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REGULARLY VARYING SOLUTIONS OF FRIEDMAN EQUATION

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Abstract. We discuss solutions of the acceleration equation, the equation associated to the Friedman equation, in the light of the theory of regularly varying functions, also known as Karamata functions. As a result we obtain that the solutions of the acceleration equation might have a multiplicative term which is a slowly varying function. Under usual assumptions for the scale factor $a(t)$, such as $a(t) = t^{\alpha}$, it appears that this slowly varying term exists. Slowly varying term may explain some phenomena in the standard models of the evolution of the Universe. This paper is an announcement of the more detailed research in this area.

1. INTRODUCTION

The Friedman acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \tag{1}$$

together with the fluid equation

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \tag{2}$$

and the Friedman equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} \tag{3}$$

determines the expansion scale factor $a(t)$ of the Universe. Here ρ is the mass density while p is the pressure of the material in the Universe. The nature of the solution of (1) strongly depends on the energy density term

$$\mathbf{E} = \rho + 3p/c^2, \tag{4}$$

particularly of the sign of $\mathbf{E}$. In order to explain the acceleration of the expansion of the Universe the cosmological constant Λ is added in (1):

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}. \tag{5}$$

The modified energy density term for $\Lambda \neq 0$

$$\mathbf{E}_\Lambda = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) + \frac{\Lambda}{3}. \tag{6}$$

admits negative values, giving more possibilities in modelling of possible scenarios of the past and the evolution of the Universe (e.g. Coles, et al., 2002; Hogan 2007, Lidle, et al., 2000; Narlikar, 2002; Peacock, 1999; Vikman 2005). We note the following two remarks in regards to Λ. First, under the transformations

$$\rho' = \rho + \Lambda/(8\pi G), p' = p - \Lambda/(8\pi G) \tag{7}$$

the equation (5) transforms into (1), but now referring it to the terms ρ' and p'. Therefore, our discussion will be concentrated further on the mathematical solutions of the equation (1). However, we shall abandon in certain situations the strong energy condition $\mathbf{E} > 0$. Secondly, it is well known that there are significant discrepancies in the predictions of what order should be the value of Λ. The reason may lay in the course tuned asymptotic description of the scale factor $a(t)$. In order to avoid this situation a better asymptotic analysis is needed. The theory of regularly varying function in the Karamata sense provides the means for such an analysis, particularly for solutions of the second order differential equations as it is (1). This theory is quite well developed (Bingham at al., 1987; Hille, 1948; Howard, et al., 1990; Marić, 2000; Marić et al, 1990; Omey, 1981; Seneta, 1976; Swanson, 1968), but it seems it has not been much applied in cosmology and in astrophysics in general. Yet, there are few to mention (Mijajlović et al., 2007; Molchanov, et al., 1997; Stern, 1997).

2. REGULAR VARIATION

In this section we shall briefly review the basic notions related to the regularly varying functions. We shall discuss also some properties of solutions of the second order differential equation of the form

$$y'' + f(x)y = 0 \tag{8}$$

related to the regularly varying functions. Observe that the acceleration equation has the form (8).

2.1. Regularly varying functions

A regular variation is related to the power-law distributions, a kind of polynomial relationship between two quantities. It exhibits the property of scale invariance represented by

$$f(x) = ax^k + o(x^k) \tag{9}$$

where a and k are constants, and $o(x^k)$ is an asymptotically small function of x^k. Here, k is the scaling exponent, i.e. a power-law function satisfies $f(\lambda x) \propto f(x)$ where λ is a constant. So, a rescaling of the function's argument changes the constant of proportionality but preserves the shape of the function itself. Power-law relations characterize a large number of natural phenomena, particularly in physics and astronomy. Examples are the Stefan-Boltzman law, gravitational potential and the scale factor $a(t)$ in various cosmological models. A particular interest in a power law can be found in the study of probability distribution and the large fluctuations that occur in the tail of the distribution – the part of the distribution representing large but rare events.

The most general form of a power law is given by

$$f(x) = L(x)x^{\alpha}, \quad \text{or} \quad f(x) \propto x^{\alpha} \tag{10}$$

where $L(x)$ is a slowly varying function i.e. $L(x)$ is positive continuous function (more generally measurable function, but in this article we are dealing anyway only with continuous functions) defined on some neighborhood $[a, \infty]$ of the infinity which satisfies

$$\lim_{x \to \infty} \frac{L(\lambda x)}{L(x)} = 1, \quad \text{for each } \lambda > 0 \tag{11}$$

The real number α is called the index of regular variation.

