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The Affine Uncertainty Principle in One and Two Dimensions

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Abstract—In this paper, we construct families of wavelets that minimize an uncertainty relation associated with square integrable representations of some canonical groups. Especially, we obtain a new interpretation of the Mexican hat function.

Keywords—Square integrable representations, Wavelet transform, Uncertainty principle, Self-adjoint operators.

1. INTRODUCTION

The starting point for this paper is the so-called windowed Fourier transform G_ψ which is defined by

$$(G_\psi f)(\omega, b) := \int_{\mathbf{R}} f(x)\psi(x-b)\, e^{-i\omega x}\, dx. \tag{1.1}$$

This integral transform is a well-known tool for various applications, e.g., in signal analysis or in quantum physics, where it is used for defining and investigating coherent states. The canonical choice for ψ is the Gaussian window function

$$\psi(x) := \pi^{-1/4}\, e^{-x^2/2}.$$

This function stands out since it minimizes the Heisenberg uncertainty principle. The aim of this paper is to determine the analogue of the Gaussian window for the wavelet transform in one and two dimensions.

In the one-dimensional case, the wavelet transform is defined by

$$(W_\psi f)(a, b) := \int_{\mathbf{R}} f(x)|a|^{-1/2}\, \overline{\psi\left(\frac{x-b}{a}\right)}\, dx. \tag{1.2}$$

The common thread between both transforms is group theory: both are derived from square integrable group representations of certain groups, see [1], where the windowed Fourier transform is related to the Weyl-Heisenberg group and the wavelet transform stems from the affine group.

This connection allows us to construct generalized Gaussian functions in the following way. Suppose we are given a locally compact group G with a square integrable irreducible representation U in a Hilbert space H. Then, first of all, we obtain an integral transform by inner

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products

$$\langle U(g)\psi, f\rangle_H\,, \qquad g \in G, \tag{1.3}$$

compare with (1.1) and (1.2). Secondly, the infinitesimal operators of the representation lead to an uncertainty principle which in turn gives rise to a canonical choice for ψ by computing minimizing functions.

In the next section, we review the basic theory on group representations and uncertainty principles as far as it is relevant for our purposes.

This machinery will then be applied to the affine group and the wavelet transform in the subsequent sections.

2. GROUP REPRESENTATIONS AND UNCERTAINTY PRINCIPLES

Let G be a locally compact group with Haar measure $d\mu$. A unitary irreducible representation U of G in a Hilbert space H is called **square integrable** if there exists a vector ψ in H such that

$$\int_G |\langle U(g)\psi, \psi\rangle_H|^2 \, d\mu(g) < \infty. \tag{2.1}$$

A vector ψ that satisfies (2.1) is called **admissible**. If ψ is admissible, then

$$T_\psi : H \to L^2(G, d\mu), \qquad f \mapsto \langle U(g)\psi, f\rangle_H \tag{2.2}$$

is a well-defined invertible map. In particular, let G be the **Weyl-Heisenberg group**, i.e.,

$$G := \{(\omega, b, \tau) \mid b,\, \omega \in \mathbf{R},\, \tau \in \mathbf{C},\, |\tau| = 1\} \tag{2.3}$$

with group law

$$(\omega, b, \tau) \circ (\omega', b', \tau') = \left(\omega + \omega', b + b', \tau\tau' e^{i(\omega b' - \omega' b)/2}\right). \tag{2.4}$$

The Weyl-Heisenberg group possesses a unitary irreducible representation $\tilde{U}$ in $L^2(\mathbf{R})$ which is given by $[\tilde{U}(\omega, b, \tau) f](x) = \tau e^{-i\omega b/2} e^{i\omega x} f(x - b)$. It can be checked that $\tilde{U}$ is square integrable and that every function ψ in $L^2(\mathbf{R})$ is admissible, see [2] for details. If we ignore the toral component of the group representation $\tilde{U}$, i.e., if we define

$$[U(\omega, b) f](x) := e^{i\omega x} f(x - b), \tag{2.5}$$

then definition (2.2) leads us to the windowed Fourier transform given by (1.1).

With the aid of (2.5), we can define two associated self-adjoint differential operators by setting

$$\begin{aligned}
(T_\omega f)(x) &:= i\frac{\partial}{\partial\omega}[U(\omega, b) f](x)\,|_{\omega=0,b=0} = -x f(x), \\
(T_b f)(x) &:= i\frac{\partial}{\partial b}[U(\omega, b) f](x)\,|_{\omega=0,b=0} = -i f'(x).
\end{aligned} \tag{2.6}$$

These operators are called the **infinitesimal operators**, they can also be used as a basis for the Lie algebra of G. The main property for our purpose is that they lead to an uncertainty principle via the following theorem. This result is well known, see, e.g., [3] or [4]; we include the proof for the reason of consistency.

