{I_2J_2K_2}\partial_\nu \phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k.$$

- **Matter-matter interactions** $\mathbb{S}_{\phi\phi}(p,k)$ (of zero or first orders):

$$Z^{I_1J_1K_1}\partial_\mu \phi Z^{I_2J_2K_2}\partial_\nu \phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k,$$

$$Z^{I_1J_1K_1}\phi Z^{I_2J_2K_2}\phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k.$$

- In view of (13.1), we analogously define $\mathbb{S}_{\psi^\sharp\psi^\sharp}(p,k)$ and $\mathbb{S}_{\psi^\flat\psi^\flat}(p,k)$.

Good semi-linear terms (including null terms).

- **Metric-metric interactions** $\mathbb{GS}_{hh}(p,k)$:

$$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a h_{\alpha\beta} Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\gamma h_{\alpha'\beta'}, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k,$$

and in the interior $\mathcal{M}^{\text{int}}$ only:

$$\frac{s^2}{t^2} Z^{I_1J_1K_1}\underline{\partial}_0 h_{\alpha\beta} Z^{I_2J_2K_2}\underline{\partial}_0 h_{\alpha'\beta'}, \qquad |I_1,J_1,K_1,I_2,J_2,K_2| \le p, \quad |J_1,K_1,J_2,K_2| \le k,$$

in which we recall that $\underline{\widetilde{\partial}}_0 = \partial_t$.

- **Curvature-curvature interactions** of first order $\mathbb{GS}_{\rho^\sharp\rho^\sharp}(p,k)$:

$$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\rho^\sharp Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\gamma\rho^\sharp, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k,$$

and in the interior $\mathcal{M}^{\text{int}}$ only:

$$\frac{s^2}{t^2}\partial^{I_1}L^{J_1}\underline{\partial}_0\rho^\sharp\partial^{I_1}L^{J_1}\underline{\partial}_0\rho^\sharp, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k.$$

- **Curvature-matter interactions** $\mathbb{GS}_{\rho^\sharp\phi}(p,k)$:

$$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\rho^\sharp Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\gamma\phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k,$$

$$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\phi Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\gamma\rho^\sharp, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k,$$

and in the interior $\mathcal{M}^{\text{int}}$ only:

$$\frac{s^2}{t^2}Z^{I_1J_1K_1}\underline{\partial}_0\rho^\sharp\partial^{I_1}L^{J_1}\underline{\partial}_0\phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k.$$

- **Matter-matter interactions** of first order $\mathbb{GS}_{\phi\phi}(p,k)$:

$$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\phi Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_\gamma\phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k,$$

and in the interior $\mathcal{M}^{\text{int}}$ only:

$$\frac{s^2}{t^2}Z^{I_1J_1K_1}\underline{\partial}_0\phi Z^{I_2J_2K_2}\underline{\partial}_0\phi, \qquad |I_1,J_1,K_1,I_2,J_2,K_2|\leq p, \quad |J_1,K_1,J_2,K_2|\leq k.$$

- Moreover, in view of (13.1), we analogously define $\mathbb{GS}_{\psi^\sharp\psi^\sharp}(p,k)$.

Higher-order terms.

- **Change of frame terms** $\mathbb{F}(p,k)$, which arise when a second-order derivative is transformed from the canonical frame to the semi-hyperboloidal frame:

$$\begin{array}{ll}
\frac{1}{t+r}Z^{I_1J_1K_1}h_{\mu\nu}Z^{I_2J_2K_2}\partial_\gamma h_{\mu'\nu'}, & \frac{1}{(t+r)^2}Z^{I_1J_1K_1}h_{\mu\nu}\partial^{I_2}L^{J_2}h_{\mu'\nu'}, \\
\frac{1}{t+r}Z^{I_1J_1K_1}\partial_\mu h_{\alpha\beta}Z^{I_2J_2K_2}\partial_\nu\rho^\sharp, & \frac{1}{t+r}Z^{I_1J_1K_1}\partial_\mu h_{\alpha\beta}Z^{I_2J_2K_2}\partial_\nu\phi, \\
\frac{1}{(t+r)^2}Z^{I_1J_1K_1}h_{\mu\nu}Z^{I_2J_2K_2}\rho^\sharp, & \frac{1}{(t+r)^2}Z^{I_1J_1K_1}h_{\mu\nu}Z^{I_2J_2K_2}\phi,
\end{array}$$

$|I_1,I_2|\leq p-k$, $|J_1,K_1,J_2,K_2|\leq k$, and

$$\frac{1}{t+r}\,\mathbb{S}_{hh}(p,k), \quad \frac{1}{t+r}\,\mathbb{S}_{\phi\phi}(p,k), \qquad \frac{1}{t+r}\,\mathbb{S}_{\phi\rho^\sharp}(p,k).$$

- **Cubic interactions** $\mathbb{C}(p,k)$. Such terms enjoy better decay in time and we will not list them in full detail. We only observe that these interaction terms include:
  - cubic (or higher-order) terms in $\rho^\flat$, and
  - cubic (or higher-order) terms in $(h_{\alpha\beta}, \rho^\sharp, \phi)$,
  - *except* cubic (nor higher-order) terms $h_{\alpha\beta}$ (which do not arise).

We have summarized our notation in Table 13.2. Furthermore, when an expression C is decomposed into a sum of terms in the above classes, *for instance,* in $\mathbb{S}_{hh}(p,k)$ and $\mathbb{F}(p,k)$, we use the **symbolic notation**

$$C \simeq \mathbb{S}_{hh}(p,k) + \mathbb{F}(p,k). \tag{13.13}$$

Quasi-linear	Metric-metric	$\mathbb{Q}_{hh}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\partial\partial h$
	Metric-curvature	$\mathbb{Q}_{h\psi^\sharp}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\partial\partial\rho^\sharp$
	Metric-matter	$\mathbb{Q}_{h\phi}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\partial\partial\phi$
Good quasi-linear	Metric-metric	$\mathbb{GQ}_{hh}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}h$
	Metric-curvature	$\mathbb{GQ}_{h\psi^\sharp}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}\rho^\sharp$
	Metric-matter	$\mathbb{GQ}_{h\phi}(p,k)$	$Z^{I_1J_1K_1}hZ^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\underline{\widetilde{\partial}}\phi$
Semi-linear	Metric-metric	$\mathbb{S}_{hh}(p,k)$	$Z^{I_1J_1K_1}\partial hZ^{I_2J_2K_2}\partial h$
	Curvature-curvature	$\mathbb{S}_{\rho^\sharp\rho^\sharp}(p,k)$	$Z^{I_1J_1K_1}\partial\rho^\sharp Z^{I_2J_2K_2}\partial\rho^\sharp$
		$\mathbb{S}_{\rho^\flat\rho^\flat}(p,k)$	$Z^{I_1J_1K_1}\rho^\flat Z^{I_2J_2K_2}\rho^\flat$
	Curvature-matter	$\mathbb{S}_{\rho^\sharp\phi}(p,k)$	$Z^{I_1J_1K_1}\partial\rho^\sharp Z^{I_2J_2K_2}\partial\phi$
	Matter-matter	$\mathbb{S}_{\phi\phi}(p,k)$	$Z^{I_1J_1K_1}\partial\phi Z^{I_2J_2K_2}\partial\phi$
			$Z^{I_1J_1K_1}\phi\partial^{I_2}L^{J_2}\phi$
Good semi-linear	Metric-metric	$\mathbb{GS}_{hh}(p,k)$	$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a hZ^{I_2J_2K_2}\underline{\widetilde{\partial}}h$
			interior only:
			$(s^2/t^2)\partial^{I_1}L^{J_1}\underline{\widetilde{\partial}}_0 hZ^{I_2J_2K_2}\underline{\widetilde{\partial}}_0 h$
	Curvature-curvature	$\mathbb{GS}_{\rho^\sharp\rho^\sharp}(p,k)$	$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\rho^\sharp Z^{I_2J_2K_2}\underline{\widetilde{\partial}}\rho^\sharp$
			interior only:
			$(s^2/t^2)\partial^{I_1}L^{J_1}\underline{\partial}_0\rho^\sharp\partial^{I_2}L^{J_2}\underline{\partial}_0\rho^\sharp$
	Curvature-matter	$\mathbb{GS}_{\rho^\sharp\phi}(p,k)$	$\partial^{I_1}L^{J_1}\underline{\widetilde{\partial}}_a\rho^\sharp Z^{I_2J_2K_2}\underline{\widetilde{\partial}}\phi$
			$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}\rho^\sharp Z^{I_2J_2K_2}\underline{\widetilde{\partial}}_a\phi$
			interior only:
			$(s^2/t^2)Z^{I_1J_1K_1}\underline{\partial}\rho^\sharp Z^{I_2J_2K_2}\underline{\partial}_a\phi$
	Matter-matter	$\mathbb{GS}_{\phi\phi}(p,k)$	$Z^{I_1J_1K_1}\underline{\widetilde{\partial}}_a\phi Z^{I_2J_2K_2}\underline{\widetilde{\partial}}\phi$
			interior only:
			$(s^2/t^2)Z^{I_1J_1K_1}\underline{\partial}_0\phi Z^{I_2J_2K_2}\underline{\partial}_0\phi$

Table: Quasi-linear and semi-linear terms (with the range of indices omitted)

14 The quasi-null structure for the null-hyperboloidal frame

14.1 The nonlinear wave equations for the metric components

We now analyze the metric equations in the frame of interest. Here and in the rest of this Monograph, we rely on a symbolic notation in order to express the nonlinearities arising in the problem and, in particular, we suppress (irrelevant) numerical constants. As already pointed out, the general algebraic structure of the field equations in the interior and exterior domains are the same, except for the weight functions that we take into account in our estimates.

We use the notation $\widetilde{\underline{h}}$ to denote either $\underline{h}$ in the interior domain or $\widetilde{\underline{h}}$ in the transition or exterior domains.

