# The distribution of Kummer sums at prime arguments

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### 1. Introduction and results

Let $\omega = e^{2\pi i/3}$ and let $(\cdot/\cdot)_3$ be the cubic residue symbol in $\mathbf{Q}(\omega)$. For $c \in \mathbf{Z}[\omega]$, $c \equiv 1\,(3)$ the cubic Gauss sum $g(c)$ is defined by

$$g(c) = \sum_{\substack{d\,(\mathrm{mod}\, c)\\ (c,d)=1}} (d/c)_3\, e(d/c)$$

where, for $z \in \mathbf{C}$,

$$e(z) = \exp\big(2\pi i(z+\bar{z})\big).$$

It is well known that (cf. [5] pp. 443—445)

$$g(c)^3 = \mu(c)\, c^2 \bar{c},$$

where $\mu$ denotes the Möbius function in $\mathbf{Q}(\omega)$. In particular, if $c$ is square-free

$$|g(c)| = |c|.$$

Now let $N$ be the norm of $\mathbf{Q}(\omega)/\mathbf{Q}$ and define

$$\tilde{g}(c) = g(c)/N(c)^{1/2};$$

one then has

$$\tilde{g}(c)^3 = \mu(c)\, c/|c|,$$

and, if $c$ is square-free

$$|\tilde{g}(c)| = 1.$$

If $c$ is not square-free then

$$g(c) = \tilde{g}(c) = 0.$$

If $\pi \equiv 1\,(3)$ is a prime in $\mathbf{Z}[\omega]$ and $p = N(\pi)$ is a prime in $\mathbf{Z}$ then one can show that

$$g(\pi) = \sum_{j=1}^{p-1} (j/\pi)_3\, e^{2\pi i j/p}$$

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(cf. [16], Appendix II). This is the more usual definition. Also one should note that many authors normalize $\pi$ by $\pi \equiv -1\,(3)$.

From the formulae quoted above one has

$$\tilde{g}(\pi)^3 = -\pi/|\pi|,$$

and, by analogy with Gauss' famous determination of the sign of the quadratic Gauss sum, one can ask whether there exists some method of selecting which cube root of $-\pi/|\pi|$ one should take. This was the problem which Kummer investigated in [10]. He found algorithms for computing $\tilde{g}(\pi)$ but they give no real theoretical insight. Having found no rule for computing $\tilde{g}(\pi)$ in terms of congruence conditions on $\pi$ Kummer examined his evaluations of $\tilde{g}(\pi)$ from a statistical point of view. This he did by dividing the primes of $\mathbf{Z}[\omega]$, $\equiv 1\,(3)$, into three classes, I, II, III according to whether $\mathrm{Re}(\tilde{g}(\pi)) < -1/2$, $-1/2 \leqq \mathrm{Re}(\tilde{g}(\pi)) < 1/2$ or $1/2 \leqq \mathrm{Re}(\tilde{g}(\pi))$. He considered, for $J = \mathrm{I}, \mathrm{II}, \mathrm{III}$

$$N_J(X) = \mathrm{Card}\{\pi\colon N(\pi) \leqq X,\ \pi \text{ of class } J\}.$$

From his calculations he found that, approximately

$$N_{\mathrm{I}}(500) : N_{\mathrm{II}}(500) : N_{\mathrm{III}}(500) :: 1:2:3,$$

and he suggested, rather cautiously, that this might be asymptotically true, [10] pp. 353—4. This was later raised to the status of a conjecture by Hasse [5] p. 453ff. who also discussed it at some length from the point of view of class-field theory.

However, later computations [14], [11], [1] showed that Kummer's original suggestion was premature and a folklore conjecture arose that the $\tilde{g}(\pi)$ were uniformly distributed on the unit circle. The numerical evidence supports this but is not very conclusive. An interesting, but unsuccessful, attempt to prove this was made in 1967 by A. I. Vinogradov [20].