THEOREM 2.1. *For a self-adjoint operator P, define*

$$\mu_f(P) := \langle Pf, f\rangle\,.$$

Let S, T be densely defined self-adjoint operators

$$S: L^2(\mathbf{R}^n) \supset \mathcal{D}(S) \to L^2(\mathbf{R}^n),$$
$$T: L^2(\mathbf{R}^n) \supset \mathcal{D}(T) \to L^2(\mathbf{R}^n).$$

Furthermore, let the commutator $[S,T]$ be given by

$$[S,T] := ST - TS.$$

1. *Then, for any function $f \in \mathcal{D}([S,T])$ with $\|f\|_2 = 1$ and any pair (μ_1, μ_2) of real numbers the following inequality holds:*

$$|\langle [S,T]f, f\rangle|^2 \leq 4\ \|(S-\mu_1)f\|^2\ \|(T-\mu_2)f\|^2. \tag{2.7}$$

2. *Equality in (2.7) holds if and only if there exists a $\lambda \in i\mathbf{R}$ such that*

$$(S-\mu_1)f \ = \ \lambda\ (T-\mu_2)f. \tag{2.8}$$

3. *Equality in (2.7) implies that*

$$\mu_1 = \mu_f(S) = \langle Sf, f\rangle, \qquad \mu_2 = \mu_f(T) = \langle Tf, f\rangle. \tag{2.9}$$

4. *Let the variances $\mathrm{var}_f(T)$, $\mathrm{var}_f(S)$ be defined by*

$$\begin{aligned} \mathrm{var}_f(T) &:= \left\langle (T-\mu_f(T))^2 f, f\right\rangle, \\ \mathrm{var}_f(S) &:= \left\langle (S-\mu_f(S))^2 f, f\right\rangle. \end{aligned} \tag{2.10}$$

   Then, any $f \in \mathcal{D}([S,T])$ with $\|f\|_2 = 1$ obeys the following inequality, called the uncertainty principle:

$$|\mu_f([S,T])|^2 \leq 4\ \mathrm{var}_f(T)\mathrm{var}_f(S). \tag{2.11}$$

PROOF. S, T are assumed to be self-adjoint and μ_1, μ_2 are real numbers, hence,

$$\begin{aligned} \langle [S,T]f, f\rangle &= \langle Tf, Sf\rangle - \langle Sf, Tf\rangle \\ &= \langle (T-\mu_2)f, (S-\mu_1)f\rangle - \langle (S-\mu_1)f, (T-\mu_2)f\rangle \\ &= 2i\ \mathrm{Im}(\langle (T-\mu_2)f, (S-\mu_1)f\rangle). \end{aligned} \tag{2.12}$$

$\mathrm{Im}(\cdot)$ denotes the imaginary part. Taking absolute values on both sides yields

$$\begin{aligned} |\langle [S,T]f, f\rangle|^2 &= 4\ |\mathrm{Im}(\langle (T-\mu_2)f, (S-\mu_1)f\rangle)|^2 \\ &\leq 4\ |\ \langle (T-\mu_2)f, (S-\mu_1)f\rangle\ |^2 \qquad (2.13) \\ &\leq 4\ \|(T-\mu_2)f\|^2 \|(S-\mu_1)f\|^2. \qquad (2.14) \end{aligned}$$

Equality in (2.14) holds iff

$$(S-\mu_1)f = \lambda\ (T-\mu_2)f.$$

In order to give equality in (2.13), λ has to be purely imaginary. This proves the first two assertions.

To prove Statement 3 of the above theorem, we compute the scalar products with f on both sides of equation (2.8), $(\mu_1, \mu_2 \in \mathbf{R}, \|f\| = 1)$:

$$\langle Sf, f\rangle - \mu_1 = \langle (S-\mu_1)f, f\rangle = \lambda\, \langle (T-\mu_2)f, f\rangle = \lambda\, \langle Tf, f\rangle - \lambda\mu_2.$$

T and S are self-adjoint, therefore $\langle Sf, f\rangle - \mu_1$ is real and $\lambda\, \langle Tf, f\rangle - \lambda\mu_2$ is imaginary. Hence, both terms have to equal zero; this gives

$$\mu_1 = \langle Sf, f\rangle, \qquad \mu_2 = \langle Tf, f\rangle.$$

Finally, the uncertainty principle of Statement 4 follows from (2.7) since $\mu_f(S)$ and $\mu_f(T)$ are real numbers. ∎

REMARK 2.1. The name uncertainty principle is justified if we interpret $|f|^2$ as the density of a probability distribution on the real line. Then, $\langle Sf, f\rangle$ amounts to computing the expectation of f with respect to the operator S. The above theorem states that the expectation of the commutator gives a lower bound for the product of the variances of S and T. If we now take the operators $S = T_\omega$ and $T = T_b$ defined in (2.6), then, in the language of quantum mechanics, the operator S is the position and T is the momentum operator. In this case, Theorem 2.1 restates the classical uncertainty principle.

As a consequence of Theorem 2.1, we obtain an uncertainty principle whenever we have a pair of noncommuting self-adjoint operators.

We are interested in functions that minimize (2.11), i.e., we want to find functions for which (2.11) is, in fact, an equality. As stated above, for the two operators of (2.6), it can be checked that a minimizing function is given by $\psi(x) = \pi^{-1/4}e^{-x^2/2}$.

