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An analytical solution of the ill–posed Cauchy problem for Lame equation in a rectangle

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Abstract. In this paper we present an analytical method for the inverse Cauchy problem of Lame equation of the elasticity theory. A rectangular domain is frequently used in engineering structures and we only consider the analytical solution in a two-dimensional rectangle, wherein a missing boundary condition is recovered from a full measurement of the stresses and displacements on an accessible boundary. The essence of the method consists in solving three independent Cauchy problems for the Laplace and Poisson equations. For each of them the Fourier series is used to formulate a first-kind Fredholm integral equation for the unknown function of data. Then, we use a Lavrentiev regularization method and the termwise separable property of kernel function allows us to obtain a closed-form regularized solution. As a result, for the displacement components, we obtain solutions in the form of a sum of series with three regularization parameters. The uniform convergence and error estimation of the regularized solutions are proved.

1. Introduction

Let us have an element of construction with a boundary $S = S_1 \cup S_2$ consisting of two parts S_1 and S_2. The part S_1 is accessible for all measurements and the part S_2 is not accessible. In engineering practice it is often necessary to reconstruct the stress-strain state of such structural element by means of measured data on S_1. In this case within an elastic model we have the Cauchy problem for the Lame equation of theory of elasticity. The Cauchy problem for elliptic equations is a so-called ill-posed problem. The main difficulty for solving such problems is numerical instability. For the Cauchy problem in linear elasticity there exist numerous ways for numerical solving. Many of these methods can be classified as the Tikhonov type regularization methods, the finite element method, the boundary element method, etc. (sf. [1, 2, 3, 4]). With regard to analytical solutions it is known only usage of Carleman's method in the elasticity theory [5, 6, 7]. In this paper we present an analytical method for the Cauchy problem solution of Lame equation in a rectangle. This method generalizes the Liu method [8] for an analytical solution of the Cauchy problem of Laplace equation in the rectangle.

2. Formulation of a problem

In numerous publications under the name of Cauchy problem for the Laplace equation there exist two types of problems. In them it is necessary to find a harmonic function in $\Omega \in \mathbb{R}^3$. At first one has a cylindrical domain $\Omega = S \times (0, T)$ with a lower base $S \in \mathbb{R}^2$ and a lateral surface S_1. On S the Cauchy conditions are given: $u = f$, $\partial u/\partial n = g$, and some boundary conditions

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are given on the lateral surface S_1, for example, $u = 0$, on an upper base there are no given boundary conditions. Such problems can be named as an initial boundary value problem for the Laplace equation (see Ch.9 in [1]). For another type we have a bounded domain $\Omega \in \mathbb{R}^3$ with a smooth boundary $S = S_1 \cup S_2$ and the Cauchy conditions are given on the part S_1 of a boundary S. We will deal with the first type of the Cauchy problem and it's analogous in the elasticity theory. It is necessary to note that the first type of Cauchy problem is not considered for analytical solutions in the elasticity theory. All analytical results in the elasticity theory concern the second type problem and use Carleman's method.

Let x, y, z be the Cartesian coordinates, for the convenience we also will simultaneously use index notations: x_i $(i = 1, 2, 3)$. An equilibrium equation in the theory of elasticity is the Lame equation

$$L\mathbf{u} \equiv \mu \triangle \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = 0. \tag{1}$$

Hooke's Law expresses connections between components σ_{ij} of stress tensor and components ε_{ij} of deformation tensor and for the linear elasticity has the form

$$\sigma_{ij} = (\lambda + 2\mu)(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right). \quad (i, j = 1, 2, 3) \tag{2}$$

Here and below, the comma and an index (or a variable) denote the corresponding partial derivative. For simplicity we will consider the plane deformation when components of displacement $\mathbf{u}$ have the form $u_x = u(x, y)$, $u_y = v(x, y)$, $u_z = 0$. In such case equilibrium equations in stresses components have the form

$$\begin{aligned} \sigma_{xx,x} + \sigma_{xy,y} = 0, \\ \sigma_{xy,x} + \sigma_{yy,y} = 0. \end{aligned} \tag{3}$$

Let Ω be the rectangle $\mathbf{r} \in (0, a) \times (0, b) \in \mathbb{R}^2$. The problem to be solved is the next:

$$\begin{aligned} &L\mathbf{u}(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega; \\ &\sigma_{yy} = u_x = u_y = 0, \quad \sigma_{xy} = \sigma(x), \quad \text{on } y = b; \\ &\sigma_{xx} = u_y = 0 \quad \text{on } x = 0 \text{ and } x = a. \end{aligned} \tag{4}$$

Thus, we have the Cauchy problem for the Lame equation with the Cauchy data on the upper base, with zero boundary conditions on the lateral sides of rectangle and without any boundary conditions on the lower base of a rectangle. On the upper base only one boundary function $\sigma(\tilde{x})$ from four is taken not equal to zero only for the simplicity.