The first real insight came from the pioneer work of T. Kubota [8], [9]. Further development of these ideas led to the determination of the asymptotic distribution of $\tilde{g}(c)$ as $c$ runs over all integers $\equiv 1\,(3)$ in $\mathbf{Z}[\omega]$, [16].

At the same time that these techniques were developed there also came available a new and powerful method, due to R. C. Vaughan [19], for bounding sums over primes; in [19] Vaughan found bounds on the "minor arcs" for certain trigonometric sums over primes which are used in the Hardy-Littlewood "circle" method. By using these two techniques it has been possible for us to demonstrate that the $\tilde{g}(\pi)$ are indeed uniformly distributed on the unit circle. To describe our results more precisely we denote by $\Lambda(c)$ the von Mangoldt function in $\mathbf{Z}[\omega]$; recall that if $c \equiv 1\,(3)$ then

$$\begin{aligned} \Lambda(c) &= \log N(\pi) \qquad (c = \pi^k,\ \pi \text{ prime},\ k > 1) \\ &= 0 \qquad\qquad\quad\ (\text{otherwise}). \end{aligned}$$

Then our main result is:

**Theorem 1.** *With the notations above we have, for any fixed* $\varepsilon > 0$,

$$\sum_{\substack{N(c) \leqq X \\ c \equiv 1\,(3)}} \tilde{g}(c)\Lambda(c)\,(\bar{c}/|c|)^l \ll X^{30/31+\varepsilon} + |l|\,X^{29/31+\varepsilon},$$

*where* $l \in \mathbf{Z}$. *The implied constant depends only on* $\varepsilon$.

From this, as we shall show in Section 2, it is easy to deduce that the $\tilde{g}(\pi)$ ($\pi$ prime, $\pi \equiv 1(3)$) are uniformly distributed in the following sense:

**Theorem 2.** *There exists an absolute constant $A>0$ so that for $0 \leqq \theta_1 \leqq \theta_2 \leqq 2\pi$*

$$\sum_{\substack{N(\pi) \leqq X \\ \theta_1 < \arg(\tilde{g}(\pi)) \leqq \theta_2}} 1 = \frac{\theta_2 - \theta_1}{2\pi} \operatorname{Li}(X) + O(X \exp(-A(\log X)^{1/2})).$$

Theorem 1 does not express what we expect to be the ultimate truth about the distribution of the $\tilde{g}(\pi)$. By a formal argument one is led to believe that much more is true; in [16] the following, in a slightly weaker form, was conjectured.

**Conjecture.** *With the notations above we have, for any $\varepsilon > 0$, $l \in \mathbf{Z}$,*

$$\sum_{\substack{N(c) \leqq X \\ c \equiv 1(3)}} \tilde{g}(c)\, \Lambda(c)\, (\bar{c}/|c|)^l = bX^{5/6} + O(X^{1/2+\varepsilon}) \qquad (l=0)$$
$$= O(X^{1/2+\varepsilon}) \qquad (l \neq 0)$$

*where*

$$b = 2(2\pi)^{2/3}/5\Gamma(2/3).$$

As was discussed in [16] Appendix II this conjecture is supported by the numerical evidence available. It expresses a bias of the $\tilde{g}(\pi)$ towards being positive and real, which "explains" the fact that from the earlier calculations it was not at all clear that the $\tilde{g}(\pi)$ would be uniformly distributed.

Theorem 1 is neither the most general or most precise result of its type, for we have preferred to give as direct and as simple an account as possible. The exponents in the right-hand side of the estimate of Theorem 1 can be improved by a more sophisticated treatment. Also, on the left-hand side one can replace the term $(\bar{c}/|c|)^l$ by any unitary Größencharakter, but then the right-hand side must be suitably modified. From this it follows that $\arg(\tilde{g}(\pi))$ cannot be determined in any simple way by congruence conditions on $\pi$.