A positive continuous function R defined on some neighborhood $[a, \infty]$ is the regularly varying of index α if and only if it satisfies

$$\lim_{x \to \infty} \frac{R(\lambda x)}{R(x)} = \lambda^{\alpha}, \quad \text{for each } \lambda > 0 \tag{12}$$

It immediately follows that the regularly varying function $R(x)$ has the form

$$R(x) = L(x)x^{\alpha} \tag{13}$$

where $L(x)$ is slowly regular.

Jovan Karamata (Karamata, 1930) introduced the conceptions of slowly varying function and regularly varying functions. He also proved the following two fundamental theorems.

Examples of slowly varying functions includes iterated logarithms, but there are more complicated examples e.g.

$$L_1(x) = \frac{1}{x} \int_a^x \frac{dt}{\ln(t)}, \tag{14}$$

$$L_2(x) = \exp\left((\ln(x)^{\frac{1}{3}} \cos(\ln(x))^{\frac{1}{3}} \right) \tag{15}$$

The second example is interesting since $L_2(X)$ oscillates infinitely many times between 0 and infinity.

3. REGULARLY VARIATION AND ACCELERATION EQUATION

We shall represent the acceleration equation (5) in the form

$$\ddot{a} + \frac{\mu(t)}{t^2} a = 0. \tag{16}$$

The Hubble parameter $H(t)$ and the deceleration parameter $q(t)$ are defined by

$$H(t) = \frac{\dot{a}(t)}{a(t)}, \quad q(t) = -\frac{\ddot{a}(t)}{a(t)} \frac{1}{H(t)^2} \tag{17}$$

where $a(t)$ is the expansion scale factor of the Universe. Therefore, the equation (16) can be written as

$$\ddot{a} + \frac{q(t)(H(t)t)^2}{t^2}\, a = 0, \tag{18}$$

hence

$$\mu(t) = q(t)(H(t)t)^2. \tag{19}$$

Observe that $\mu(t)$ is a dimensionless parameter. Let us remind that the density parameter $\Omega(t)$ and the density parameter for the cosmological constant Λ are defined by

$$\Omega = \frac{\rho}{\rho_c}, \quad \Omega_\Lambda = \frac{\Lambda}{3H^2}$$

where ρ_c is the critical density.

The Friedman acceleration equation (1) has two different fundamental solutions that satisfy a power law if and only if the limit

$$\gamma = \lim_{x\to\infty} x \int_x^\infty \frac{\mu(t)}{t^2} dt \tag{20}$$

exists and $\gamma < \frac{1}{4}$. According to the theory of of regularly varying functions and differential equations, if $-\infty < \gamma < 1/4$ and $\alpha_1 < \alpha_2$ are two roots of the equation

$$\alpha^2 - \alpha + \gamma = 0. \tag{21}$$

then there exist two linearly independent regularly varying solutions of the equation $y'' + f(x)y = 0$ of the form

$$y_i(x) = x^{\alpha_i} L_i(x), i = 1, 2, \tag{22}$$

if and only if $\lim_{x\to\infty} x \int_x^\infty f(t)dt = \gamma$. Here, L_i, i=1,2 denote two normalized slowly varying functions. We see that this observation directly applies to the Friedman acceleration equation (16). Thus, the expansion scale factor $a(t)$ satisfies the power law if and only if $\gamma < \frac{1}{4}$.

We discuss only the existence and the possible values of the limit $\lim_{t\to\infty} \mu(t)$. The existence of this limit is the sufficient condition for the existence of (20) and in this case

$\gamma = \lim_{t\to\infty} \mu(t)$. It appears that the values of the constant γ determine the asymptotical behavior at the infinity of the solutions of the acceleration equation, i.e. of the expansion scale factor $a(t)$ of the Universe. If the matter-dominated evolution of the Universe is assumed, that is, dominated by some form of pressureless material since the certain time moment t_0 then the expression $H(t)t$ depends solely on the parameter Ω. In this case we will be able to estimate the possible values of γ. We discuss also the status of the constant γ and the related asymptotic behavior of $a(t)$ for the flat Universe including the cosmological constant Λ and the open Universe with $\Lambda = 0$. Detailed proofs of will be published somewhere else.

4. CONCLUSION

A new constant γ is introduced by (20) related to the Friedman acceleration equation (1). The values of the constant γ determine the asymptotical behavior at the infinity of the solutions of the acceleration equation, i.e. of the expansion scale factor $a(t)$ of the Universe. The instance $\gamma < \frac{1}{4}$ is appropriate for the both cases, the flat and open Universe, and gives the sufficient and necessary condition that the solutions of the acceleration equation are in the Karamata class of functions; more specifically that they satisfy a power law. This property of the acceleration equation is formulated as the *power law conclusion* in the subsection 3.3. As power law functions are the most frequently occurring type of the solutions of the Friedman equation, the study of the constant γ and the related function $\mu(t)$ defined by (19) might be of a particular interest.

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