ON THE ZEROS OF CERTAIN TRIGONOMETRIC INTEGRALS

G. Pólya.*

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1. What properties of the function $F(u)$ are sufficient to secure that the integral

$$2\int_0^\infty F(u)\cos zu\,du = G(z) \tag{1}$$

has only real zeros? The origin of this rather artificial question is the Riemann hypothesis concerning the Zeta-function. If we put

$$F(u) = \sum_{n=1}^{\infty} \left(4\pi^2 n^4 e^{\frac{9}{2}u} - 6\pi n^2 e^{\frac{5}{2}u}\right) e^{-n^2\pi e^{2u}}, \tag{2}$$

$G(z)$ becomes Riemann's function $\xi(z)$.

I have found some criteria answering the proposed question, but I will not give them here, because they are of a rather unsystematic and tentative character. I prefer to enumerate a few cases in which I have proved that $G(z)$ represents an integral function having only real zeros:

(i) $F(u) = (1-u^{2n})^{a-1} \quad (0 < u < 1), \quad F(u) = 0 \quad (u > 1),$

(ii) $F(u) = e^{-u^{2n} - a u^{4n}},$

(iii) $F(u) = e^{-2a\cosh u},$

(iv) $F(u) = 8\pi^2 \cosh \frac{9}{2}u\, e^{- \quad \cosh 2u}$

(v) $F(u) = \left\{8\pi^2 \cosh \frac{9}{2}u - 12\pi \cosh \frac{5}{2}u\right\} e^{-2\pi\cosh 2u}.$

In these formulae a denotes a positive number and n a positive integer. Putting in case (i) $n = 1$, we get the well known theorem that the Bessel function $J_{a-\frac{1}{2}}(z)$ has only real zeros. Some other special cases are also known. Observe that the function of case (iv) differs little, and that of case (v) differs yet less, from the leading term of the series (ii), corresponding to $n = 1$, as u tends to $+\infty$.

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For comments on this paper[90], see p. 424.

2. The existence of an infinity of real zeros is generally easier to establish than the non-existence of complex zeros. A method of Hardy leads to the following very convenient criterion :

Suppose that $F(u)$ is an even function, analytic and real for real values of u, and such that

$$\lim_{u \to +\infty} F^{(n)}(u)\, u^2 = 0 \quad (n = 0, 1, 2, \ldots).$$

If the function $G(z)$ has only a finite number of real zeros, then there is an integer N such that $|F^{(n)}(it)|$ is a steadily increasing function of t if $n > N$ and $0 < t < T$, iT being the singular point of $F(u)$ which is next to the origin [$T = +\infty$ if $F(u)$ is an integral function].

This criterion, applied to the cases (ii), (iii), (iv), and (v) shows at once that the corresponding function $G(z)$ has an infinity of real zeros. In order to apply it to the function $\xi(z)$ put

$$\sqrt[4]{x} \sum_{n=-\infty}^{+\infty} e^{-n^2 \pi x} = \vartheta(x) \tag{3}$$

(which does not agree with the usual notation), and write the function (2) in the more convenient form

$$F(u) = \frac{d^2 \vartheta(x)}{du^2} - \tfrac{1}{4}\vartheta(x) \quad (x = e^{2u}). \tag{2'}$$

From the well known property of the function (3), namely that $\vartheta(x^{-1}) = \vartheta(x)$, it is easy to deduce the following properties of the function (2′) :

$$F(-u) = F(u), \tag{a}$$

$$\lim_{t \to \frac{1}{4}\pi} F^{(n)}(it) = 0 \quad (n = 0, 1, 2, \ldots;\ t \text{ real}). \tag{b}$$

Now property (b) is, in virtue of the general criterion, inconsistent with the assumption that $\xi(z)$ has only a finite number of real zeros. Thus we get a proof for the existence of an infinity of real zeros of $\xi(z)$, which is analogous to the original proof due to Hardy, but different from it, and, I think, a very simple one.