Lemma 14.1. *In terms of their components in the null-semi-hyperboloidal frame, the field equations of modified gravity take the following form*[1]

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger} \widetilde{\underline{h}}_{00} &\simeq \widetilde{\underline{P}}_{00} + \mathcal{O}(1)\Big(\widetilde{\underline{\partial}}_0 \rho^\sharp \widetilde{\underline{\partial}}_0 \rho^\sharp + \mathcal{O}(1) \widetilde{\underline{\partial}}_0 \phi \widetilde{\underline{\partial}}_0 \phi\Big) \\
&\quad + \mathcal{O}(1)\Big(\rho^{\flat 2} + \phi^2\Big) \widetilde{\underline{g}}_{M,00} + \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0), \\
\widehat{\Box}_{g^\dagger} \widetilde{\underline{h}}_{0a} &\simeq \widetilde{\underline{P}}_{0a} + \frac{1}{t+r} \widetilde{\underline{\partial}}_a h_{00} + \mathcal{O}\Big(\frac{r}{(t+r)^3}\Big) h_{00} + \mathcal{O}(1)\Big(\widetilde{\underline{\partial}}_a \rho^\sharp \widetilde{\underline{\partial}}_0 \rho^\sharp + \widetilde{\underline{\partial}}_a \phi \widetilde{\underline{\partial}}_0 \phi\Big) \\
&\quad + \mathcal{O}(1)\Big(\rho^{\flat 2} + \phi^2\Big) \widetilde{\underline{g}}_{M,a0} + \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0),
\end{aligned}
\tag{14.1}
$$

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger} \widetilde{\underline{h}}_{aa} &\simeq \widetilde{\underline{P}}_{aa} + \mathcal{O}\Big(\frac{r}{(t+r)^2}\Big) \widetilde{\underline{\partial}}_a h_{00} + \mathcal{O}\Big(\frac{1}{(t+r)^2}\Big) h_{00} + \frac{1}{t+r} \widetilde{\underline{\partial}}_a h_{0a} + \mathcal{O}\Big(\frac{r}{(t+r)^3}\Big) h_{0a} \\
&\quad + \mathcal{O}(1)\Big(\widetilde{\underline{\partial}}_a \rho^\sharp \widetilde{\underline{\partial}}_a \rho^\sharp + \widetilde{\underline{\partial}}_a \phi \widetilde{\underline{\partial}}_a \phi\Big) + \mathcal{O}(1)\Big(\rho^{\flat 2} + \phi^2\Big) \widetilde{\underline{g}}_{M,aa} \\
&\quad + \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) \qquad (a = 1, 2, 3),
\end{aligned}
\tag{14.2}
$$

[1] It is convenient to keep here the components $h_{\alpha\beta}$ in the right-hand sides.

and

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger}\underline{\widetilde{h}}_{ab} \simeq \underline{\widetilde{P}}_{ab} + +\mathbb{GS}_{hh}(0,0) + \mathcal{O}(1)\Big(\underline{\widetilde{\partial}}_a\rho^\sharp\underline{\widetilde{\partial}}_b\rho^\sharp + \underline{\widetilde{\partial}}_a\phi\underline{\widetilde{\partial}}_b\phi\Big) \\
+ \mathcal{O}\Big(\frac{r}{(t+r)^2}\Big)\big(\underline{\widetilde{\partial}}_a h_{00} + \underline{\widetilde{\partial}}_b h_{00}\big) + \mathcal{O}\Big(\frac{1}{t+r}\Big)\big(\underline{\widetilde{\partial}}_a h_{0b} + \underline{\widetilde{\partial}}_a h_{0a}\big) \\
+ \mathcal{O}\Big(\frac{r^2}{(t+r)^4}\Big)h_{00} + \mathcal{O}\Big(\frac{r}{(t+r)^3}\Big)\big(h_{0a} + h_{0b}\big) \\
+ \mathcal{O}(1)\Big(\rho^{\flat 2} + \phi^2\Big)\underline{\widetilde{g}}_{M,ab} + \mathbb{C}(0,0) \qquad (a \neq b).
\end{aligned}
\tag{14.3}
$$

A similar statement holds for any derivative $\partial^I L^J \underline{\widetilde{h}}_{\alpha\beta}$ of the null-semi-hyperboloidal components.

Proof. The general identity

$$
\widehat{\Box}_{g^\dagger}(uv) = u\,\widehat{\Box}_{g^\dagger}v + v\,\widehat{\Box}_{g^\dagger}u + 2\,{g^\dagger}^{\alpha\beta}\partial_\alpha u \partial_\beta v
$$

allows us to decompose the modified wave operator $\widehat{\Box}_{g^\dagger}$ in the conformal metric, as follows:

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger}\underline{h}_{\alpha\beta} &= \widehat{\Box}_{g^\dagger}\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}h_{\alpha'\beta'}\big) \\
&= \underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}\widehat{\Box}_{g^\dagger}h_{\alpha'\beta'} + h_{\alpha'\beta'}\widehat{\Box}_{g^\dagger}\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}\big) + 2\,{g^\dagger}^{\mu\nu}\,\partial_\mu\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\Phi}_\beta^{\beta'}\big)\partial_\nu h_{\alpha'\beta'},
\end{aligned}
$$

in which we are going to use the expressions of the transition matrices in (7.3) and (7.8). In the interior domain, we have

$$
\begin{aligned}
&\underline{\Phi}_0^0\underline{\Phi}_0^0 = 1, && \underline{\Phi}_a^0\underline{\Phi}_0^0 = x_a/t, && \underline{\Phi}_a^a\underline{\Phi}_0^0 = 1, \\
&\underline{\Phi}_a^0\underline{\Phi}_b^0 = x_a x_b/t^2, && \underline{\Phi}_a^0\underline{\Phi}_b^b = x_a/t, && \underline{\Phi}_a^a\underline{\Phi}_b^b = 1,
\end{aligned}
\tag{14.4}
$$

which are bounded functions in $\mathcal{M}^{\text{int}}$ and, more precisely, are coefficients of degree zero. On the other hand, in the transition and exterior domains we have chosen the null frame and we have

$$
\begin{aligned}
&\widetilde{\Phi}_0^0\widetilde{\Phi}_0^0 = 1, && \underline{\Phi}_a^0\widetilde{\Phi}_0^0 = x_a/r, && \widetilde{\Phi}_a^a\widetilde{\Phi}_0^0 = 1, \\
&\widetilde{\Phi}_a^0\widetilde{\Phi}_b^0 = x_a x_b/r^2, && \widetilde{\Phi}_a^0\widetilde{\Phi}_b^b = x_a/r, && \widetilde{\Phi}_a^a\widetilde{\Phi}_b^b = 1,
\end{aligned}
\tag{14.5}
$$

which also are bounded functions in $\mathcal{M}^{\text{tran}} \cup \mathcal{M}^{\text{ext}}$ and, more precisely, are coefficients of degree zero.

Consider first the component $\underline{\widetilde{h}}_{00}$. From the wave-Klein-Gordon system (13.4)-(13.5) and in view of the decomposition of $F_{\alpha\beta}$ given in Proposition 13.1, we find

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger}\underline{\widetilde{h}}_{00} &= \underline{\widetilde{P}}_{00} + \underline{\widetilde{\Phi}}_0^{\alpha'}\underline{\widetilde{\Phi}}_0^{\beta'}Q_{\alpha'\beta'} + \mathcal{O}(1)\underline{\widetilde{\partial}}_0\rho^\sharp\underline{\widetilde{\partial}}_0\rho^\sharp + \mathcal{O}(1)e^{-\rho^\sharp}\underline{\widetilde{\partial}}_0\phi\underline{\widetilde{\partial}}_0\phi \\
&\quad + \Big(\mathcal{O}(1)\rho^{\flat 2} + \mathcal{O}(1)e^{-2\rho^\sharp}\phi^2\Big)\underline{\widetilde{g}}^\dagger_{00} \\
&= \underline{\widetilde{P}}_{00} + \mathbb{GS}_{hh}(0,0) + \mathcal{O}(1)\underline{\widetilde{\partial}}_0\rho^\sharp\underline{\widetilde{\partial}}_0\rho^\sharp + \mathcal{O}(1)e^{-\rho^\sharp}\underline{\widetilde{\partial}}_0\phi\underline{\widetilde{\partial}}_0\phi \\
&\quad + \Big(\mathcal{O}(1)\rho^{\flat 2} + \mathcal{O}(1)e^{-2\rho^\sharp}\phi^2\Big)\underline{\widetilde{g}}_{M,00} + \mathbb{C}(0,0),
\end{aligned}
$$

in which the contribution of the exponential factors can be absorbed in cubic terms.