A rather remarkable feature of Theorem 1 is that the saving over the trivial estimate is a power of $X$. This is in marked contrast to the present state of knowledge concerning the generalized prime-number theorem. It is, in fact, the comparitive weakness of the generalized prime-number theorem which is responsible for the difference between the error terms of Theorems 1 and 2.

The results above are purely "statistical"; they give no insight into the determination of the individual $g(c)$. Following an entirely different line of thought, J. W. S. Cassels [2], [3] has proposed an explicit formula for $g(c)$ as a product of division values of an elliptic function which has recently been proved by C. R. Matthews [13]. By means of a technique of A. J. McGetterick these can be expressed in "elementary" terms (cf. [2], [13]) which are closely related to some further conjectures of J. H. Loxton [12]. Analogues of these exist for biquadratic Gauss sums. It would be of considerable interest to understand the connection between this circle of ideas and ours.

Finally it is worth noting that our results are not restricted to cubic Gauss sums. In an algebraic number field $k$ containing the $n^{th}$ roots of unity one can form Gauss sums of order $n$, although the definition requires more care when $k$ no longer has class-number 1. The techniques of this paper are again applicable as long as $n > 2$. The one point of apparent difficulty is in the use of the results of [15] which we quote in Section 4. However, we need only [15] Theorem 6. 1 although we also use Theorem 9. 1 for simplicity. But Theorem 6. 1 is a consequence of the theory of Eisenstein series on metaplectic groups and this is generally valid although it has not been discussed in detail in print.

## 2. First steps

In this section we shall first state two theorems from which we shall deduce Theorem 1. Then we shall show how Theorem 2 follows from Theorem 1.

To formulate our results it is convenient to introduce the function

$$\tilde{g}_l(c) = \tilde{g}(c)\,(\bar{c}/|c|)^l \qquad (c \equiv 1(3)).$$

Then we define

$$H_l(X) = \sum_{\substack{N(c) \leqq X \\ c \equiv 1(3)}} \tilde{g}_l(c)\,\Lambda(c),$$

and, for $\alpha \in \mathbf{Z}[\omega]$,

$$F_l(X, \alpha) = \sum_{\substack{N(c) \leqq X \\ c \equiv 1(3) \\ c \equiv 0(\alpha)}} \tilde{g}_l(c).$$

**Theorem 3.** *With the notations above, and* $u \leqq X^{1/2}$, $\varepsilon > 0$, *we have*

$$H_l(2X) - H_l(X) \ll X^{\varepsilon}\Big(Xu^{-1/9} + \sum_{\substack{N(\alpha) \leqq u^2 \\ \alpha \equiv 1(3)}} \operatorname*{Max}_{X \leqq z \leqq 2X} |F_l(z, \alpha)|\Big).$$

*The implied constant depends only on* $\varepsilon$.

**Theorem 4.** *If* $\alpha \in \mathbf{Z}[\omega]$ *satisfies* $N(\alpha) \leqq X^{2/3}$ *and* $\varepsilon > 0$ *then*

$$F_l(X, \alpha) \ll \delta_l X^{5/6} N(\alpha)^{-1} + N(\alpha)^{-5/8} X^{3/4+\varepsilon} + |l|\, N(\alpha)^{-1/4} X^{1/2+\varepsilon},$$

*where* $\delta_l = 1$ $(l = 0)$, $= 0 (l \neq 0)$. *The implied constant depends only on* $\varepsilon$.

The proofs of these two theorems are based on different ideas and it is for this reason that we have separated them.

