THE BEST BOUNDS OF HARMONIC SEQUENCE

CHAO-PING CHEN AND FENG QI

Abstract. For any natural number $n \in \mathbb{N}$,

$$\frac{1}{2n+\frac{1}{1-\gamma}-2} \le \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}}, \tag{1}$$

where $\gamma = 0.57721566490153286\cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma}-2$ and $\frac{1}{3}$ are the best possible.

As by-products, two double inequalities of the digamma and trigamma functions are established.

1. Introduction

Let n be a natural number, then we have

$$\frac{1}{2n} - \frac{1}{8n^2} < \sum_{i=1}^{n} \frac{1}{i} - \ln n - \gamma < \frac{1}{2n}, \tag{2}$$

where $\gamma = 0.57721566\cdots$ is Euler's constant.

The inequality (2) is called in literature Franel's inequality [5, Ex. 18]. Because of the well known importance of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$, there exists a very rich literature on inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$. For example, [1, 3], [4, pp. 68–78] and references therein.

L. Tóth and S. Mare in [6, p. 264] proposed the following problems:

(1) Prove that for every positive integer n we have

$$\frac{1}{2n+\frac{2}{5}} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n - \gamma < \frac{1}{2n+\frac{1}{3}}, \tag{3}$$

where γ is Euler's constant.

(2) Show that $\frac{2}{5}$ can be replaced by a slightly smaller number, but that $\frac{1}{3}$ cannot be replaced by a slightly larger number.

In [10], basing on improving of Euler-Maclaurin summation formula, some general inequalities of the harmonic sequence $\sum_{i=1}^{n} \frac{1}{i}$ are established, including recovery of inequality (3). In 1999, Sh.-R. Wei and B.-Ch. Yang in [8] verified inequality (3) again by calculus.

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In 1997, K. Wu and B.-Ch. Yang in [9] proved the second problem due to Tóth and Mare by using the following

$$\sum_{i=1}^{n}\frac{1}{i}=\gamma+\ln n+\frac{1}{2n}-\sum_{i=1}^{q-1}\frac{B_{2i}}{2in^{2i}}-\int_{n}^{\infty}\frac{B_{2q}(x)}{x^{2q}}\,\mathrm{d}x \tag{4}$$

and

$$\int_{n}^{\infty}\frac{B_{2q-1}(x)}{x^{2q}}\,\mathrm{d}x<\frac{(-1)^{q}B_{2q}}{2qn^{2q}}, \tag{5}$$

where $B_i(x)$ is Bernoulli's polynomial, $B_{2i}=B_{2i}(n)$ is Bernoulli's number for $i\in\mathbb{N}$, and n and q are positive integers,.

In this short note, through establishing, by exploiting the well known first Binet's formula, two double inequalities of the digamma and trigamma functions and by utilizing the approximating expansion of digamma function ψ, the best lower and upper bounds of the sequence $\sum_{i=1}^{n}\frac{1}{i}-\ln n-\gamma$ are given. Thses results refine inequality (3) and solve affirmatively the second problem mentioned above.

Theorem 1. *For any natural number $n\in\mathbb{N}$, we have*

$$\frac{1}{2n+\frac{1}{1-\gamma}-2}\le\sum_{i=1}^{n}\frac{1}{i}-\ln n-\gamma<\frac{1}{2n+\frac{1}{3}}, \tag{6}$$

where $\gamma=0.57721566490153286\cdots$ denotes Euler's constant. The constants $\frac{1}{1-\gamma}-2$ and $\frac{1}{3}$ are the best possible.

2. Lemma

In order to prove inequality (3), the following lemma is necessary.

Lemma 1. *For $x>0$, we have*

$$\frac{1}{2x}-\frac{1}{12x^2}<\psi(x+1)-\ln x<\frac{1}{2x} \tag{7}$$

and

$$\frac{1}{2x^2}-\frac{1}{6x^3}<\frac{1}{x}-\psi'(x+1)<\frac{1}{2x^2}-\frac{1}{6x^3}+\frac{1}{30x^5}, \tag{8}$$

where $\psi=\frac{\Gamma'}{\Gamma}$ is the logarithmic derivative of the gamma function

$$\Gamma(x)=\int_{0}^{\infty}t^{x-1}e^{-t}\,\mathrm{d}t. \tag{9}$$

Proof. It is a well known fact ([1] and [7, p. 103]) that for $x>0$ and a nonnegative integer m,