We treat similarly, the terms

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger}\underline{\widetilde{h}}_{0a} &= \underline{\widetilde{P}}_{0a} + \underline{\widetilde{\Phi}}_0^{\alpha'}\underline{\widetilde{\Phi}}_a^{\beta'}Q_{\alpha'\beta'} + \mathcal{O}(1)\underline{\widetilde{\partial}}_a\rho^\sharp\underline{\widetilde{\partial}}_0\rho^\sharp + \mathcal{O}(1)e^{-\rho^\sharp}\underline{\widetilde{\partial}}_a\phi\underline{\widetilde{\partial}}_0\phi \\
&\quad + \Big(\mathcal{O}(1)\rho^{\flat 2} + \mathcal{O}(1)e^{-2\rho^\sharp}\phi^2\Big)\underline{\widetilde{g}}^\dagger_{a0} + \mathcal{O}(1)\frac{2}{t+r}\underline{\widetilde{\partial}}_a h_{00} - \mathcal{O}(1)\frac{2x_a}{(t+r)^3}h_{00} \\
&\simeq \underline{\widetilde{P}}_{0a} + \mathbb{GS}_{hh}(0,0) + \mathcal{O}(1)\Big(\underline{\widetilde{\partial}}_a\rho^\sharp\underline{\widetilde{\partial}}_0\rho^\sharp + \underline{\widetilde{\partial}}_a\phi\underline{\widetilde{\partial}}_0\phi\Big) \\
&\quad + \Big(\mathcal{O}(1)\rho^{\flat 2} + \mathcal{O}(1)e^{-2\rho^\sharp}\phi^2\Big)\underline{\widetilde{g}}_{M,a0} + \mathbb{C}(0,0) + \frac{\mathcal{O}(1)}{t+r}\underline{\widetilde{\partial}}_a h_{00} + \mathcal{O}\Big(\frac{r}{(t+r)^3}\Big)h_{00},
\end{aligned}
$$

where cubic terms are denoted with the symbolic notation $\mathbb{C}(0,0)$. Next, for any $a = 1,2,3$ we compute

$$
\begin{aligned}
\widehat{\Box}_{g^\dagger}\underline{\widetilde{h}}_{aa} &\simeq \underline{\widetilde{P}}_{aa} + \underline{\widetilde{\Phi}}_a^{\alpha'}\underline{\widetilde{\Phi}}_a^{\beta'}Q_{\alpha'\beta'} + \mathcal{O}(1)\underline{\widetilde{\partial}}_a\rho^\sharp\underline{\widetilde{\partial}}_a\rho^\sharp + \mathcal{O}(1)\underline{\widetilde{\partial}}_a\phi\underline{\widetilde{\partial}}_a\phi \\
&\quad + \big(\mathcal{O}(1)\rho^{\flat 2} + \mathcal{O}(1)e^{-2\rho^\sharp}\phi^2\big)\underline{\widetilde{g}}^\dagger_{aa} \\
&\quad + \frac{4x_a}{(t+r)^2}\underline{\widetilde{\partial}}_a h_{00} + \frac{4}{T}\underline{\widetilde{\partial}}_a h_{0a} - \frac{4x_a}{(t+r)^3}h_{0a} + \Big(\frac{2}{(t+r)^2} - \frac{6|x_a|^2}{(t+r)^4}\Big)h_{00} \\
&\simeq \underline{\widetilde{P}}_{aa} + \mathbb{GS}_{hh}(0,0) + \mathcal{O}(1)\Big(\underline{\widetilde{\partial}}_a\rho^\sharp\underline{\widetilde{\partial}}_a\rho^\sharp + \underline{\widetilde{\partial}}_a\phi\underline{\widetilde{\partial}}_a\phi\Big) \\
&\quad + \mathcal{O}(1)\Big(\rho^{\flat 2} + \phi^2\Big)\underline{g}_{M,aa} + \mathbb{C}(0,0) \\
&\quad + \mathcal{O}\Big(\frac{r}{(t+r)^2}\Big)\underline{\widetilde{\partial}}_a h_{00} + \Big(\frac{1}{(t+r)^2} + \mathcal{O}\Big(\frac{r^2}{(t+r)^4}\Big)\Big)h_{00} \\
&\quad + \mathcal{O}\Big(\frac{1}{t+r}\Big)\underline{\widetilde{\partial}}_a h_{0a} + \mathcal{O}\Big(\frac{r}{(t+r)^3}\Big)h_{0a}.
\end{aligned}
$$

Finally, for $a \neq b$ we write similarly

$$\begin{aligned}
\widehat{\Box}_g \underline{\widetilde{h}}_{ab} = & \, \underline{\widetilde{P}}_{ab} + \underline{\widetilde{\Phi}}_a^{\alpha'} \underline{\widetilde{\Phi}}_b^{\beta'} Q_{\alpha'\beta'} + \mathcal{O}(1) \underline{\widetilde{\partial}}_a \rho^\sharp \underline{\widetilde{\partial}}_b \rho^\sharp + \mathcal{O}(1) \underline{\widetilde{\partial}}_a \phi \underline{\widetilde{\partial}}_b \phi \\
& + \mathcal{O}(1) \Big(\rho^{\flat 2} + \phi^2 \Big) \underline{\widetilde{g}}^\dagger_{ab} + \frac{2x}{T^2} \Big(x_b \underline{\widetilde{\partial}}_a h_{00} + x_a \underline{\widetilde{\partial}}_b h_{00} \Big) \\
& + \mathcal{O}(1) \frac{2}{t+r} \underline{\widetilde{\partial}}_a h_{0b} + \frac{2}{t+r} \underline{\widetilde{\partial}}_b h_{0a} \\
& - \frac{6x_a x_b}{(t+r)^4} h_{00} - \frac{2x_a}{(t+r)^3} h_{0b} - \frac{2x_b}{(t+r)^3} h_{0a} + \mathbb{C}(0,0) \\
\simeq & \, \underline{P}_{ab} + + \mathbb{GS}_{hh}(0,0) + \mathcal{O}(1) \underline{\widetilde{\partial}}_a \rho^\sharp \underline{\widetilde{\partial}}_b \rho^\sharp + \mathcal{O}(1) \underline{\widetilde{\partial}}_a \phi \underline{\widetilde{\partial}}_b \phi \\
& + \mathcal{O}\Big(\frac{r}{(t+r)^2} \Big) \underline{\widetilde{\partial}}_a h_{00} + \mathcal{O}\Big(\frac{r}{(t+r)^2} \Big) \underline{\widetilde{\partial}}_b h_{00} + \mathcal{O}\Big(\frac{r^2}{(t+r)^4} \Big) h_{00} \\
& + \mathcal{O}\Big(\frac{1}{t+r} \Big) \underline{\widetilde{\partial}}_a h_{0b} + \mathcal{O}\Big(\frac{r}{(t+r)^3} \Big) h_{0b} + \mathcal{O}\Big(\frac{1}{t+r} \Big) \underline{\widetilde{\partial}}_a h_{0a} + \mathcal{O}\Big(\frac{r}{(t+r)^3} \Big) h_{0a} \\
& + \mathcal{O}(1) \Big(\rho^{\flat 2} + \phi^2 \Big) \underline{\widetilde{g}}_{M,ab} + \mathbb{C}(0,0) \quad (a \neq b).
\end{aligned}$$

This completes the proof of Lemma 14.1. □

14.2 Reduction of the expression of quasi-null terms

The quasi-null terms in the Euclidian–semi-hyperboloidal frame are

$$\underline{\widetilde{P}}_{\alpha\beta} = \frac{1}{4} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{\widetilde{\partial}}_\alpha h_{\gamma\delta} \underline{\widetilde{\partial}}_\beta h_{\gamma'\delta'} - \frac{1}{2} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{\widetilde{\partial}}_\alpha h_{\gamma\gamma'} \underline{\widetilde{\partial}}_\beta h_{\delta\delta'}. \tag{14.6}$$

Clearly, the components $\underline{\widetilde{P}}_{\alpha\beta}$ (but not the component $\underline{\widetilde{P}}_{00}$) are null terms since they involve at least one derivative tangential to the hyperboloids in the interior domain and tangential to the light cone in the transition and exterior domains. Hence, the following statement is immediate.

Lemma 14.2. *The (a,β)-components of the quasi-null terms have the property*

$$\underline{\widetilde{P}}_{a\beta} \simeq \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) \tag{14.7}$$

and, more generally,

$$\begin{aligned}
& \partial^I L^J \Omega^K \underline{\widetilde{P}}_{a\beta} \simeq \mathbb{GS}_{hh}(p,k) + \mathbb{C}(p,k), \\
& p = |I| + |J| + |K|, \quad |J| + |K| = k.
\end{aligned} \tag{14.8}$$

Consequently, in the equations satisfied by the components $\underline{\widetilde{h}}_{a\beta}$ (but excluding $\underline{\widetilde{h}}_{00}$), the relevant quasi-null component $\underline{\widetilde{P}}_{a\beta}$ are null terms and, consequently, we will be able to establish a good decay estimate for these components. On the other hand, the component $\underline{\widetilde{h}}_{00}$, whose evolution equation involves $\underline{\widetilde{P}}_{00}$, requires special analysis (based on the next section).

Lemma 14.3. *The component $\underline{\widetilde{P}}_{00}$ satisfies, up to irrelevant multiplicative coefficients of degree ≤ 0,*

$$\begin{aligned}\underline{\widetilde{P}}_{00} \simeq & \, \underline{\widetilde{g}}_M^{\gamma\gamma'} \underline{\widetilde{g}}_M^{\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\gamma'} \partial_t \underline{\widetilde{h}}_{\delta\delta'} + \underline{\widetilde{g}}_M^{\gamma\gamma'} \underline{\widetilde{g}}_M^{\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\delta} \partial_t \underline{\widetilde{h}}_{\gamma'\delta'} \\ & + \mathbb{GS}_{hh}(0,0) + \mathbb{F}(0,0) + \mathbb{C}(0,0).\end{aligned} \tag{14.9}$$

Proof. By changes of frame formulas, such as

$$\begin{aligned}& g^{\dagger\gamma\gamma'} g^{\delta\delta'} \partial_t h_{\gamma\delta} \partial_t h_{\gamma'\delta'} \\ & = \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\delta} \partial_t \underline{\widetilde{h}}_{\gamma'\delta'} + \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \underline{\widetilde{h}}_{\gamma''\delta''} \partial_t \big(\underline{\widetilde{\Psi}}_\gamma^{\gamma''} \underline{\widetilde{\Psi}}_\delta^{\delta''}\big) \partial_t \big(\underline{\widetilde{\Psi}}_{\gamma'}^{\gamma'''} \underline{\widetilde{\Psi}}_{\delta'}^{\delta'''}\big) \underline{\widetilde{h}}_{\gamma'''\delta'''} \\ & \quad + \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \underline{\widetilde{\Psi}}_\gamma^{\gamma''} \underline{\widetilde{\Psi}}_\delta^{\delta''} \partial_t \underline{\widetilde{h}}_{\gamma''\delta''} \partial_t \big(\underline{\widetilde{\Psi}}_{\gamma'}^{\gamma'''} \underline{\widetilde{\Psi}}_{\delta'}^{\delta'''}\big) \underline{\widetilde{h}}_{\gamma'''\delta'''} \\ & \quad + \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \partial_t \big(\underline{\widetilde{\Psi}}_\gamma^{\gamma''} \underline{\widetilde{\Psi}}_\delta^{\delta''}\big) \underline{\widetilde{h}}_{\gamma''\delta''} \underline{\widetilde{\Psi}}_{\gamma'}^{\gamma''} \underline{\widetilde{\Psi}}_{\delta'}^{\delta''} \partial_t \underline{\widetilde{h}}_{\gamma'''\delta'''},\end{aligned}$$

we can put $\underline{\widetilde{P}}_{00}$ in the form

$$\begin{aligned}\underline{\widetilde{P}}_{00} & = \frac{1}{4} \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\delta} \partial_t \underline{\widetilde{h}}_{\gamma'\delta'} - \frac{1}{2} \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\gamma'} \partial_t \underline{\widetilde{h}}_{\delta\delta'} \\ & \simeq \frac{1}{4} \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\delta} \partial_t \underline{\widetilde{h}}_{\gamma'\delta'} - \frac{1}{2} \underline{\widetilde{g}}^{\dagger\gamma\gamma'} \underline{\widetilde{g}}^{\dagger\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\gamma'} \partial_t \underline{\widetilde{h}}_{\delta\delta'} + \mathbb{F}(0,0).\end{aligned}$$