*Proof of Theorem* 1. We can apply the result of Theorem 4 immediately to the bound for $H_l$ given in Theorem 3; thus if $u \leqq X^{1/2}$ and $u^2 \leqq X^{2/3}$ we have

$$H_l(2X) - H_l(X) \ll X^{\varepsilon}(Xu^{-1/9} + X^{3/4}u^{3/4} + |l| X^{1/2} u^{3/2} + \delta_l X^{5/6} \log u).$$

We take $u = X^{9/31}$ which yields

$$H_l(2X) - H_l(X) \ll X^{\varepsilon}(X^{30/31} + |l| X^{29/31}).$$

Finally one deduces Theorem 1 by replacing $X$ by $X/2$, $X/4, \dots$ and summing.

*Proof of Theorem* 2. We shall begin this proof by showing that there exists an absolute constant $A>0$ so that, if $n \in \mathbf{Z}$, $n \neq 0$, $|n| \leqq \exp((\log X)^{1/2})$ then

$$\sum_{N(\pi)\leqq X} \tilde{g}(\pi)^n \ll X \exp(-A(\log X)^{1/2}),$$

where $\pi$ denotes a prime of $\mathbf{Z}[\omega]$, $\pi \equiv 1(3)$. To prove this one considers separately the cases $n \equiv 0(3)$, $n \equiv 1(3)$ and $n \equiv -1(3)$.

Suppose then that $n \equiv 0(3)$ so that $n = 3l$ ($l \in \mathbf{Z}$). Then as $\tilde{g}(\pi)^3 = -\pi/|\pi|$ one has

$$\sum_{N(\pi)\leqq X} \tilde{g}(\pi)^n = (-1)^l \sum_{N(\pi)\leqq X} (\pi/|\pi|)^l.$$

By Kubilyus [7] Lemma 4 there exists an absolute constant $A'>0$ so that, if $l \neq 0$

$$\sum_{N(\pi)\leqq X} (\pi/|\pi|)^l \ll X \exp(-A' \log X/(\log|l| + (\log X)^{1/2})).$$

From this the asserted estimate, when $n \equiv 0(3)$, follows immediately.

Suppose next that $n \equiv 1(3)$; then there exists $l \in \mathbf{Z}$ so that $n = 1-3l$ and hence

$$\sum_{N(\pi)\leqq X} \tilde{g}(\pi)^n = (-1)^l \sum_{N(\pi)\leqq X} \tilde{g}(\pi)\,(\bar{\pi}/|\pi|)^l$$
$$= (-1)^l \sum_{N(\pi)\leqq X} \tilde{g}_l(\pi).$$

But from Theorem 1 and partial summation one has, for fixed $\varepsilon>0$,

$$\sum_{N(c)\leqq X} \tilde{g}_l(c)\, \Lambda(c)/\log N(c) \ll X^{\varepsilon}(X^{30/31} + |l| X^{29/31}).$$

In this one notes that $\Lambda(c)=0$ unless $c$ is of the form $\pi^k$ ($k>0$), and $\tilde{g}(\pi^k)=0$ unless $k=1$. Thus

$$\sum_{N(\pi)\leqq X} \tilde{g}_l(\pi) \ll X^{\varepsilon}(X^{30/31} + |l| X^{29/31}).$$

The asserted estimate is merely a weakened form of this.

Only the case $n \equiv -1(3)$ remains, but this can be reduced to the previous case, $n \equiv 1(3)$, by observing that

$$g(\bar{c}) = \overline{g(c)}.$$

We shall now apply the theorem of Erdös-Turán [4]. This is a quantitative version of Weyl's criterion for uniform distribution; it states that if we have a sequence $\zeta_1, \zeta_2, \ldots$ of complex numbers so that $|\zeta_k|=1$, and if we define, for $Y \geqq 1$, $0<\theta\leqq 2\pi$,

$$N(\theta, Y) = \sum_{\substack{k \leqq Y \\ 0<\arg(\zeta_k)\leqq\theta}} 1,$$

then the dispersion

$$D(Y) = \operatorname*{Sup}_{\theta} |N(\theta, Y) - Y\theta/(2\pi)|$$

satisfies

$$D(Y) \ll \left( Ym^{-1} + \sum_{k=1}^{m} k^{-1} \left| \sum_{j\leqq Y} \zeta_j^k \right| \right)$$