3. Method of solution

3.1. Cauchy problem for the auxiliary harmonic function

Let us introduce an auxiliary function proportional to the divergence of a displacement vector

$$f = (\lambda + 2\mu)\nabla \cdot \mathbf{u}. \tag{5}$$

It is well known that this function is harmonic. From Hook's Low (2) we have

$$\begin{aligned} \sigma_{xx} = \lambda\nabla \cdot \mathbf{u} + 2\mu\varepsilon_{xx} = \lambda\nabla \cdot \mathbf{u} + 2\mu u_{x,x} = \lambda\nabla \cdot \mathbf{u} + 2\mu(\nabla \cdot \mathbf{u} - u_{y,y}), \\ \sigma_{yy} = \lambda\nabla \cdot \mathbf{u} + 2\mu\varepsilon_{yy} = \lambda\nabla \cdot \mathbf{u} + 2\mu u_{y,y} = \lambda\nabla \cdot \mathbf{u} + 2\mu(\nabla \cdot \mathbf{u} - u_{x,x}), \end{aligned}$$

and we have the next expressions for our auxiliary function (5)

$$
\begin{aligned}
f &= \sigma_{xx} + 2\mu u_{y,y}, \\
f &= \sigma_{yy} + 2\mu u_{x,x}.
\end{aligned} \tag{6}
$$

Then, after differentiating of these expressions and using equilibrium equations (3) and (1) we obtain

$$
\begin{aligned}
f_{,y} &= \sigma_{yy,y} + 2\mu u_{x,xy} = -\sigma_{xy,x} + 2\mu(\nabla\cdot\mathbf{u} - u_{y,y})_{,y} = -\sigma_{xy,x} + 2\mu\nabla\cdot\mathbf{u}_{,y} - 2\mu u_{y,yy} \\
&= -\sigma_{xy,x} + 2\mu\nabla\cdot\mathbf{u}_{,y} - 2[-\mu u_{y,xx} - (\lambda+\mu)\nabla\cdot\mathbf{u}_{,y}] = -\sigma_{xy,x} + 2\mu u_{y,xx} + 2f_{,y},
\end{aligned}
$$

from here we have

$$
f_{,y} = \sigma_{xy,x} - 2\mu u_{y,xx}. \tag{7}
$$

Analogically one can obtain the next similar formula

$$
f_{,x} = \sigma_{xy,y} - 2\mu u_{x,yy}. \tag{8}
$$

Let the boundary conditions of the Cauchy problem (4) provide of a solution smoothness $\mathbf{u} \in C^2(\overline{\Omega})$. In this case all formulas (6)–(8) are valid up to the boundary of a rectangle. Then from the formulas (6)–(8) and boundary conditions from (4) we obtain boundary conditions for the function f and we have Cauchy problem for the harmonic function f:

$$
\begin{aligned}
&\Delta f = 0, \quad \mathbf{r} \in \Omega, \\
&f = 0, \quad f_{,y} = -h(x) \quad \text{on } y = b, \\
&f = 0 \quad \text{on } x = 0 \text{ and } x = a,
\end{aligned} \tag{9}
$$

where the function $h(x)$ is expressed in terms of the initial boundary functions from (4)

$$
h(x) = -\sigma'(x).
$$

The regularized solution f^α of this problem is obtained by using the Liu [8] method in the next form:

$$
\begin{aligned}
f^\alpha(x,y) &= \sum_{k=1}^{\infty} a_k^\alpha \frac{\sinh \dfrac{k\pi(b-y)}{a}}{\sinh \dfrac{k\pi b}{a}} \sin\frac{k\pi x}{a}, \\
a_k^\alpha &= \frac{k\pi\sigma_k \sinh \dfrac{k\pi b}{a}}{k\pi + \alpha a \sinh \dfrac{k\pi b}{a}}, \\
\sigma_k &= \frac{2}{a}\int_0^a \sigma(\xi)\cos\frac{k\pi\xi}{a} d\xi,
\end{aligned} \tag{10}
$$

where α is a parameter of regularization. In this method the integral equation of the first kind is solved by Lavrent'ev's regularization method using the Fourier method. The kernel in this integral equation has a termwise separable property. This property is the main reason for a solution to be obtained in a closed form.