Moreover, modulo cubic terms, the curved metric can be replaced by the flat one, i.e.

$$\frac{1}{4} \underline{\widetilde{g}}_M^{\gamma\gamma'} \underline{\widetilde{g}}_M^{\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\delta} \partial_t \underline{\widetilde{h}}_{\gamma'\delta'} - \frac{1}{2} \underline{\widetilde{g}}_M^{\gamma\gamma'} \underline{\widetilde{g}}_M^{\dagger\delta\delta'} \partial_t \underline{\widetilde{h}}_{\gamma\gamma'} \partial_t \underline{\widetilde{h}}_{\delta\delta'} + \mathbb{F}(0,0) + \mathbb{C}(0,0).$$

□

We again rely on the wave gauge condition and in view of the decomposition (14.9) of $\underline{\widetilde{P}}_{00}$ and thanks to Proposition 14.5, we arrive at the following conclusion, which shows that it will remain only to control the term $\partial_t \underline{\widetilde{h}}_{a\alpha} \partial_t \underline{\widetilde{h}}_{b\beta}$ (and its derivatives $\partial^I L^J$), which is not a null term and will require an improved decay estimate on $\partial \underline{\widetilde{h}}_{a\alpha}$.

Proposition 14.4 (Quasi-null interactions in the null-semi-hyperboloidal frame)**.** *The quasi-null terms satisfy*

$$\begin{aligned}\underline{\widetilde{P}}_{00} &\simeq \partial_t\underline{\widetilde{h}}_{a\alpha}\partial_t\underline{\widetilde{h}}_{b\beta} + \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) + \mathbb{F}(0,0),\\ \underline{\widetilde{P}}_{a\beta} &\simeq \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) + \mathbb{F}(0,0).\end{aligned} \tag{14.10}$$

More generally, this statement also holds for their derivatives:

$$\begin{aligned}\partial^I L^J\underline{\widetilde{P}}_{00} &\simeq \partial^I L^J\big(\partial_t\underline{\widetilde{h}}_{a\alpha}\partial_t\underline{\widetilde{h}}_{b\beta}\big) + \mathbb{GS}_{hh}(p,k) + \mathbb{C}(p,k) + \mathbb{F}(p,k),\\ \partial^I L^J\underline{\widetilde{P}}_{a\beta} &\simeq \mathbb{GS}_{hh}(p,k) + \mathbb{C}(p,k) + \mathbb{F}(p,k),\end{aligned} \tag{14.11}$$

with $p = |I|$ and $k = |J|$, provided $|\partial^{I'}L^{J'}h| + |\partial^{I'}L^{J'}\partial h|$ is sufficiently small for all $|I'| + |J'| \leq p + k$.

Proof. **Step 1**. We first claim that

$$\begin{aligned}&\underline{\widetilde{g}}^{\dagger\alpha\alpha'}\underline{\widetilde{g}}^{\dagger\beta\beta'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\alpha\alpha'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\beta\beta'}\\ &\simeq \underline{\widetilde{g}}^{\dagger 0a}\underline{\partial}_0\underline{\widetilde{g}}^{\dagger}_{0a}\underline{g}^{\dagger 0b}\underline{\partial}_0\underline{g}^{\dagger}_{0b} + \mathbb{GS}_{hh}(0,0) + \mathbb{F}(0,0) + \mathbb{C}(0,0).\end{aligned} \tag{14.12}$$

Indeed, the wave gauge condition

$$g^{\dagger\alpha\beta}\partial_\alpha h_{\beta\gamma} = \frac{1}{2}g^{\dagger\alpha\beta}\partial_\gamma h_{\alpha\beta} \tag{14.13}$$

reads, in the null-semi-hyperboloidal frame,

$$\underline{\widetilde{g}}^{\dagger\alpha\beta}\underline{\partial}_\alpha\underline{\widetilde{h}}_{\beta\gamma} + \underline{\widetilde{\Phi}}_\gamma^{\gamma'}\underline{\widetilde{g}}^{\dagger\alpha\beta}\partial_\alpha\Big(\underline{\widetilde{\Psi}}_\beta^{\beta'}\underline{\widetilde{\Psi}}_{\gamma'}^{\gamma''}\Big)\underline{\widetilde{h}}_{\beta'\gamma''} = \frac{1}{2}\underline{\widetilde{g}}^{\dagger\alpha\beta}\underline{\widetilde{\partial}}_\gamma\underline{\widetilde{h}}_{\alpha\beta} + \frac{1}{2}\underline{\widetilde{g}}^{\dagger\alpha\beta}\underline{\widetilde{\partial}}_\gamma\Big(\underline{\widetilde{\Psi}}_\alpha^{\alpha'}\underline{\widetilde{\Psi}}_\beta^{\beta'}\Big)\underline{\widetilde{h}}_{\alpha'\beta'}.$$

Letting[1] $\gamma = 0$, we find

$$\underline{\widetilde{g}}^{\dagger\alpha\beta}\partial_t\underline{\widetilde{h}}_{\alpha\beta} = 2\,\underline{\widetilde{g}}^{\dagger\alpha\beta}\underline{\widetilde{\partial}}_\alpha\underline{h}_{0\beta} + 2\,\underline{\widetilde{\Phi}}_0^{\gamma'}g^{\alpha\beta}\partial_\alpha\Big(\underline{\widetilde{\Psi}}_\beta^{\beta'}\underline{\widetilde{\Psi}}_{\gamma'}^{\gamma''}\Big)\underline{\widetilde{h}}_{\beta'\gamma''} - \underline{\widetilde{g}}^{\dagger\alpha\beta}\partial_t\Big(\underline{\widetilde{\Psi}}_\alpha^{\alpha'}\underline{\widetilde{\Psi}}_\beta^{\beta'}\Big)\underline{\widetilde{h}}_{\alpha'\beta'},$$

which we put in the form

$$\begin{aligned}\underline{\widetilde{g}}^{\dagger\alpha\beta}\partial_t\underline{h}_{\alpha\beta} = {}& 2\,\underline{\widetilde{g}}_M^{\alpha\beta}\underline{\partial}_\alpha\underline{\widetilde{h}}_{\beta 0} + 2\underline{\widetilde{h}}^{\alpha\beta}\underline{\partial}_\alpha\underline{\widetilde{h}}_{\beta 0} + 2\,\underline{\widetilde{\Phi}}_0^{\gamma'}\underline{\widetilde{g}}_M^{\alpha\beta}\partial_\alpha\Big(\underline{\widetilde{\Psi}}_\beta^{\beta'}\underline{\widetilde{\Psi}}_{\gamma'}^{\gamma''}\Big)\underline{\widetilde{h}}_{\beta'\gamma''}\\ &- \underline{\widetilde{g}}_M^{\alpha\beta}\partial_t\Big(\underline{\widetilde{\Psi}}_\alpha^{\alpha'}\underline{\widetilde{\Psi}}_\beta^{\beta'}\Big)\underline{\widetilde{h}}_{\alpha'\beta'}\\ &+ 2\,\underline{\Phi}_0^{\gamma'}h^{\alpha\beta}\partial_\alpha\Big(\underline{\Psi}_\beta^{\beta'}\underline{\Psi}_{\gamma'}^{\gamma''}\Big)\underline{h}_{\beta'\gamma''} - \underline{\widetilde{h}}^{\alpha\beta}\partial_t\Big(\underline{\Psi}_\alpha^{\alpha'}\underline{\Psi}_\beta^{\beta'}\Big)\underline{h}_{\alpha'\beta'}.\end{aligned}$$

[1]Observe that, in contrast, γ is replaced by $a = 1, 2, 3$ in the proof of Proposition 14.5.

In the right-hand side, except for the first term, we have at least quadratic terms or terms containing an extra decay factor such as $\partial_\alpha\Big(\underline{\widetilde{\Psi}}_\beta^{\beta'}\,\underline{\widetilde{\Psi}}_\gamma^{\gamma''}\Big)$. So, in the expression of interest $\underline{\widetilde{g}}^{\dagger\alpha\alpha'}\underline{\widetilde{g}}^{\dagger\beta\beta'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\alpha\alpha'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\beta\beta'}$, the only term to be treated is

$$\underline{\widetilde{g}}_M^{\alpha\alpha'}\underline{\widetilde{g}}_M^{\beta\beta'}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}_{\alpha'0}\underline{\widetilde{\partial}}_\beta\underline{\widetilde{h}}_{\beta'0}. \tag{14.14}$$

We also emphasize that, provided $|\underline{\widetilde{h}}|$ is sufficiently small, $\underline{\widetilde{h}}^{\alpha\beta}$ can be expressed as a power series in $\underline{\widetilde{h}}_{\alpha\beta}$ (without zero-order terms), which is itself a linear combination of $h_{\alpha\beta}$ (with homogeneous coefficients of degree ≤ 0). So, similarly, $\underline{\widetilde{h}}^{\alpha\beta}$ can be expressed as a power series in $h_{\alpha\beta}$.