where $m$ is any positive integer and the implied constant is absolute. In our case we arrange the primes of $\mathbf{Z}[\omega]$, $\equiv 1\,(3)$, in order of increasing norm; if two primes have the same norm then their order is immaterial. Then set

$$\zeta_j = \tilde{g}(\pi_j).$$

We also let $X$ be a parameter and let

$$Y(X) = \mathrm{Card}\{j\colon N(\pi_j) \leqq X\}.$$

Then, by the Erdös-Turán theorem and the preceding estimate, we have, if $m \leqq \exp((\log X)^{1/2})$,

$$D(Y(X)) \ll Y(X) m^{-1} + Y(X) \exp(-A(\log X)^{1/2}) \log m.$$

From the generalised prime-number theorem we know that

$$Y(X) \sim \mathrm{Li}(X)$$

and thus, taking $m = \exp((\log X)^{1/2})$, we see that

$$D(Y(X)) \ll X \exp(-A''(\log X)^{1/2})$$

where

$$A'' = \mathrm{Min}(1, A).$$

From the definition of $D(Y)$ it follows that

$$\sum_{\substack{N(\pi) \leqq X \\ 0 < \arg(\tilde{g}(\pi)) \leqq \theta}} 1 = \theta Y(X)/(2\pi) + O(X \exp(-A''(\log X)^{1/2})).$$

If we use the generalised prime-number theorem again we see that there exists a constant $A_0 > 0$ so that the right-hand side becomes

$$\theta\, \mathrm{Li}(X)/(2\pi) + O(X \exp(-A_0(\log X)^{1/2})).$$

To obtain Theorem 2 now one merely notes that

$$\sum_{\substack{N(\pi) \leqq X \\ \theta_1 < \arg(\tilde{g}(\pi)) \leqq \theta_2}} 1 = \sum_{\substack{N(\pi) \leqq X \\ 0 < \arg(\tilde{g}(\pi)) \leqq \theta_2}} 1 - \sum_{\substack{N(\pi) \leqq X \\ 0 < \arg(\tilde{g}(\pi)) \leqq \theta_1}} 1.$$

## 3. Proof of Theorem 3

The proof of Theorem 3 will be based on an identity due to R. C. Vaughan [19] which allows us to replace a summation involving the von Mangoldt function by certain multiple sums. These sums can then be estimated by appropriate methods.

We shall now define these sums, denoted by $\Sigma_j(X, u)$ $(0 \leqq j \leqq 4)$ where $X, u$ are positive real numbers, by

$$\Sigma_j(X, u) = \sum_{a, b, c} \Lambda(a)\, \mu(b)\, \tilde{g}_l(abc)$$

where $a, b, c$ run through square-free integers of $\mathbf{Z}[\omega]$, $a, b, c \equiv 1(3)$, and the range of summation is

| | | |
|---|---|---|
| if $j=0$ | $X < N(abc) \leqq 2X,$ | $N(bc) \leqq u,$ |
| $j=1$ | $X < N(abc) \leqq 2X,$ | $N(b) \leqq u,$ |
| $j=2$ | $X < N(abc) \leqq 2X,$ | $N(a) \leqq u,\ N(b) \leqq u,$ |
| $j=3$ | $X < N(abc) \leqq 2X,$ | $N(a) > u,\ N(bc) > u,\ N(b) \leqq u,$ |
| $j=4$ | $X < N(abc) \leqq 2X,$ | $N(bc) \leqq u,\ N(a) \leqq u.$ |

Then Vaughan's identity is the following combinatorial statement,

$$\Sigma_0(X, u) + \Sigma_2(X, u) + \Sigma_3(X, u) = \Sigma_1(X, u) + \Sigma_4(X, u).$$