The aim of this paper is to study the minimizing functions for the **wavelet transform** instead of the windowed Fourier transform. In one dimension, the wavelet transform is defined by means of the so-called **affine** group $\mathcal{A}$ given by

$$\mathcal{A} := \{(a,b) \mid (a,b) \in \mathbf{R}^2,\ a \neq 0\}, \tag{2.15}$$

with group law

$$(a,b) \circ (a',b') = (aa', ab' + b). \tag{2.16}$$

$\mathcal{A}$ is a locally compact topological group with left Haar measure $(da\,db)/a^2$. It can be checked that

$$[U(a,b)f](x) := |a|^{-1/2} f\left(\frac{x-b}{a}\right) \tag{2.17}$$

is a unitary irreducible representation of $\mathcal{A}$ in $L^2(\mathbf{R})$. Furthermore, U is square integrable and every function ψ satisfying

$$\int_{\mathbf{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|}\, d\xi < \infty \tag{2.18}$$

is admissible. Therefore, the wavelet transform

$$W_\psi(f)(a,b) := \int_{\mathbf{R}} f(x)|a|^{-1/2}\overline{\psi\left(\frac{x-b}{a}\right)}\, dx \tag{2.19}$$

is well-defined. Definition (2.19) can be generalized to higher dimensions. Following Antoine *et al.* [5,6], we consider the group

$$\mathcal{B} := \mathbf{R}^+ \times \mathbf{R}^2 \times SO(2), \tag{2.20}$$

with group law

$$(a, b, \tau_\Theta) \circ (a', b', \tau_{\Theta'}) = (aa', b + a\tau_\Theta b', \tau_{\Theta+\Theta'}) \tag{2.21}$$

and identity element

$$e = (1, 0, I), \tag{2.22}$$

where I is the identity matrix and Θ is the rotation angle. The rotation $\tau_\Theta \in SO(2)$ acts on $b = (b_1, b_2)$ as usual:

$$\tau_\Theta(b_1, b_2) = (b_1 \cos\Theta - b_2 \sin\Theta, b_1 \sin\Theta + b_2 \cos\Theta), \qquad 0 \le \Theta < 2\pi. \tag{2.23}$$

$\mathcal{B}$ has a unitary irreducible representation in $L^2(\mathbf{R}^2)$ given by

$$[U(a,b,\Theta)f](x) = \frac{1}{a} f\left(\tau_{-\Theta}\left(\frac{x-b}{a}\right)\right). \tag{2.24}$$

Once again, this representation is square integrable and every vector ψ satisfying

$$\int_{\mathbf{R}^2} \frac{|\hat{\psi}(\xi)|^2}{\|\xi\|^2}\, d\xi < \infty \tag{2.25}$$

is admissible, so that the two-dimensional wavelet transform

$$[W_\psi f](a,b,\Theta) := \int_{\mathbf{R}^2} f(x) \frac{1}{a}\, \overline{\psi\left(\tau_{-\Theta}\left(\frac{x-b}{a}\right)\right)}\, dx \tag{2.26}$$

is well-defined.

In this paper, we set up the uncertainty relations for both, the 1-D and the 2-D continuous wavelet transform, and construct the associated families of minimizing functions.

3. THE 1-D CASE

To set up the uncertainty principle for the 1-D wavelet transform, we have to determine the associated self-adjoint operators. To obtain these operators, we have to compute the derivatives of the representation (2.17) at the identity element, compare with (2.6). We get

$$\begin{aligned}
\tilde{T}_a f(x) &:= \frac{\partial}{\partial a} \left[U(a,b) f\right](x)\big|_{a=1,b=0}, \\
&= \frac{\partial}{\partial a} \left(|a|^{-1/2} f\left(\frac{x-b}{a}\right)\right)\bigg|_{a=1,b=0} = -\frac{1}{2} f(x) - x f'(x), \\
\tilde{T}_b f(x) &:= \frac{\partial}{\partial b} \left[U(a,b) f\right](x)\,|_{a=1,b=0} \\
&= \frac{\partial}{\partial b} \left(|a|^{-1/2} f\left(\frac{x-b}{a}\right)\right)\bigg|_{a=1,b=0} = -f'(x).
\end{aligned} \tag{3.1}$$

The operators $\tilde{T}_a$ and $\tilde{T}_b$ are not self-adjoint, but after the modification

$$T_a := i\tilde{T}_a, \qquad T_b := i\tilde{T}_b, \tag{3.2}$$

we are in business. Using these operators, the affine uncertainty principle reads as follows.

LEMMA 3.1. *Let T_a and T_b be the self-adjoint operators defined by* (3.1) *and* (3.2). *Then for all $f \in \mathcal{D}([T_a, T_b])$ with $\|f\|^2 = 1$ the following relation holds:*

$$|\mu_f([T_a,T_b])|^2 = |\mu_f(T_b)|^2 \le 4(\|f'\|^2 - \mu_f(T_b)^2)\left(\|xf'\|^2 - \frac{1}{2}\|f\|^2 - \mu_f(T_a)^2\right). \tag{3.3}$$

PROOF. One has

$$([T_a,T_b]f)(x) = f'(x) = i(T_b f)(x),$$

which implies that

$$\mu_f([T_a,T_b]) = i\mu_f(T_b)\,. \tag{3.4}$$

Partial integration yields

$$\begin{aligned}
\mu_f(T_a) &= -i \int_{\mathbf{R}} \left(\frac{1}{2}|f(x)|^2 + x f'(x) \bar{f}(x)\right) dx, \\
\mu_f(T_b) &= -i \int_{\mathbf{R}} f'(x) \bar{f}(x)\, dx.
\end{aligned} \tag{3.5}$$

Employing (3.5) and using the fact that f is normalized leads to

$$\begin{aligned} \operatorname{var}_f(T_a) &= \|xf'\|^2 - \frac{1}{2}\|f\|^2 - \mu_f(T_a)^2, \\ \operatorname{var}_f(T_b) &= \|f'\|^2 - \mu_f(T_b)^2. \end{aligned} \tag{3.6}$$

Combining (3.4) and (3.6), formula (3.3) now follows immediately by Theorem 2.1. ∎

We want to find the minimizing functions, i.e., those functions for which we have equality in formula (3.3).