We deduce that all the terms within the key product $\underline{\widetilde{g}}^{\dagger\alpha\alpha'}\underline{\widetilde{g}}^{\dagger\beta\beta'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\alpha\alpha'}\partial_t\underline{\widetilde{g}}^{\dagger}_{\beta\beta'}$ belong to $\mathbb{C}(0,0)$ or $\mathbb{F}(0,0)$, with the possible exception of the term

$$\underline{\widetilde{g}}_M^{\alpha\alpha'}\underline{\widetilde{g}}_M^{\beta\beta'}\underline{\partial}_\alpha\underline{\widetilde{h}}_{\alpha'0}\underline{\partial}_\beta\underline{\widetilde{h}}_{\beta'0}. \tag{14.15}$$

We therefore focus on this term and write

$$\begin{aligned}
&\big(\underline{\widetilde{g}}_M^{\alpha\alpha'}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}_{\alpha'0}\big)\big(\underline{\widetilde{g}}_M^{\beta\beta'}\underline{\widetilde{\partial}}_\beta\underline{\widetilde{h}}_{\beta'0}\big)\\
&= \Big(\underline{\widetilde{g}}_M^{a\alpha'}\underline{\widetilde{\partial}}_a\underline{\widetilde{h}}_{\alpha'0}+\underline{\widetilde{g}}_M^{00}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{00}+\underline{\widetilde{g}}_M^{0a'}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{0a'}\Big)\Big(\underline{\widetilde{g}}_M^{b\beta'}\underline{\widetilde{\partial}}_b\underline{\widetilde{h}}_{\beta'0}+\underline{\widetilde{g}}_M^{00}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{00}+\underline{\widetilde{g}}_M^{0b}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{0b}\Big)\\
&= \Big(\underline{\widetilde{g}}_M^{a\alpha'}\underline{\widetilde{\partial}}_a\underline{\widetilde{h}}_{\alpha'0}+\underline{\widetilde{g}}_M^{00}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{00}\big)\big(\underline{\widetilde{g}}_M^{b\beta'}\underline{\widetilde{\partial}}_b\underline{\widetilde{h}}_{\beta'0}+\underline{\widetilde{g}}_M^{00}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{00}+\underline{\widetilde{g}}_M^{0b}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{0b}\Big)\\
&\quad+\underline{\widetilde{g}}_M^{0a'}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{0a'}\Big(\underline{\widetilde{h}}^{b\beta'}\underline{\widetilde{\partial}}_b\underline{\widetilde{h}}_{\beta'0}+\underline{\widetilde{g}}_M^{00}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{00}\Big)+\underline{\widetilde{g}}_M^{0a'}\underline{\partial}_0\underline{\widetilde{h}}_{0a'}\underline{\widetilde{g}}_M^{0b}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}_{0b}.
\end{aligned}$$

The last term is already listed in (14.12). The remaining terms are null quadratic terms (recall that $\underline{g}_M^{00}=s^2/T^2$).

Step 2. The term $\underline{\widetilde{g}}^{\dagger\gamma\gamma'}\underline{\widetilde{g}}^{\dagger\delta\delta'}\partial_t\underline{\widetilde{h}}_{\gamma\gamma'}\partial_t\underline{\widetilde{h}}_{\delta\delta'}$ being treated in Step 1, it remains to discuss the term

$$\underline{\widetilde{g}}_M^{\gamma\gamma'}\underline{\widetilde{g}}_M^{\delta\delta'}\partial_t\underline{\widetilde{h}}_{\gamma\delta}\partial_t\underline{\widetilde{h}}_{\gamma'\delta'}. \tag{14.16}$$

- In the interior domain, we have

$$\big|\partial^I L^J\underline{g}_M^{00}\big|\lesssim\frac{s^2}{T^2},\qquad\big|\underline{g}_M^{\alpha\beta}\big|\lesssim 1,\qquad \underline{g}_M^{00}=\frac{s^2}{T^2}\quad\text{in }\mathcal{M}^{\text{int}}. \tag{14.17}$$

– On one hand, by letting $(\gamma,\gamma')=(0,0)$ or $(\delta,\delta')=(0,0)$, the corresponding term $\underline{g}_M^{\gamma\gamma'}\underline{g}_M^{\delta\delta'}\partial_t\underline{h}_{\gamma\delta}\partial_t\underline{h}_{\gamma'\delta'}$ is a good term, since it contains the favorable factor $(s/t)^2$.
– On the other hand, with $(\gamma,\gamma')\neq(0,0)$ and $(\delta,\delta')\neq(0,0)$, we can introduce the notation $(\gamma,\gamma')=(a,\alpha)$ and $(\delta,\delta')=(b,\beta)$, and we conclude that $\underline{g}_M^{\gamma\gamma'}\underline{g}_M^{\delta\delta'}\partial_t\underline{h}_{\gamma\delta}\partial_t\underline{h}_{\gamma'\delta'}$ is a linear combination of $\partial_t\underline{h}_{a\alpha}\partial_t\underline{h}_{b\beta}$ with homogeneous coefficients of degree zero.

- In the transition and exterior domain

$$\partial^I L^J \widetilde{g}_M^{00}=0 \qquad \big|\underline{g}_M^{\alpha\beta}\big|\lesssim 1 \quad \text{in } \mathcal{M}^{\text{ext}}. \tag{14.18}$$

– On one hand, by letting $(\gamma,\gamma')=(0,0)$ or $(\delta,\delta')=(0,0)$, the corresponding term $\widetilde{g}_M^{\gamma\gamma'}\widetilde{g}_M^{\delta\delta'}\partial_t\widetilde{h}_{\gamma\delta}\partial_t\widetilde{h}_{\gamma'\delta'}$ vanishes.
– On the other hand, with $(\gamma,\gamma')\neq(0,0)$ and $(\delta,\delta')\neq(0,0)$, we can introduce the notation $(\gamma,\gamma')=(a,\alpha)$ and $(\delta,\delta')=(b,\beta)$, and we obtain that $\widetilde{g}_M^{\gamma\gamma'}\widetilde{g}_M^{\delta\delta'}\partial_t\widetilde{h}_{\gamma\delta}\partial_t\widetilde{h}_{\gamma'\delta'}$ is a linear combination of $\partial_t\widetilde{h}_{a\alpha}\partial_t\widetilde{h}_{b\beta}$ with homogeneous coefficients of degree zero.

□

14.3 The null-semi-hyperboloidal $(0,0)$-component

Next, we rely on the notation

$$\widetilde{\underline{h}}^{\alpha\beta}=h^{\alpha'\beta'}\underline{\widetilde{\Psi}}_{\alpha'}^{\alpha}\underline{\widetilde{\Psi}}_{\beta'}^{\beta}, \qquad \widetilde{\underline{h}}_{\alpha\beta}=h_{\alpha'\beta'}\underline{\widetilde{\Phi}}_{\alpha}^{\alpha'}\underline{\widetilde{\Phi}}_{\beta}^{\beta'}, \tag{14.19}$$

and

$$\begin{aligned} h^{\alpha\beta}&=g^{\dagger\alpha\beta}-g_M^{\alpha\beta}, & h_{\alpha\beta}&=g^{\dagger}_{\alpha\beta}-g_{M,\alpha\beta},\\ \widetilde{\underline{h}}^{\alpha\beta}&=\widetilde{\underline{g}}^{\dagger\alpha\beta}-\widetilde{\underline{g}}_M^{\alpha\beta}, & \widetilde{\underline{h}}_{\alpha\beta}&=\widetilde{\underline{g}}^{\dagger}_{\alpha\beta}-\widetilde{\underline{g}}_{M,\alpha\beta}. \end{aligned} \tag{14.20}$$

Proposition 14.5. *The metric component $\partial_t\widetilde{\underline{h}}^{00}$ in the semi-hyperboloidal frame can be decomposed as follows:*

$$\partial_t\widetilde{\underline{h}}^{00}\simeq\widetilde{\underline{\partial}}\widetilde{\underline{h}}+\frac{1}{t+r}\widetilde{\underline{h}}+\widetilde{\underline{h}}\,\partial\widetilde{\underline{h}}+\frac{1}{t+r}h\,\widetilde{\underline{h}}+\begin{cases}\frac{s^2}{t^2}\partial\widetilde{\underline{h}} & \text{in } \mathcal{M}^{int},\\ 0 & \text{in } \mathcal{M}^{tran}\cup\mathcal{M}^{ext}.\end{cases} \tag{14.21}$$

Proof. The wave gauge condition $g^{\dagger\alpha\beta}\Gamma^{\dagger\gamma}_{\alpha\beta} = 0$ is equivalent to

$$g^\dagger{}_{\beta\gamma}\partial_\alpha g^{\dagger\alpha\beta} = \frac{1}{2} g^\dagger_{\alpha\beta}\partial_\gamma g^{\dagger\alpha\beta} \tag{14.22}$$

and therefore, in the null-semi-hyperboloidal frame, reads

$$\underline{\widetilde{g}}^\dagger_{\beta\gamma}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} + g^\dagger_{\beta'\gamma'}\underline{\widetilde{\Phi}}_\gamma^{\gamma'}\underline{\widetilde{h}}^{\alpha\beta}\partial_{\alpha'}\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}\big) = \frac{1}{2}\underline{\widetilde{g}}^\dagger_{\alpha\beta}\underline{\widetilde{\partial}}_\gamma\underline{\widetilde{h}}^{\alpha\beta} + \frac{1}{2}\underline{\widetilde{g}}^\dagger_{\alpha\beta}\underline{\widetilde{h}}^{\alpha'\beta'}\underline{\widetilde{\partial}}_\gamma\big(\underline{\widetilde{\Phi}}_{\alpha'}^{\alpha}\underline{\widetilde{\Phi}}_{\beta'}^{\beta}\big)$$

or, equivalently,

$$\begin{aligned}\underline{\widetilde{g}}_{M,\beta\gamma}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} =& \frac{1}{2}\underline{\widetilde{g}}^\dagger_{\alpha\beta}\underline{\widetilde{\partial}}_\gamma\underline{\widetilde{h}}^{\alpha\beta} + \frac{1}{2}\underline{\widetilde{g}}^\dagger_{\alpha\beta}\underline{\widetilde{h}}^{\alpha'\beta'}\underline{\widetilde{\partial}}_\gamma\big(\underline{\widetilde{\Phi}}_{\alpha'}^{\alpha}\underline{\widetilde{\Phi}}_{\beta'}^{\beta}\big)\\ &- \underline{\widetilde{g}}^\dagger_{\beta'\gamma'}\underline{\widetilde{\Phi}}_\gamma^{\gamma'}\underline{\widetilde{h}}^{\alpha\beta}\partial_{\alpha'}\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}\big) - \underline{\widetilde{h}}_{\beta\gamma}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta}.\end{aligned} \tag{14.23}$$