Note that our conventions are not quite the same as those of Vaughan. In this identity $\Sigma_0(X, u)$ will turn out to be the quantity which we want to estimate, $\Sigma_1(X, u)$, $\Sigma_2(X, u)$ can be estimated in terms of the $F_l(z, \alpha)$, and $\Sigma_3(X, u)$, the term which causes most trouble, is estimated by the classical method of I. M. Vinogradov. If, as we shall henceforth assume,

$$1 \leqq u \leqq X^{1/2},$$

then, as the range of summation is empty,

$$\Sigma_4(X, u) = 0.$$

**Proposition 1.** (i) $\Sigma_0(X, u) = H_l(2X) - H_l(X)$,

(ii) $|\Sigma_1(X, u)| \leqq 3 \log(2X) \sum_{N(\alpha) \leqq u} \operatorname{Max}_{X \leqq z \leqq 2X} |F_l(z, \alpha)|$,

(iii) $|\Sigma_2(X, u)| \leqq 4 \log u \sum_{N(\alpha) \leqq u^2} \operatorname{Max}_{X \leqq z \leqq 2X} |F_l(z, \alpha)|$.

*Proof.* (i) We have, on writing $d = bc$,

$$\Sigma_0(X, u) = \sum_{\substack{X < N(ad) \leqq 2X \\ N(d) \leqq u}} \Lambda(a)\, \tilde{g}_l(ad) \sum_{b \mid d} \mu(b).$$

The inner sum is zero unless $d = 1$; hence

$$\Sigma_0(X, u) = \sum_{X < N(a) \leqq 2X} \Lambda(a)\, \tilde{g}_l(a) = H_l(2X) - H_l(X).$$

(ii) For $\Sigma_1(X, u)$ we set $d = ac$ which yields

$$\Sigma_1(X, u) = \sum_{\substack{X < N(bd) \leqq 2X \\ N(b) \leqq u}} \mu(b)\, \tilde{g}_l(bd) \sum_{a \mid d} \Lambda(a).$$

Now the inner sum is $\log N(d)$. On replacing $bd$ by $d$ we have

$$|\Sigma_1(X,u)| \leqq \sum_{N(b)\leqq u} \left| \sum_{\substack{X<N(d)\leqq 2X \\ d\equiv 0(b)}} \tilde{g}_l(d)\,(\log N(d)-\log N(b)) \right|.$$

But with $F_l(z,\alpha)$ defined as above we find on summing by parts

$$\sum_{\substack{X<N(d)\leqq 2X \\ d\equiv 0(b)}} \tilde{g}_l(d)\log N(d) = F_l(2X,b)\log 2X - F_l(X,b)\log X - \int_X^{2X} F_l(x,b)x^{-1}\,dx.$$

Hence

$$\left| \sum_{\substack{X<N(d)\leqq 2X \\ d\equiv 0(b)}} \tilde{g}_l(d)\log N(d) \right| \leqq 2\log(2X) \operatorname*{Max}_{X\leqq z\leqq 2X} |F_l(z,b)|.$$

We also have

$$\left| \sum_{\substack{X<N(d)\leqq 2X \\ d\equiv 0(b)}} \tilde{g}_l(d)\log N(b) \right| \leqq 2\log u \operatorname*{Max}_{X\leqq z\leqq 2X} |F_l(z,b)|,$$

whence (ii) follows as $u\leqq X^{1/2}$.