THEOREM 3.1. *For a given set of numbers* $\mu_1, \mu_2 \in \mathbf{R}$, $\lambda \in i\mathbf{R}$ *satisfying* $0 > -i\lambda\mu_2$ *let* f *be defined by*

$$f(x) := c(x - \lambda)^\alpha, \tag{3.7}$$

where α *is given by*

$$\alpha := -\frac{1}{2} - i\lambda\mu_2 + i\mu_1.$$

If the constant c *is chosen such that*

$$\|f\|_2^2 = 1,$$

then $\mu_f(T_a) = \mu_1$, $\mu_f(T_b) = \mu_2$ *and* f *is a minimizing function of* (3.3).

PROOF. We apply Theorem 2.1 with the operators $S = T_a$ and $T = T_b$. In this theorem, the minimizing functions of an uncertainty principle are characterized as the solutions of

$$(S - \mu_1)f = \lambda(T - \mu_2)f,$$

where $\mu_1, \mu_2, i\lambda$ are real numbers. For our choice of operators, we have to solve the differential equation

$$-\frac{1}{2}iy - ixy' - \mu_1 y = -i\lambda y' - \lambda\mu_2 y.$$

Separation of variables yields

$$\frac{i\,dx}{\lambda - x} = \frac{dy}{y\,(\lambda\mu_2 - \mu_1 - i/2)}$$

which has the solution

$$f(x) = c(x - \lambda)^\alpha,$$

with

$$\alpha = -\frac{1}{2} - i\lambda\mu_2 + i\mu_1.$$

We need to check whether $f \in L^2(\mathbf{R})$. This requires $\operatorname{Re}(\alpha) < -1/2$ or equivalently $-i\lambda\mu_2 < 0$.

Assume that c is chosen such that $\|f\|_2 = 1$. Then, since f is a minimizing function, Part 3 of Theorem 2.1 states that automatically

$$\mu_1 = \mu_f(T_a) = \langle T_a f, f \rangle, \qquad \mu_2 = \mu_f(T_b) = \langle T_b f, f \rangle.$$ ∎

In order to obtain a wavelet, we also have to check whether the solution f satisfies the admissibility condition. The Fourier transform of $f(x) = c(x - \lambda)^\alpha$ is given by

$$\hat{f}(\xi) = \begin{cases} 0, & \text{for } \xi < 0, \\ \tilde{c} \cdot \xi^{-1/2 + i\lambda\mu_2 - i\mu_1} e^{-i\lambda\xi}, & \text{for } \xi > 0, \end{cases}$$

with the constant $\tilde{c} := c \cdot 2\pi i^\alpha \Gamma(-\alpha)^{-1}$, provided that $i\lambda\mu_2 > -1/2$ and $i\lambda > 0$, see formula 3.2.3 in [7]. The admissibility condition restricts the choice for λ and μ_2 to

$$i\lambda\mu_2 > 1.$$

The case $i\lambda < 0$ can be treated similarly.

The minimizing functions for $\lambda = \pm i$ are in a certain sense the canonical choices since they satisfy the following interesting variant of the affine uncertainty principle.

COROLLARY 3.1. *The minimizing functions for $\lambda = \pm i$ satisfy*

$$|\mu_f(T_b)| = \mathrm{var}_f(T_a) + \mathrm{var}_f(T_b). \tag{3.8}$$

PROOF. If we combine the formulas (3.4) and (2.14), we see that a minimizing function satisfies

$$|\mu_f(T_b)| = 2\|(T_b - \mu_2)f\|\|(T_a - \mu_1)f\|.$$

Recall that

$$2\alpha\beta \leq \alpha^2 + \beta^2, \qquad \alpha, \beta \geq 0.$$

Equality holds if $\alpha = \beta$, and if we take into account the expression (2.8), we obtain

$$\begin{aligned} |\mu_f(T_b)| &= \|(T_b - \mu_2)f\|^2 + \|(T_a - \mu_1)f\|^2 \\ &= \mathrm{var}_f(T_b) + \mathrm{var}_f(T_a). \end{aligned} \tag{3.9}$$

∎

REMARK 3.1. It can be checked that the Gaussian window function satisfies an uncertainty principle of the same type as (3.8) with respect to the operators defined in (2.6); see [3] for details.