We replace the index γ by a Latin index $c = 1, 2, 3$ and obtain

$$\begin{aligned}\underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} =& \frac{1}{2}\underline{\widetilde{g}}_{M,\alpha\beta}\underline{\widetilde{\partial}}_c\underline{\widetilde{h}}^{\alpha\beta} + \frac{1}{2}\underline{\widetilde{h}}_{\alpha\beta}\underline{\widetilde{\partial}}_c\underline{\widetilde{h}}^{\alpha\beta} + \frac{1}{2}g^\dagger_{\alpha\beta}\underline{\widetilde{h}}^{\alpha'\beta'}\underline{\widetilde{\partial}}_c\big(\underline{\widetilde{\Phi}}_{\alpha'}^{\alpha}\underline{\widetilde{\Phi}}_{\beta'}^{\beta}\big)\\ &- \underline{\widetilde{g}}^\dagger_{\beta'\gamma'}\underline{\widetilde{\Phi}}_c^{\gamma'}\underline{\widetilde{h}}^{\alpha\beta}\partial_{\alpha'}\big(\underline{\widetilde{\Phi}}_\alpha^{\alpha'}\underline{\widetilde{\Phi}}_\beta^{\beta'}\big) - \underline{\widetilde{h}}_{\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta},\end{aligned} \tag{14.24}$$

which is of the form $\underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} = \frac{1}{2}\underline{\widetilde{g}}_{M,\alpha\beta}\underline{\widetilde{\partial}}_c\underline{\widetilde{h}}^{\alpha\beta} + \text{l.o.t.}$, in which l.o.t. are terms of the form given in the right-hand side of (14.21).

The left-hand side of (14.24) admits the following decomposition:

$$\underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} = \underline{\widetilde{g}}_{M,0c}\underline{\partial}_0\underline{h}^{00} + \underline{\widetilde{g}}_{M,bc}\underline{\widetilde{\partial}}_0\underline{h}^{0b} + \underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_a\underline{\widetilde{h}}^{a\beta}$$

or, equivalently,

$$\underline{\widetilde{g}}_{M,0c}\underline{\partial}_0\underline{\widetilde{h}}^{00} = \underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} - \underline{\widetilde{g}}_{M,bc}\underline{\widetilde{\partial}}_0\underline{h}^{0b} - \underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_a\underline{h}^{a\beta}.$$

After multiplication by $\underline{\widetilde{g}}^{0c}_M$, we obtain the identity:

$$\underline{\widetilde{g}}^{0c}_M\underline{\widetilde{g}}_{M,0c}\underline{\partial}_0\underline{\widetilde{h}}^{00} = \underline{\widetilde{g}}^{0c}_M\underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_\alpha\underline{\widetilde{h}}^{\alpha\beta} - \underline{\widetilde{g}}^{0c}_M\underline{\widetilde{g}}_{M,bc}\underline{\widetilde{\partial}}_0\underline{\widetilde{h}}^{0b} - \underline{\widetilde{g}}^{0c}_M\underline{\widetilde{g}}_{M,\beta c}\underline{\widetilde{\partial}}_a\underline{\widetilde{h}}^{a\beta'}. \tag{14.25}$$

- Next, within the interior domain $\mathcal{M}^{\text{int}}$, we have

$$\underline{g}^{0c}_M\underline{g}_{M,0c} = (r^2/t^2), \qquad \underline{g}^{0c}_M\underline{g}_{M,bc} = -(s/t)^2(x_b/t) \quad \text{in } \mathcal{M}^{\text{int}}, \tag{14.26}$$

  and we rewrite (14.25) as

$$(r/t)^2\underline{\partial}_0\underline{h}^{00} = (s/t)^2\frac{x_b}{t}\underline{\partial}_0\underline{h}^{0b} - \underline{g}^{0c}_M\underline{g}_{M,\beta c}\underline{\partial}_a\underline{h}^{a\beta} + \underline{g}^{0c}_M\underline{g}_{M,\beta c}\underline{\partial}_\alpha\underline{h}^{\alpha\beta} \qquad \text{in } \mathcal{M}^{\text{int}}. \tag{14.27}$$

- Still in the interior domain, by replacing in (14.27) the term $\underline{g}_M^{0c}\underline{g}_{M,\beta c}\underline{\partial}_\alpha\underline{h}^{\alpha\beta}$ by its expression given by the wave condition (14.24), we arrive at

$$
\begin{aligned}
\frac{r^2}{t^2}\underline{\partial}_0\underline{h}^{00} =& \frac{s^2}{t^2}\frac{x_b}{t}\underline{\partial}_0\underline{h}^{0b} - \underline{g}_M^{0c}\underline{g}_{M,\beta c}\underline{\partial}_a\underline{h}^{a\beta'} \\
&+ \underline{g}_M^{0c}\Big(\frac{1}{2}\underline{g}^\dagger_{\alpha\beta}\underline{\partial}_c\underline{h}^{\alpha\beta} + \frac{1}{2}g^\dagger_{\alpha\beta}\underline{h}^{\alpha'\beta'}\underline{\partial}_c\big(\underline{\Phi}^\alpha_{\alpha'}\underline{\Phi}^\beta_{\beta'}\big) \\
&\qquad - \widetilde{\underline{g}}^\dagger_{\beta'\gamma'}\underline{\Phi}^{\gamma'}_c\underline{h}^{\alpha\beta}\partial_{\alpha'}\big(\underline{\Phi}^{\alpha'}_\alpha\underline{\Phi}^{\beta'}_\beta\big) - \underline{h}_{\beta c}\underline{\partial}_\alpha\underline{h}^{\alpha\beta}\Big) \qquad \text{in } \mathcal{M}^{\text{int}}.
\end{aligned}
$$

We therefore have (in symbolic notation)

$$
\frac{r^2}{t^2}\partial_t\underline{h}^{00} \simeq \frac{s^2}{t^2}\partial_\alpha\underline{h}^{\beta\gamma} + \underline{\partial}_a\underline{h}^{\beta\gamma} + \frac{1}{t}\underline{h}^{\alpha\beta} + \underline{h}^{\alpha\beta}\partial_\gamma\underline{h}^{\alpha'\beta'} + \frac{1}{t}h_{\alpha\beta}\underline{h}^{\alpha'\beta'} \qquad \text{in } \mathcal{M}^{\text{int}},
$$

and we conclude with $1 = (r/t)^2 + (s/t)^2$ in $\mathcal{M}^{\text{int}}$.

- On the other hand, in the transition and exterior domains, we use the relations

$$
\widetilde{g}_M^{0c}\,\widetilde{g}_{M,0c} = 1, \qquad \widetilde{g}_M^{0c}\,\widetilde{g}_{M,bc} = 0. \tag{14.28}
$$

We now rewrite (14.25) as

$$
\widetilde{\partial}_0\widetilde{h}^{00} = -\widetilde{g}_M^{0c}\widetilde{g}_{M,\beta c}\widetilde{\partial}_a\widetilde{h}^{a\beta} + \widetilde{g}_M^{0c}\widetilde{g}_{M,\beta c}\widetilde{\partial}_\alpha\widetilde{h}^{\alpha\beta} \qquad \text{in } \mathcal{M}^{\text{tran}} \cup \mathcal{M}^{\text{ext}}. \tag{14.29}
$$

□

We use the short-hand notation $|\underline{\widetilde{h}}|, |\partial\underline{\widetilde{h}}|, |\underline{\widetilde{\partial}}\underline{\widetilde{h}}|$ to denote components of the metric, its derivatives, and tangential derivatives, respectively. Recall that within our bootstrap argument, all the component of h and its derivatives are small. The "bad" derivative of $\underline{\widetilde{h}}^{00}$ is now shown to be bounded by tangential derivatives or terms with better decay —thanks to the favorable factor s^2/t^2 in the interior. For clarity, in the following statement we distinguish between the interior, transition, and exterior domains.