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For the solution (10) are valid the next results presented in [8], including an error estimation:

Theorem 1. If the Neumann datum $h(x)$ is bounded in the interval $x \in [0, a]$, then for any $\alpha > 0$ and $y_0 > 0$ for all $x \in [0, a]$ and $y \in [y_0, b]$ the regularized solution (10) converges uniformly and this series can be termwise differentiable any number of times.

Theorem 2. Assume that the Neumann datum $h(x) \in L^2(0, a)$. Then the sufficient and necessary condition that the Cauchy problem (9) has a solution is that

$$\sum_{k=1}^{\infty} \frac{\sinh^2 \dfrac{k\pi b}{a}}{k^2\pi^2} \left(\int_0^a h(\xi) \sin \frac{k\pi\xi}{a} d\xi \right)^2 < \infty. \tag{11}$$

Theorem 3. If the Neumann datum $h(x)$ satisfies condition (11) and there exists an $\varepsilon \in (0, 1)$, such that

$$4\sum_{k=1}^{\infty} \left[\frac{a^2 \sinh^2 \dfrac{k\pi b}{a}}{k^2\pi^2} \right]^{1+\varepsilon} \left(\int_0^a h(\xi) \sin \frac{k\pi\xi}{a} d\xi \right)^2 := M^2(\varepsilon) < \infty, \tag{12}$$

then for any $\alpha > 0$ the regularized solution $f^\alpha(x, y)$ satisfies the following error estimation:

$$\|f^\alpha(x, y) - f(x, y)\|_{L^2(0,a)} \leq \alpha^\varepsilon M(\varepsilon).$$

3.2. Cauchy problems for the components of displacements

After substitution $f^\alpha(x, y)$ in place of the $f(x, y)$ in the equilibrium equations (1) for the components of a displacement we have Poisson equations. It can be shown that for the each components $u_x(x, y)$ and $u_y(x, y)$ we can find Cauchy boundary conditions. Thus for the $u_x(x, y)$ we have the next Cauchy problem for the Poisson equations:

$$\begin{aligned}
&\Delta u_x = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)} f^\alpha_{,x}, \quad \mathbf{r} \in \Omega, \\
&u_x = 0, \quad u_{x,y} = \frac{1}{\mu}\sigma(x) \quad \text{on } y = b, \\
&u_{x,x} = 0 \quad \text{on } x = 0 \text{ and } x = a.
\end{aligned}$$

For the component $u_y(x, y)$ we have another Cauchy problem:

$$\begin{aligned}
&\Delta u_y = -\frac{\lambda + \mu}{\mu(\lambda + 2\mu)} f^\alpha_{,y}, \quad \mathbf{r} \in \Omega, \\
&u_y = u_{y,y} = 0 \quad \text{on } y = b, \\
&u_y = 0 \quad \text{on } x = 0 \text{ and } x = a.
\end{aligned}$$

These problems are solved by the method similar to Liu. Thus, for the solution u_x and u_y of the Cauchy problem (4) we have regularized expressions similar to (10), but with additional regularization parameters β and γ. Now we give the final formulas for these solutions:

$$\begin{aligned}
u_x^{\alpha,\beta}(x, y) &= w_x^\alpha(x, y) + v_x^{\alpha,\beta}(x, y), \\
u_y^{\alpha,\gamma}(x, y) &= w_y^\alpha(x, y) + v_y^{\alpha,\gamma}(x, y);
\end{aligned}$$

$$w_x^\alpha(x,y) = \sum_{k=1}^{\infty} w_{xk}^\alpha(y) \cos\frac{k\pi x}{a},$$

$$w_{xk}^\alpha(x,y) = \frac{\lambda+\mu}{\mu(\lambda+2\mu)} \frac{a_k^\alpha}{\sinh(\frac{k\pi b}{a})} \left[y\cosh\frac{k\pi(b-y)}{a} - b\frac{\sinh\frac{k\pi y}{a}}{\sinh\frac{k\pi b}{a}} \right];$$