REMARK 3.2. The operators constructed by the equations (3.1) and (3.2) can be interpreted as a canonical basis for the Lie algebra. Consequently, the formulation of the uncertainty relation in Lemma 3.1 seems to be the most natural one. Nevertheless, the minimizing functions of Theorem 3.1 are in a certain sense invariant with respect to a change of the basis. More precisely, let S and T be defined as nontrivial linear combinations of T_a and T_b, i.e.,

$$S := c_1 T_a + c_2 T_b, \qquad T := d_1 T_a + d_2 T_b, \qquad c_1, c_2, d_1, d_2 \in \mathbf{R}.$$

A minimizing function with respect to these operators has to be a solution of the equation

$$AT_a f - \tilde{\mu} f = BT_b f,$$

where

$$A := c_1 - \lambda d_1, \qquad B := \lambda d_2 - c_2, \qquad \tilde{\mu} := \mu_f(S) - \lambda \mu_f(T),$$

compare with (2.8). Separation of variables yields

$$\frac{i\,dx}{A((B/A) - x)} = \frac{dy}{y\,(-\tilde{\mu} - (A/2)i)},$$

which has the solution

$$f(x) = c\left(x - \frac{B}{A}\right)^{i(\tilde{\mu}/A) - 1/2}.$$

This function is indeed of the same form as the minimizing function given by (3.7).

REMARK 3.3. By employing a quite different approach, the relations between wavelet analysis and uncertainty principles were studied before by T. Paul and K. Seip, see [8] for details. Paul and Seip are interested in problems arising from quantum mechanics, namely from the description of the hydrogen atom. To this end, they study the eigenfunctions of the operator

$$\overline{Z} = i\frac{d}{dx} - \frac{il}{x}, \qquad l \in \mathbf{Z}, \quad l > 0.$$

These eigenfunctions are called the affine coherent states and they are given by

$$\psi_{\overline{z}}(x) = x^l e^{-i\overline{z}x}, \qquad \mathrm{Im}\, z > 0,$$

i.e., their structure is very similar to Fourier transform of our solution. It turns out that these functions minimize the uncertainty relation with respect to the operators $P = -i\frac{d}{dx}$ and $Q = \frac{l}{x}$. A similar family of functions has also been introduced in the appendix of [9].

4. THE 2-D CASE

In this section, we study the uncertainty principle associated with the unitary irreducible representation defined by (2.24).

Once again, we have to compute the partial derivatives of the representation at the identity element. We obtain, $\left(\nabla f = (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2})^t\right)$,

$$\frac{\partial}{\partial \theta}\,[U(a,b,\theta)f](x)|_{(1,0,0)} = \nabla f \cdot \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} =: [\hat{T}_\theta(f)](x), \tag{4.1}$$

$$\frac{\partial}{\partial a}\,[U(a,b,\theta)f](x)|_{(1,0,0)} = -f(x) - \nabla f \cdot x =: [\hat{T}_a(f)](x), \tag{4.2}$$

$$\frac{\partial}{\partial b_1}\,[U(a,b,\theta)]f(x)|_{(1,0,0)} = -\frac{\partial f}{\partial x_1} =: [\hat{T}_{b_1}(f)](x), \tag{4.3}$$

$$\frac{\partial}{\partial b_2}\,[U(a,b,\theta)]f(x)|_{(1,0,0)} = -\frac{\partial f}{\partial x_2} =: [\hat{T}_{b_2}(f)](x). \tag{4.4}$$

Again, the operators defined by (4.1)–(4.4) are not self-adjoint. Therefore, we perform the modification

$$T_a := i\hat{T}_a, \quad T_\theta := i\hat{T}_\theta, \quad T_{b_1} := i\hat{T}_{b_1}, \quad T_{b_2} := i\hat{T}_{b_2}\,. \tag{4.5}$$

Fortunately, some of the operators defined by (4.5) commute. More precisely, we have

$$[T_a, T_\theta] = 0, \qquad [T_{b_1}, T_{b_2}] = 0, \tag{4.6}$$

$$[T_\theta, T_{b_1}] = -iT_{b_2}, \qquad [T_a, T_{b_1}] = iT_{b_1}, \tag{4.7}$$

$$[T_\theta, T_{b_2}] = -iT_{b_1}, \qquad [T_a, T_{b_2}] = iT_{b_2}. \tag{4.8}$$

Therefore, we have four uncertainty relations induced by the formulas (4.7) and (4.8).

LEMMA 4.1. *Let T_a, T_{b_1}, T_{b_2} and T_θ be the self-adjoint operators defined by (4.5). Then for all suitable functions f the following relations hold:*

$$\mu_f^2(T_{b_1}) \le 4\left(\left\|\frac{\partial f}{\partial x_1}\right\|_2^2 - \mu_f^2(T_{b_1})\right)\left(\|\nabla f \cdot x\|_2^2 - \|f\|_2^2 - \mu_{T_a}^2\right), \tag{4.9}$$

$$\mu_f^2(T_{b_2}) \le 4\left(\left\|\frac{\partial f}{\partial x_2}\right\|_2^2 - \mu_f^2(T_{b_2})\right)\left(\|\nabla f \cdot x\|_2^2 - \|f\|_2^2 - \mu_{T_a}^2\right), \tag{4.10}$$