$$\psi(x+1)=\psi(x)+\frac{1}{x} \tag{10}$$

and

$$\frac{m!}{x^{m+1}}=\int_{0}^{\infty}t^{m}e^{-xt}\,\mathrm{d}t. \tag{11}$$

The first Binet's formula ([1] and [7, p. 106]) states that for $x>0$

$$\ln\Gamma(x)=\left(x-\frac{1}{2}\right)\ln x-x+\ln\sqrt{2\pi}-\int_{0}^{\infty}\left(\frac{1}{2}+\frac{1}{t}-\frac{1}{1-e^{-t}}\right)\frac{e^{-xt}}{t}\,\mathrm{d}t. \tag{12}$$

Differentiating (12), integrating by part and using formulas (11) and (10), it is deduced that

$$\psi(x+1)-\ln x=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}\right)e^{-xt}\,\mathrm{d}t. \tag{13}$$

Using formulas (11) and (13) and the series expansion of e^x at $x=0$ yields

$$\begin{aligned}
&\psi(x+1)-\ln x-\frac{1}{2x}+\frac{1}{12x^2}\\
&=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}+\frac{1}{12}t\right)e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\frac{12(e^t-1)-12t-6t(e^t-1)+t^2(e^t-1)}{12t(e^t-1)}e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\left[\frac{1}{12t(e^t-1)}\sum_{n=5}^\infty\frac{(n-3)(n-4)}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\
&>0
\end{aligned} \tag{14}$$

and

$$\begin{aligned}
\psi(x+1)-\ln x-\frac{1}{2x}&=\int_0^\infty\left(\frac{1}{t}-\frac{1}{e^t-1}-\frac{1}{2}\right)e^{-xt}\,\mathrm{d}t\\
&=-\int_0^\infty\left[\frac{1}{2t(e^t-1)}\sum_{n=3}^\infty\frac{n-2}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\
&<0.
\end{aligned} \tag{15}$$

Hence, inequality (7) follows.

Differentiation of (13) immediately produces

$$\frac{1}{x}-\psi'(x+1)=\int_0^\infty\left(1-\frac{t}{e^t-1}\right)e^{-xt}\,\mathrm{d}t. \tag{16}$$

Exploiting formulas (11) and (16) and the series expansion of e^x at $x=0$ yields

$$\begin{aligned}
&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}\\
&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2\right)e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\left[\frac{1}{12(e^t-1)}\sum_{n=5}^\infty\frac{(n-3)(n-4)}{n!}t^n\right]e^{-xt}\,\mathrm{d}t\\
&>0
\end{aligned} \tag{17}$$

and

$$\begin{aligned}
&\frac{1}{x}-\psi'(x+1)-\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}\\
&=\int_0^\infty\left(1-\frac{t}{e^t-1}-\frac{1}{2}t+\frac{1}{12}t^2-\frac{1}{720}t^4\right)e^{-xt}\,\mathrm{d}t\\
&=\int_0^\infty\left[\frac{1}{720(e^t-1)}\sum_{n=7}^\infty\left(\frac{720}{n!}-\frac{360}{(n-1)!}+\frac{60}{(n-2)!}-\frac{1}{(n-4)!}\right)t^n\right]e^{-xt}\,\mathrm{d}t.
\end{aligned} \tag{18}$$

Noticing that for $n \geq 7$,

$$\begin{aligned} &\frac{720}{n!} - \frac{360}{(n-1)!} + \frac{60}{(n-2)!} - \frac{1}{(n-4)!} \\ &= -\frac{120 + 218(n-7) + 119(n-7)^2 + 22(n-7)^3 + (n-7)^4}{n!} < 0, \end{aligned} \tag{19}$$

we obtain

$$\frac{1}{x} - \psi'(x+1) - \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} < 0. \tag{20}$$

Therefore, inequality (8) holds. The proof is complete. □

3. Proof of Theorem 1

In [1], [2, p. 593] and [7, p. 104] it is given that $\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma$. Thus, inequality (6) can be rearranged as

$$\frac{1}{3} < \frac{1}{\psi(n+1) - \ln n} - 2n \leq \frac{1}{1-\gamma} - 2. \tag{21}$$

Define for $x > 0$

$$\phi(x) = \frac{1}{\psi(x+1) - \ln x} - 2x. \tag{22}$$

Differentiating ϕ and utilizing (7) and (8) reveals that for $x > \frac{12}{5}$,

$$\begin{aligned} &(\psi(x+1) - \ln x)^2 \phi'(x) \\ &= \frac{1}{x} - \psi'(x+1) - 2(\psi(x+1) - \ln x)^2 \\ &< \frac{1}{2x^2} - \frac{1}{6x^3} + \frac{1}{30x^5} - 2\left(\frac{1}{2x} - \frac{1}{12x^2}\right)^2 \\ &= \frac{12 - 5x}{360x^5} < 0, \end{aligned} \tag{23}$$

and $\phi(x)$ decreases with $x > \frac{12}{5}$.