$$v_x^{\alpha,\beta}(x,y) = b_0(b-y) + \sum_{k=1}^{\infty} b_k^{\alpha,\beta} \frac{\sinh\frac{k\pi(b-y)}{a}}{\sinh\frac{k\pi b}{a}} \cos\frac{k\pi x}{a},$$

$$b_k^{\alpha,\beta} = \frac{a\sinh\frac{k\pi b}{a}}{\mu\left(k\pi + \beta\sinh\frac{k\pi b}{a}\right)} \left[\frac{\lambda+\mu}{2(\lambda+2\mu)} \frac{a_k^\alpha}{\sinh\frac{k\pi b}{a}} \left(1 - \frac{k\pi b}{a}\coth\frac{k\pi b}{a}\right) - \sigma_k \right];$$

$$w_y^\alpha(x,y) = -\frac{\lambda+\mu}{\mu(\lambda+2\mu)} \sum_{k=1}^{\infty} a_k^\alpha y \frac{\sinh\frac{k\pi(b-y)}{a}}{\sinh\frac{k\pi b}{a}} \sin\frac{k\pi x}{a};$$

$$v_y^{\alpha,\gamma}(x,y) = \sum_{k=1}^{\infty} c_k^{\alpha,\gamma} \frac{\sinh\frac{k\pi(b-y)}{a}}{\sinh\frac{k\pi b}{a}} \sin\frac{k\pi x}{a},$$

$$c_k^{\alpha,\gamma} = \frac{\lambda+\mu}{2\mu(\lambda+2\mu)} \cdot \frac{k\pi b}{k\pi + \gamma\sinh\frac{k\pi b}{a}} a_k^\alpha,$$

$$b_0 = -\frac{1}{\mu}\sigma_0 = -\frac{1}{a\mu}\int_0^a \sigma(\xi)d\xi.$$

If the boundary function $\sigma(x)$ in (4) is bounded in closed interval, then all of these regularized solutions converge uniformly and can be differentiated termwise any number of times. When the parameters $\alpha, \beta, \gamma \to 0$ then the regularized solutions $u_x^{\alpha,\beta}(x,y), u_y^{\alpha,\gamma}(x,y)$ tend to the exact solutions.

4. Error estimation

For the regularized auxiliary function $f^\alpha(x,y)$ we have error estimation according of the Theorem 3. For the component of a displacement $u_x^{\alpha,\beta}(x,y)$ the next error estimation theorem can be proved:

Theorem 4. If the Neumann datum $h(x)$ satisfies condition (11) and there exist $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, such that conditions (12) and the next

$$4\sum_{k=1}^{\infty} \left[\frac{a^2\sinh^2\frac{k\pi b}{a}}{k^2\pi^2} \right]^{1+\delta} \left(\int_0^a q(\xi)\sin\frac{k\pi\xi}{a}d\xi \right)^2 := N^2(\delta) < \infty,$$

$$q(x) = -\frac{1}{\mu}\sigma(x) + w_{,y}^\alpha(x,b)$$

are hold, then for any $\alpha > 0$, $\beta > 0$ the regularized solution $u_x^{\alpha,\beta}(x, y)$ satisfies the following error estimation:

$$\|u_x(x, y) - u_x^{\alpha,\beta}(x, y))\|_{L^2(0,a)} \leq \sqrt{\frac{b}{2}}\alpha^{\varepsilon} M(\varepsilon) + \beta^{\delta} N(\delta).$$

For the component of a displacement $u_y^{\alpha,\gamma}(x, y)$ an analogical error estimation theorem also can be formulated. In the paper of C.S. Liu [8] the examples of effective numerical calculations by means of presented formula for the $f^{\alpha}(x, y)$ are presented.

5. Conclusion

The Liu method for solving the Cauchy problem for the Laplace equation is generalized for the Cauchy problem of the linear elasticity. This allows us to obtain the regularized solution of the problem in the closed form by using the Fourier and Lavrent'ev methods. The obtained regularized solution converges uniformly and the error estimation of regularized solution is proved. For simplicity we only consider a two–dimensional rectangular domain. The method can be used in the cases of other boundary conditions, it can be generalized for a three–dimensional case. Also it is interesting to note that the used auxiliary function $f = (\lambda + 2\mu)\nabla \cdot \mathbf{u}$ is a scalar part of a regular quaternion function connected with the solution $\mathbf{u}$ of the Lame equation (1) [9, 10, 11]. Thus it may be possible to use presented method in the problem of the regular extension into a domain of the quaternion function defined on a part of its boundary.

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