$$\mu_f^2(T_{b_1}) \le 4\left(\left\|\frac{\partial f}{\partial x_1}\right\|_2^2 - \mu_f^2(T_{b_2})\right) \times \left(\left\|\frac{\partial f}{\partial x_1}x_2\right\|_2^2 + \left\|\frac{\partial f}{\partial x_2}x_1\right\|_2^2 - 2\,\mathrm{Re}\left\langle \frac{\partial f}{\partial x_2}x_2, \frac{\partial f}{\partial x_1}x_1\right\rangle - \mu_{T_\theta}^2\right), \tag{4.11}$$

$$\mu_f^2(T_{b_2}) \le 4\left(\left\|\frac{\partial f}{\partial x_1}\right\|_2^2 - \mu_f^2(T_{b_1})\right) \times \left(\left\|\frac{\partial f}{\partial x_1}x_2\right\|_2^2 + \left\|\frac{\partial f}{\partial x_2}x_1\right\|_2^2 - 2\,\mathrm{Re}\left\langle \frac{\partial f}{\partial x_2}x_2, \frac{\partial f}{\partial x_1}x_1\right\rangle - \mu_{T_\theta}^2\right). \tag{4.12}$$

REMARK 4.1. A function is suitable in our setting if it is in the domain of $[T_a, T_{b_1}]$, $[T_a, T_{b_2}]$, $[T_\theta, T_{b_1}]$ or $[T_\theta, T_{b_2}]$ and satisfies $\|f\|_2 = 1$. For such a function, at least one of the relations in Lemma 4.1 holds.

PROOF OF LEMMA 4.1. We want to apply Theorem 2.1. Therefore, we have to compute the variances. By partial integration, we obtain

$$\begin{aligned}
\mathrm{var}_f(T_\theta) &= \langle (T_\theta - \mu_f(T_\theta))^2 f, f\rangle \\
&= \langle T_\theta^2 f, f\rangle - \mu_{T_\theta}^2 \\
&= -\int_{\mathbf{R}^2} \left(\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1\right) x_2 - \frac{\partial}{\partial x_2}\left(\frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1\right) x_1 \right) \bar{f}\,dx - \mu_{T_\theta}^2 \\
&= \int_{\mathbf{R}^2} \left(\left(\frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1\right) x_2 \frac{\partial \bar{f}}{\partial x_1} - \left(\frac{\partial f}{\partial x_1}x_2 - \frac{\partial f}{\partial x_2}x_1\right) x_1 \frac{\partial \bar{f}}{\partial x_2} \right) dx - \mu_{T_\theta}^2 \\
&= \left\|\frac{\partial f}{\partial x_1}x_2\right\|_2^2 + \left\|\frac{\partial f}{\partial x_2}x_1\right\|_2^2 - \left\langle \frac{\partial f}{\partial x_2}x_2, \frac{\partial f}{\partial x_1}x_1\right\rangle - \left\langle \frac{\partial f}{\partial x_1}x_1, \frac{\partial f}{\partial x_2}x_2\right\rangle - \mu_{T_\theta}^2 \\
&= \left\|\frac{\partial f}{\partial x_1}x_2\right\|_2^2 + \left\|\frac{\partial f}{\partial x_2}x_1\right\|_2^2 - 2\,\mathrm{Re}\left\langle \frac{\partial f}{\partial x_2}x_2, \frac{\partial f}{\partial x_1}x_1\right\rangle - \mu_{T_\theta}^2
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{var}_f(T_a) &= \langle T_a^2 f, f\rangle - \mu_{T_a}^2 \\
&= -\int_{\mathbf{R}^2} (f + \nabla f\cdot x + \nabla f \cdot x + \nabla(\nabla f\cdot x)\cdot x \bar{f})\,dx - \mu_{T_a}^2 \\
&= -\|f\|_2^2 - 2\int_{\mathbf{R}^2} \nabla f\cdot x\bar{f}\,dx - \int_{\mathbf{R}^2}\left(\frac{\partial}{\partial x_1}(\nabla f\cdot x)x_1\bar{f} + \frac{\partial}{\partial x_2}(\nabla f\cdot x)x_2\bar{f}\right) dx - \mu_{T_a}^2 \\
&= -\|f\|_2^2 + \int_{\mathbf{R}^2}\left((\nabla f\cdot x)x_1\frac{\partial\bar{f}}{\partial x_1} + (\nabla f\cdot x)x_2\frac{\partial \bar{f}}{\partial x_2}\right) dx - \mu_{T_a}^2 \\
&= -\|f\|_2^2 + \|\nabla f\cdot x\|_2^2 - \mu_{T_a}^2.
\end{aligned}$$

Similarly, one has

$$\mathrm{var}_f(T_{b_1}) = \left\|\frac{\partial f}{\partial x_1}\right\|_2^2 - \mu_{T_{b_1}}^2, \qquad \mathrm{var}_f(T_{b_2}) = \left\|\frac{\partial f}{\partial x_2}\right\|_2^2 - \mu_{T_{b_2}}^2.$$

Now an application of Theorem 2.1 combined with (4.7) and (4.8) yields the result. ∎