Corollary 14.6. *The following estimate holds:*

$$
\big|\underline{\widetilde{\partial}_t}\underline{\widetilde{h}}^{00}\big| \lesssim
\begin{cases}
\frac{s^2}{t^2}|\partial\underline{h}| + |\underline{\partial h}| + \frac{1}{t}|\underline{h}| + |\partial\underline{h}|\,|\underline{h}| & \textit{in } \mathcal{M}^{int}, \\
\\
|\widetilde{\partial}\widetilde{h}| + \frac{1}{r}|\widetilde{h}| + |\partial\widetilde{h}|\,|\widetilde{h}| & \textit{in } \mathcal{M}^{tran}\cup\mathcal{M}^{ext},
\end{cases} \tag{14.30}
$$

More generally, one has

$$
\begin{aligned}
&\big|\partial^I L^J \partial_t \underline{\widetilde{h}}^{00}\big| + \big|\partial_t \partial^I L^J \underline{\widetilde{h}}^{00}\big| \\
&\lesssim \sum_{\substack{|I'|+|J'|\leq|I|+|J| \\ |J'|\leq|J|}} \Big(\big|\partial^{I'} L^{J'} \underline{\widetilde{\partial}\widetilde{h}}\big| + \frac{1}{t+r}\big|\partial^{I'} L^{J'} \underline{\widetilde{h}}\big| \Big) + \sum_{\substack{|I_1|+|I_2|\leq|I| \\ |J_1|+|J_2|\leq|J|}} \big|\partial^{I_1} L^{J_1} \underline{\widetilde{h}}\big| \, \big|\partial \partial^{I_2} L^{J_2} \underline{\widetilde{h}}\big| \\
&\quad + \begin{cases} \sum_{\substack{|I'|+|J'|\leq|I|+|J| \\ |J'|\leq|J|}} \frac{s^2}{t^2} \big|\partial \partial^{I'} L^{J'} \underline{\widetilde{h}}\big|, & \textit{in } \mathcal{M}^{int}, \\ 0, & \textit{in } \mathcal{M}^{tran} \cup \mathcal{M}^{ext}. \end{cases}
\end{aligned}
\tag{14.31}
$$

Proof. By Proposition 14.5, the expression $\partial_t \underline{\widetilde{h}}^{00}$ is a linear combination of the terms in (14.21) with homogeneous coefficients of degree ≤ 0. Hence, $\partial^I L^J \partial_t \underline{\widetilde{h}}^{00}$ is again a linear combination with coefficients of degree $\leq |I|$ of:

$$
\begin{aligned}
&\partial^{I'} L^{J'} \big((s/t)^2 \partial_\alpha \underline{\widetilde{h}}^{\beta\gamma} \big) \quad \text{in the interior) only,)} \\
&\partial^{I'} L^{J'} \big(\underline{\widetilde{\partial}_a \widetilde{h}}^{\beta\gamma} \big), \qquad \frac{1}{t+r} \partial^{I'} L^{J'} \big(\underline{\widetilde{h}}^{\alpha\beta} \big), \\
&\partial^{I'} L^{J'} \big(\underline{h}^{\alpha\beta} \partial_\gamma \underline{\widetilde{h}}^{\alpha'\beta'} \big), \\
&\frac{1}{t+r} \partial^{I'} L^{J'} \big(h_{\alpha\beta} \underline{\widetilde{h}}^{\alpha'\beta'} \big)
\end{aligned}
$$

with $|I'| \leq |I|$ and $|J'| \leq |J|$. We observe that

$$
\Big|\partial^{I'} L^{J'} \Big(\frac{s^2}{t^2} \partial_\alpha \underline{h}^{\beta\gamma} \Big)\Big| \lesssim \frac{s^2}{t^2} \sum_{\substack{|I''|\leq|I'| \\ |J''|\leq|J'|}} \big|\partial^{I''} L^{J''} \big(\partial_\alpha \underline{h}^{\beta\gamma} \big)\big| \qquad \text{in } \mathcal{M}^{\text{int}}.
$$

These terms are bounded by the commutator estimates in Sections 11 and 12 and we thus control $\partial^I L^J \partial_t \underline{\widetilde{h}}^{00}$. The statement for $\partial_t \partial^I L^J \underline{\widetilde{h}}^{00}$ similarly follows from the commutator estimates. □

15 Ricci curvature in conformal wave coordinates

15.1 Expression of the curvature

We give here a proof of Proposition 13.1. For simplicity, we write $g, \Gamma, \ldots$ instead of $g^\dagger, \Gamma^\dagger, \ldots$ throughout this proof. Consider the first two terms in the expression of the

Ricci curvature $R_{\alpha\beta}$, which involve second-order derivatives of the metric. Writing

$$\begin{aligned}\partial_\lambda \Gamma^\lambda_{\alpha\beta} =& -\frac{1}{2} g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + \frac{1}{2} g^{\lambda\delta}\partial_\lambda\partial_\alpha g_{\beta\delta} + \frac{1}{2} g^{\lambda\delta}\partial_\lambda\partial_\beta g_{\alpha\delta} \\ &+ \frac{1}{2}\partial_\lambda g^{\lambda\delta}\big(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\big)\end{aligned}$$

and

$$\partial_\alpha \Gamma^\lambda_{\beta\lambda} = \frac{1}{2}\partial_\alpha\partial_\beta g_{\lambda\delta} + \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta},$$

we obtain the identity

$$\begin{aligned}\partial_\lambda \Gamma^\lambda_{\alpha\beta} - \partial_\alpha \Gamma^\lambda_{\beta\lambda} =& -\frac{1}{2} g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + \frac{1}{2} g^{\lambda\delta}\partial_\alpha\partial_\lambda g_{\delta\beta} + \frac{1}{2} g^{\lambda\delta}\partial_\beta\partial_\lambda g_{\delta\alpha} - \frac{1}{2} g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} \\ &- \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\alpha g_{\beta\delta} + \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\beta g_{\alpha\delta} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta}.\end{aligned} \tag{15.1}$$

The first line contains second-order terms while the second line contains quadratic products of first-order terms.

15.2 Higher-order contributions

On the other hand, let us compute the combination $\partial_\alpha \Gamma_\beta + \partial_\beta \Gamma_\alpha$ which appears in our decomposition. We obtain

$$\begin{aligned}\Gamma^\gamma = g^{\alpha\beta}\Gamma^\gamma_{\alpha\beta} &= \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta}\big(\partial_\alpha g_{\beta\delta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\big) \\ &= g^{\gamma\delta} g^{\alpha\beta}\partial_\alpha g_{\beta\delta} - \frac{1}{2} g^{\alpha\beta} g^{\gamma\delta}\partial_\delta g_{\alpha\beta}\end{aligned}$$

and therefore

$$\Gamma_\lambda = g_{\lambda\gamma}\Gamma^\gamma = g^{\alpha\beta}\partial_\alpha g_{\beta\lambda} - \frac{1}{2} g^{\alpha\beta}\partial_\lambda g_{\alpha\beta},$$

so that

$$\begin{aligned}\partial_\alpha \Gamma_\beta &= \partial_\alpha\big(g^{\delta\lambda}\partial_\delta g_{\lambda\beta}\big) - \frac{1}{2}\partial_\alpha\big(g^{\lambda\delta}\partial_\beta g_{\lambda\delta}\big) \\ &= g^{\delta\lambda}\partial_\alpha\partial_\delta g_{\lambda\beta} - \frac{1}{2} g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta} + \partial_\alpha g^{\delta\lambda}\partial_\delta g_{\lambda\beta}.\end{aligned}$$

We have obtained the identity

$$\begin{aligned}\partial_\alpha\Gamma_\beta + \partial_\beta\Gamma_\alpha = {} & g^{\lambda\delta}\partial_\alpha\partial_\lambda g_{\delta\beta} + g^{\lambda\delta}\partial_\beta\partial_\lambda g_{\delta\alpha} - g^{\lambda\delta}\partial_\alpha\partial_\beta g_{\lambda\delta} \\ & + \partial_\alpha g^{\lambda\delta}\partial_\delta g_{\lambda\beta} + \partial_\beta g^{\lambda\delta}\partial_\delta g_{\lambda\alpha} - \frac{1}{2}\partial_\beta g^{\lambda\delta}\partial_\alpha g_{\lambda\delta} - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta}.\end{aligned} \tag{15.2}$$

Next, oobserve that the last term of (15.2) coincides with the last term of (15.1). The second-order terms in $\partial_\alpha\Gamma_\beta + \partial_\beta\Gamma_\alpha$ are three of the (four) second-order terms that arise in $\partial_\lambda\Gamma^\lambda_{\alpha\beta} - \partial_\alpha\Gamma^\lambda_{\beta\lambda}$. Hence, we find

$$\begin{aligned}\partial_\lambda\Gamma^\lambda_{\alpha\beta} - \partial_\alpha\Gamma^\lambda_{\beta\lambda} = {} & -\frac{1}{2}g^{\lambda\delta}\partial_\lambda\partial_\delta g_{\alpha\beta} + \frac{1}{2}\big(\partial_\alpha\Gamma_\beta + \partial_\beta\Gamma_\alpha\big) \\ & - \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\alpha g_{\beta\delta} + \frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\beta g_{\alpha\delta} \\ & - \frac{1}{2}\partial_\alpha g^{\lambda\delta}\partial_\delta g_{\lambda\beta} - \frac{1}{2}\partial_\beta g^{\lambda\delta}\partial_\delta g_{\lambda\alpha} - \frac{1}{4}\partial_\alpha g^{\lambda\delta}\partial_\beta g_{\lambda\delta} + \frac{1}{4}\partial_\beta g^{\lambda\delta}\partial_\alpha g_{\lambda\delta},\end{aligned}$$

therefore, using also the relation

$$\partial_\alpha g^{\lambda\delta} = -g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'},$$

we have

$$\begin{aligned}\partial_\lambda\Gamma^\lambda_{\alpha\beta} - \partial_\alpha\Gamma^\lambda_{\beta\lambda} = {} & -\frac{1}{2}\partial_\lambda g^{\lambda\delta}\partial_\delta g_{\alpha\beta} + \frac{1}{2}\big(\partial_\alpha\Gamma_\beta + \partial_\beta\Gamma_\alpha\big) \\ & + \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\delta g_{\alpha\beta} - \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\alpha g_{\beta\delta} \\ & - \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\beta g_{\alpha\delta} + \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\beta g_{\lambda\delta}} \\ & + \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} + \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha} - \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\alpha g_{\lambda\delta}}.\end{aligned}$$

Remarkably, the two underlined terms cancel each other, and the quadratic terms in $\partial_\lambda\Gamma^\lambda_{\alpha\beta} - \partial_\alpha\Gamma^\lambda_{\beta\lambda}$ are the following ones:

$$\begin{aligned}&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\delta g_{\alpha\beta}, \quad -\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\alpha g_{\beta\delta}, \quad -\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\beta g_{\alpha\delta}, \\ &\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta}, \quad \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha}.\end{aligned} \tag{15.3}$$