Straightforward calculation produces

$$\phi(1) = \frac{1}{1-\gamma} - 2 = 0.36527211862544155\cdots, \tag{24}$$

$$\phi(2) = \frac{1}{\frac{3}{2} - \gamma - \ln 2} - 4 = 0.35469600731465752\cdots, \tag{25}$$

$$\phi(3) = \frac{1}{\frac{11}{6} - \gamma - \ln 3} - 6 = 0.34898948531361115\cdots. \tag{26}$$

Therefore, the sequence

$$\phi(n) = \frac{1}{\psi(n+1) - \ln n} - 2n, \quad n \in \mathbb{N} \tag{27}$$

is decreasing strictly, and for $n \in \mathbb{N}$

$$\lim_{n\to\infty} \phi(n) < \phi(n) \leq \phi(1) = \frac{1}{1-\gamma} - 2. \tag{28}$$

Making use of approximating expansion of ψ in [1], [2, p. 594], or [7, p. 108] gives

$$\psi(x) = \ln x - \frac{1}{2x} - \frac{1}{12x^2} + O(x^{-4}) \quad (x \to \infty), \tag{29}$$

and then

$$\lim_{n\to\infty}\phi(n)=\lim_{x\to\infty}\phi(x)=\lim_{x\to\infty}\frac{\frac{1}{3}+O(x^{-2})}{1+O(x^{-1})}=\frac{1}{3}. \tag{30}$$

The proof is complete.

References

[1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tebles*, National Bureau of Standards, Applied Mathematics Series **55**, 4th printing, Washington, 1965, 1972.

[2] Group of compilation, *Shùxué Shǒucè* (*Handbook of Mathematics*), The People's Education Press, Beijing, China, 1979. (Chinese)

[3] B.-N. Guo and F. Qi, *An inductive proof for an identity involving* $\binom{n}{k}$ *and the partial sums of some series*, Internat. J. Math. Ed. Sci. Tech. **33** (2002), no. 2, 249–253.

[4] J.-Ch. Kuang, *Chángyòng Bùděngshì* (*Applied Inequalities*), 2nd edition, Hunan Education Press, Changsha City, Hunan Province, China, 1993. (Chinese)

[5] G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. I and II, Springer-Verlag, Berlin, Heidelberg, 1972.

[6] L. Tóth and S. Mare, *E 3432*, Amer. Math. Monthly **98** (1991), no. 3, 264.

[7] Zh.-X. Wang and D.-R. Guo, *Tèshū Hánshù Gàilùn* (*Introduction to Special Function*), The Series of Advanced Physics of Peking University, Peking University Press, Beijing, China, 2000. (Chinese)

[8] Sh.-R. Wei and B.-Ch. Yang, *A refinement on the Franel inequality*, Zhōngyāng Mínzú Dàxué Xuébào (J. Central Univ. Nationalities Natur. Sci. Ed.) **8** (1999), no. 1, 66–68. (Chinese)

[9] K. Wu and B.-Ch. Yang, *Some refinements of Franel's inequality*, Huánán Shīfàn Dàxué Xuébào (J. South China Normal Univ. Natur. Sci. Ed.) (1997), no. 3, 5–8. (Chinese)

[10] B.-Ch. Yang and G.-Q. Wang, *Some inequalities on harmonic series*, Shùxué Yánjiū (J. Math. Study) **29** (1996), no. 3, 90–97. (Chinese)

(Ch.-P. Chen) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

(F. Qi) Department of Applied Mathematics and Informatics, Jiaozuo Institute of Technology, Jiaozuo City, Henan 454000, CHINA

E-mail address: qifeng@jzit.edu.cn, fengqi618@member.ams.org

URL: http://rgmia.vu.edu.au/qi.html