Statement 2 of Theorem 2.1 implies that the minimizing functions for the uncertainty relations stated in Lemma 4.1 are given as the solutions of the following partial differential equations

$$i\frac{\partial f}{\partial x_1}x_2 - i\frac{\partial f}{\partial x_2}x_1 - \mu_f(T_\theta)f = -i\lambda_1\frac{\partial f}{\partial x_1} - \lambda_1\mu_f(T_{b_1})f, \tag{4.13}$$

$$i\frac{\partial f}{\partial x_1}x_2 - i\frac{\partial f}{\partial x_2}x_1 - \mu_f(T_\theta)f = -i\lambda_2\frac{\partial f}{\partial x_2} - \lambda_2\mu_f(T_{b_2})f, \tag{4.14}$$

$$-if(x) - i\frac{\partial f}{\partial x_1}x_1 - i\frac{\partial f}{\partial x_2}x_2 - \mu_f(T_a)f = -i\lambda_3\frac{\partial f}{\partial x_1} - \lambda_3\mu_f(T_{b_1})f, \tag{4.15}$$

$$-if(x) - i\frac{\partial f}{\partial x_1}x_1 - i\frac{\partial f}{\partial x_2}x_2 - \mu_f(T_a)f = -i\lambda_4\frac{\partial f}{\partial x_2} - \lambda_4\mu_f(T_{b_2})f. \tag{4.16}$$

It would be optimal to find a function that satisfies all these equations simultaneously with nontrivial values of λ_i, $i = 1, \ldots, 4$. But it can be checked that this is not possible. For instance, suppose that we have found a function f that satisfies (4.15) and (4.16) simultaneously. Subtracting these two equations leads to

$$-i\lambda_3\frac{\partial f}{\partial x_1} + i\lambda_4\frac{\partial f}{\partial x_2} = \lambda_3\,\mu_f(T_{b_1})f - \lambda_4\,\mu_f(T_{b_2})f.$$

Applying Fourier transform yields

$$\hat{f}(\xi)(\lambda_3\xi_1 - \lambda_4\xi_2 - \lambda_3\,\mu_f(T_{b_1}) + \lambda_4\,\mu_f(T_{b_2})) = 0,$$

implying $f = 0$. Therefore, it seems natural to try to find the minimizing functions in the kernel of T_a and T_θ, respectively, which leads to uncertainty relations with $\lambda_i = 0$. The following theorem shows that this strategy is indeed a suitable one.

THEOREM 4.1. *Every sufficiently smooth L^2-function f of the form*

$$f(x_1, x_2) = g\left(\sqrt{x_1^2 + x_2^2}\right) = g(r), \tag{4.17}$$

where $r := \sqrt{x_1^2 + x_2^2}$, satisfies (4.13) *and* (4.14) *simultaneously. A canonical solution is given by the Mexican hat function*

$$\psi(x) = \left[2 - 2\beta r^2\right] e^{-\beta r^2} \tag{4.18}$$

with

$$\beta = -\frac{i}{2\lambda}.$$

PROOF. We want to show that (4.13) and (4.14) are satisfied with

$$\lambda_1 = \lambda_2 = 0.$$

Using

$$\frac{\partial g}{\partial x_1} = \frac{dg}{dr}\frac{x_1}{r}, \qquad \frac{\partial g}{\partial x_2} = \frac{dg}{dr}\frac{x_2}{r}, \tag{4.19}$$

we obtain

$$T_\theta g = i\,\frac{\partial g}{\partial x_1}\,x_2 - i\,\frac{\partial g}{\partial x_2}\,x_1 = i\,\frac{dg}{dr}\left(\frac{x_2x_1 - x_1x_2}{r}\right) = 0. \tag{4.20}$$

Formula (4.20) obviously implies

$$\mu_g(T_\theta) = 0,$$

so that (4.13) and (4.14) are indeed satisfied.

One could try to find a function $g(r)$ that satisfies also (4.15) and (4.16) with $\lambda_3 = \lambda_4 = 0$. But inserting (4.17) into (4.15) and (4.16) and performing the calculations yields polynomial solutions which are not in $L^2(\mathbf{R}^2)$. Thus, it is not possible to find L^2-functions in the kernel of T_θ which minimize also the uncertainty principle related to the dilation and translation operators. However, if we take a combination of T_{b_1} and T_{b_2} given by

$$\hat{T}(f) = (T_{b_1}^2 + T_{b_2}^2)f = -\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial^2 f}{\partial x_2^2}, \tag{4.21}$$

then we can find functions $g(r)$ that minimize also an uncertainty principle with respect to T_a and $\hat{T}$. For a function $g(r)$, T_a is of the form

$$(T_a g)(r) = -ig(r) - ir\,\frac{dg}{dr},$$

and $\hat{T}$ is given by

$$(\hat{T}g)(r) = -\frac{\partial}{\partial x_1}\left(\frac{dg}{dr}\frac{x_1}{r}\right) - \frac{\partial}{\partial x_2}\left(\frac{dg}{dr}\frac{x_2}{r}\right) = -\frac{d^2g}{dr^2} - \frac{1}{r}\frac{dg}{dr}.$$