15.3 Quadratic contributions

Returning now to the expression of the Ricci curvature, we can consider the two products

$$\begin{aligned}\Gamma^\lambda_{\alpha\beta}\Gamma^\delta_{\lambda\delta} =& \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\lambda g_{\delta\delta'}\partial_\alpha g_{\beta\lambda'} + \partial_\beta g_{\alpha\lambda'}\partial_\lambda g_{\delta\delta'} - \partial_{\lambda'} g_{\alpha\beta}\partial_\lambda g_{\delta\delta'}\big),\\
\Gamma^\lambda_{\alpha\delta}\Gamma^\delta_{\beta\lambda} =& \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\delta\lambda'}\partial_\beta g_{\lambda\delta'} + \partial_\alpha g_{\delta\lambda'}\partial_\lambda g_{\beta\delta'} - \partial_\alpha g_{\delta\lambda'}\partial_{\delta'} g_{\beta\lambda}\\
&+ \partial_\delta g_{\alpha\lambda'}\partial_\beta g_{\lambda\delta'} + \partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\alpha\lambda'}\partial_{\delta'} g_{\beta\lambda}\\
&- \partial_{\lambda'} g_{\alpha\delta}\partial_\beta g_{\lambda\delta'} - \partial_{\lambda'} g_{\alpha\delta}\partial_\lambda g_{\beta\delta'} + \partial_{\lambda'} g_{\alpha\delta}\partial_{\delta'} g_{\beta\lambda}\big)\end{aligned}$$

and we arrive at

$$\begin{aligned}&\Gamma^\lambda_{\alpha\beta}\Gamma^\delta_{\lambda\delta} - \Gamma^\lambda_{\alpha\delta}\Gamma^\delta_{\beta\lambda}\\
&= -\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_{\lambda'} g_{\alpha\beta}\partial_\lambda g_{\delta\delta'} + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_{\delta'} g_{\beta\lambda} + \frac{1}{4}g^{\lambda\lambda'}\partial_{\lambda'} g_{\alpha\delta}\partial_\lambda g_{\beta\delta'}\\
&\quad - \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\delta\lambda'}\partial_\beta g_{\lambda\delta'}\\
&\quad + \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\delta\delta'}\partial_\alpha g_{\beta\lambda'}} + \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\delta\delta'}\partial_\beta g_{\alpha\lambda'}} - \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}.\end{aligned} \tag{15.4}$$

The first three terms in the right-hand side are null terms, but the fourth term fails to be a null term and is included as a quasi-null term. The two underlined terms will cancel out with the two underlined terms in the equation (15.7), which we derive below. Hence, only the last term still remains to be considered.

We can therefore now focus on the following six terms:

$$\begin{aligned}&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\delta g_{\alpha\beta}, \qquad -\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\alpha g_{\beta\delta}, \qquad -\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\beta g_{\alpha\delta},\\
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta}, \qquad \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha}, \qquad -\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}.\end{aligned} \tag{15.5}$$

The identities

$$g^{\alpha\beta}\partial_\alpha g_{\beta\delta} - \frac{1}{2}g^{\alpha\beta}\partial_\delta g_{\alpha\beta} = \Gamma_\delta, \qquad g_{\beta\delta}\partial_\alpha g^{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\partial_\delta g^{\alpha\beta} = \Gamma_\delta, \tag{15.6}$$

allow us to decompose the first three terms in (15.5) as follows:

$$\begin{aligned}
\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\delta g_{\alpha\beta} &= \frac{1}{2}g^{\delta\delta'}\partial_\delta g_{\alpha\beta}\Gamma_{\delta'} + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\beta}\partial_{\delta'} g_{\lambda\lambda'} \\
-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\alpha g_{\beta\delta} &= -\frac{1}{2}g^{\delta\delta'}\partial_\alpha g_{\beta\delta}\Gamma_{\delta'} - \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_{\delta'} g_{\lambda\lambda'}\partial_\alpha g_{\beta\delta}} \\
-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\lambda g_{\lambda'\delta'}\partial_\beta g_{\alpha\delta} &= -\frac{1}{2}g^{\delta\delta'}\partial_\beta g_{\alpha\delta}\Gamma_{\delta'} - \underline{\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_{\delta'} g_{\lambda\lambda'}\partial_\beta g_{\alpha\delta}}.
\end{aligned} \tag{15.7}$$

In the first line,the last term is exactly one of our quasi-null terms stated. The two underlined terms cancel out with the two underlined terms in (15.4), as we explained earlier.

Next, we treat the fourth term in (15.5) by writing

$$\begin{aligned}
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} \\
&= \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'} \\
&= \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{2}g^{\lambda\lambda'}\partial_\alpha g_{\lambda\beta}\Gamma_{\lambda'} + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda\beta}\partial_{\lambda'} g_{\delta\delta'},
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} \\
&= \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda\beta}\partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'}\partial_{\lambda'} g_{\lambda\beta}\big) \\
&\quad + \frac{1}{2}g^{\lambda\lambda'}\partial_\alpha g_{\lambda\beta}\Gamma_{\lambda'} + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\delta\delta'}\partial_{\lambda'} g_{\lambda\beta},
\end{aligned}$$

and so

$$\begin{aligned}
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} \\
&= \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda\beta}\partial_{\lambda'} g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'}\partial_{\lambda'} g_{\lambda\beta}\big) \\
&\quad + \frac{1}{2}g^{\lambda\lambda'}\partial_\alpha g_{\lambda\beta}\Gamma_{\lambda'} + \frac{1}{4}g^{\delta\delta'}\partial_\alpha g_{\delta\delta'}\Gamma_\beta + \frac{1}{8}g^{\delta\delta'}g^{\lambda\lambda'}\partial_\alpha g_{\delta\delta'}\partial_\beta g_{\lambda\lambda'}.
\end{aligned}$$

For the fifth term in the expression (15.5), we have

$$\begin{aligned}
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha}\\
&=\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha}-\partial_\beta g_{\lambda\alpha}\partial_\delta g_{\lambda'\delta'}\right)+\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\beta g_{\lambda\alpha}\partial_{\lambda'}g_{\delta\delta'}-\partial_\beta g_{\delta\delta'}\partial_{\lambda'}g_{\lambda\alpha}\right)\\
&\quad+\frac{1}{2}g^{\lambda\lambda'}\partial_\beta g_{\lambda\alpha}\Gamma_{\lambda'}+\frac{1}{4}g^{\delta\delta'}\partial_\beta g_{\delta\delta'}\Gamma_\alpha+\frac{1}{8}g^{\delta\delta'}g^{\lambda\lambda'}\partial_\beta g_{\delta\delta'}\partial_\alpha g_{\lambda\lambda'}.
\end{aligned}$$

Finally, we deal with the last term in (15.5) by writing

$$\begin{aligned}
&-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}\\
&=-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}-\partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\right)-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\\
&=-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}-\partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\right)-\frac{1}{2}g^{\lambda\lambda'}\partial_\lambda g_{\alpha\lambda'}\Gamma_\beta-\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\beta g_{\delta\delta'}\partial_\lambda g_{\alpha\lambda'}
\end{aligned}$$

therefore

$$\begin{aligned}
&-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}\\
&=-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}-\partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\right)-\frac{1}{2}g^{\lambda\lambda'}\partial_\lambda g_{\alpha\lambda'}\Gamma_\beta-\frac{1}{4}g^{\delta\delta'}\partial_\beta g_{\delta\delta'}\Gamma_\alpha\\
&\quad-\frac{1}{8}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda\lambda'}\partial_\beta g_{\delta\delta'},
\end{aligned}$$

and so

$$\begin{aligned}
&-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}\\
&=-\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\left(\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'}-\partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\right)-\frac{1}{2}\Gamma_\alpha\Gamma_\beta-\frac{1}{4}g^{\delta\delta'}\partial_\alpha g_{\delta\delta'}\Gamma_\beta-\frac{1}{4}g^{\delta\delta'}\partial_\beta g_{\delta\delta'}\Gamma_\alpha\\
&\quad-\frac{1}{8}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\lambda\lambda'}\partial_\beta g_{\delta\delta'}.
\end{aligned}$$

15.4 Concluding the calculation

In conclusion, the quadratic nonlinearities in the expression of the Ricci curvature $R_{\alpha\beta}$ are

$$\begin{aligned}
&\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\delta g_{\alpha\lambda'}\partial_{\delta'}g_{\beta\lambda} - \frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\delta g_{\alpha\lambda'}\partial_\lambda g_{\beta\delta'} - \partial_\delta g_{\beta\delta'}\partial_\lambda g_{\alpha\lambda'}\big)\\
&+\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda'\delta'}\partial_\delta g_{\lambda\beta} - \partial_\alpha g_{\lambda\beta}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\alpha g_{\lambda\beta}\partial_{\lambda'}g_{\delta\delta'} - \partial_\alpha g_{\delta\delta'}\partial_{\lambda'}g_{\lambda\beta}\big)\\
&+\frac{1}{2}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\beta g_{\lambda'\delta'}\partial_\delta g_{\lambda\alpha} - \partial_\beta g_{\lambda\alpha}\partial_\delta g_{\lambda'\delta'}\big) + \frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\big(\partial_\beta g_{\lambda\alpha}\partial_{\lambda'}g_{\delta\delta'} - \partial_\beta g_{\delta\delta'}\partial_{\lambda'}g_{\lambda\alpha}\big)\\
&-\frac{1}{4}g^{\lambda\lambda'}g^{\delta\delta'}\partial_\alpha g_{\delta\lambda'}\partial_\beta g_{\lambda\delta'} + \frac{1}{8}g^{\delta\delta'}g^{\lambda\lambda'}\partial_\beta g_{\delta\delta'}\partial_\alpha g_{\lambda\lambda'} + \frac{1}{2}g^{\delta\delta'}\partial_\delta g_{\alpha\beta}\Gamma_{\delta'} - \frac{1}{2}\Gamma_\alpha\Gamma_\beta.
\end{aligned}$$

This completes the proof of Proposition 13.1.

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