Therefore, a minimizing function $g(r)$ with respect to both of these operators has to satisfy the differential equation

$$-ig(r) - ir\frac{dg}{dr} - \mu_g(T_a)g = \lambda\left(-\frac{d^2g}{dr^2} - \frac{1}{r}\frac{dg}{dr} - \mu_g(\hat{T})g\right),$$

which is equivalent to

$$g''(r) + \left(\frac{1}{r} - \frac{i}{\lambda}r\right) g'(r) + cg(r) = 0 \tag{4.22}$$

with

$$c := \left(-\lambda^{-1}(i + \mu_g(T_a)) + \mu(\hat{T})\right).$$

We want to find L^2-solutions of (4.22), therefore, we set

$$g(r) := q(r)\, e^{-\beta r^2}. \tag{4.23}$$

Inserting (4.23),

$$g'(r) = (q'(r) - 2\beta r q(r))\, e^{-\beta r^2}$$

and

$$g''(r) = (q''(r) - 4\beta r q'(r) + (4\beta^2 r^2 - 2\beta) q(r))\, e^{-\beta r^2}$$

into (4.22) yields the following differential equation for $q(r)$:

$$q''(r) + \left(\left(-4\beta - \frac{i}{\lambda}\right) r + \frac{1}{r}\right) q'(r) + \left(\left(4\beta^2 + \frac{i2\beta}{\lambda}\right) r^2 + (c - 4\beta)\right) q(r) = 0. \tag{4.24}$$

We want to find "nice" solutions $q(r)$. The best would be polynomial solutions. Therefore, we set

$$\beta := \frac{-i}{2\lambda};$$

thus, (4.24) reduces to

$$q''(r) + \left(\frac{i}{\lambda}r + \frac{1}{r}\right) q'(r) + \tilde{c}q(r) = 0, \tag{4.25}$$

with

$$\tilde{c} = c + \frac{2i}{\lambda}.$$

Depending on the coefficients $\tilde{c}$ and λ, (4.25) can have polynomial solutions. Especially, if we choose $\tilde{c}$ and λ such that

$$\frac{2i}{\lambda} + \tilde{c} = 0,$$

then each polynomial of the form

$$q(r) = a_0 + a_2 r^2, \qquad a_0, a_2 \in \mathbf{C},$$

is a solution of (4.25) if the coefficients a_0 and a_2 are related by

$$4a_2 + \tilde{c}a_0 = 4a_2 - \frac{2i}{\lambda}\, a_0 = 0.$$

For

$$a_2 = -2\beta = \frac{i}{\lambda},$$

we obtain

$$a_0 = 2$$

and the canonical solution of (4.22) is given by

$$g(r) = (2 - 2\beta r^2)\, e^{-\beta r^2},$$

which is indeed the Mexican hat. ∎

REMARK 4.2. In its most general form, the Mexican hat wavelet reads

$$\psi(x) = [2 - (x \cdot Ax)]\, e^{-(1/2)(x \cdot Ax)}.$$

This wavelet was intensively used in image analysis by means of the continuous wavelet transform, see [6] for details. Our analysis shows that the rotation invariant form with

$$A = 2\beta I = -\frac{i}{\lambda} I$$

can be interpreted as a minimizing function of an uncertainty principle.

REMARK 4.3. Once again, the operators defined by (4.5) can be interpreted as a canonical basis for the Lie algebra. In contrary to the 1-D case, the weak invariance of the minimizing functions with respect to nontrivial linear combinations in the sense of Remark 3.2 is only partially preserved. For instance, take the combinations

$$S_1 := c_1 T_\theta + c_2 T_{b_1}, \qquad T_1 := d_1 T_\theta + d_2 T_{b_2}\ .$$

A straightforward calculation shows that a minimizing function with respect to S_1 and T_1 which is of the special form $f(x_1, x_2) = g(r)$ has to satisfy the equation

$$c_2(T_{b_1}(f) - \mu_f(T_{b_1})f) = \lambda d_2(T_{b_1}(f) - \mu_f(T_{b_1})f)$$

which implies $c_2 = 0$. This means that the weak invariance property does not fit together with the attempt to find the canonical minimizing functions in the kernel of T_θ. Nevertheless, the weak invariance is preserved for the uncertainty relation between T_a and $\hat{T}$. Indeed, take a linear combination of the form

$$S_2 := c_1 T_a + c_2 \hat{T}, \qquad T_2 := d_1 T_a + d_2 \hat{T}.$$

A similar calculation as in the proof of Theorem 4.1 shows that a minimizing function $g(r)$ with respect to S_2 and T_2 has to satisfy the equation

$$g''(r) + \left(\frac{1}{r} - \frac{i}{\hat{\lambda}} r\right) g'(r) + \hat{c} g(r) = 0, \tag{4.26}$$

with

$$\hat{\lambda} := \frac{\lambda d_2 - c_2}{c_1 - \lambda d_1},$$
$$\hat{c} := (\lambda d_2 - c_2)^{-1} \left(\lambda d_1\, \mu_g(T_a) + \lambda d_2\, \mu_g(\hat{T}) - c_1\, \mu_g(T_a) - c_2\, \mu_g(\hat{T}) - (c_1 - \lambda d_1) i\right).$$

Following once again the lines of the proof of Theorem 4.1, it can be checked that (4.26) admits the solution

$$g(r) = (2 - 2\hat{\beta} r^2) e^{-\hat{\beta} r^2}$$

with

$$\hat{\beta} := -\frac{i}{2\hat{\lambda}}.$$

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