(iii) This time we set $d=ab$. For the moment, let

$$h(d) = \sum_{\substack{a|d,\, N(a)\leqq u \\ N(d/a)\leqq u}} \mu(a)\,\Lambda(d/a),$$

and note that

$$|h(d)| \leqq \sum_{a|d} \Lambda(d/a) = \log N(d).$$

Now

$$\Sigma_2(X,u) = \sum_{\substack{X<N(cd)\leqq 2X \\ N(d)\leqq u^2}} h(d)\,\tilde{g}_l(cd) = \sum_{N(d)\leqq u^2} h(d) \sum_{\substack{X<N(c)\leqq 2X \\ c\equiv 0(d)}} \tilde{g}_l(c),$$

from which it follows that

$$|\Sigma_2(X,u)| \leqq \sum_{N(d)\leqq u^2} |h(d)| \left| \sum_{\substack{X<N(c)\leqq 2X \\ c\equiv 0(d)}} \tilde{g}_l(c) \right|$$
$$\leqq 4\log u \sum_{N(d)\leqq u^2} \operatorname*{Max}_{X\leqq z\leqq 2X} |F_l(z,d)|.$$

This is assertion (iii) and Proposition 1 is now proved.

**Proposition 2.** *If* $\varepsilon>0$ *then*

$$\Sigma_3(X,u) \ll X^{\varepsilon}(X^{7/8}+Xu^{-1/9}),$$

*where the implied constant depends only on* $\varepsilon$.

*Proof of Theorem* 3. Since $u\leqq X^{1/2}$ one has $X^{7/8}\leqq Xu^{-1/9}$; hence Theorem 3 follows from Vaughan's identity, Propositions 1 and 2. It only remains to prove Proposition 2.

*Proof of Proposition* 2. We set

$$A(a) = \Lambda(a)\,\tilde{g}_l(a),$$

$$D_u(d) = \Big(\sum_{\substack{a\mid d \\ N(b)\leqq u}} \mu(b)\Big)\,\tilde{g}_l(d).$$

Then, since we have ([5] pp. 443—5)

$$\begin{aligned}\tilde{g}_l(ad) &= (a/d)_3\,(d/a)_3\,\tilde{g}_l(a)\,\tilde{g}_l(d) \qquad &&(\text{if } (a,d)=1)\\ &= 0 &&(\text{otherwise}),\end{aligned}$$

it follows that on writing $d = bc$, we can express $\Sigma_3(X, u)$ in the form

$$\Sigma_3(X, u) = \sum_{\substack{N(a),\, N(d) > u \\ X < N(ad) \leqq 2X \\ (a,d)=1}} A(a)\, D_u(d)\, (a/d)_3\, (d/a)_3 .$$

We now define

$$M_1(X) = \operatorname*{Max}_{N(a) \leqq 2X} |A(a)|$$

$$M_2(X, u) = \operatorname*{Max}_{N(d) \leqq 2X} |D_u(d)|,$$

and we note that, if $\varepsilon > 0$,

$$M_1(X) \ll X^{\varepsilon}, \quad M_2(X, u) \ll X^{\varepsilon},$$

where the implied constants depend only on $\varepsilon$.

We now divide the range of summation into those over the sets

$$\{a\colon 2^i u < N(a) \leqq 2^{i+1} u\}, \quad \{d\colon 2^j u < N(d) \leqq 2^{j+1} u\},$$

where $i, j$ are non-negative integers and $a$, $d$ are restricted to square-free integers, $\equiv 1\,(3)$. In the summation above, as $X < N(ad) \leqq 2X$, we must have

$$\log(X/u^2) - 2 < (i+j)\log 2 < \log(X/u^2) + 1 \tag{1}$$

and so there are at most $8(2 + \log(X/u^2))$ relevant pairs $(i, j)$.

We now split the sum further. For each pair $(i, j)$ we let $Q(i, j)$ be a positive integer satisfying

$$Q(i, j) \leqq \operatorname{Min}(2^i u, 2^j u);$$

it is a new parameter to be chosen later. Then, for $q$ satisfying $1 \leqq q \leqq Q(i, j)$ we introduce intervals $I^0(i, j, q)$, $J^0(i, j, q)$; if $i \geqq j$ then these are defined by

$$I^0(i, j, q) = \{x\colon 2^i u < x \leqq 2^{i+1} u\},$$

$$J^0(i, j, q) = \left\{y\colon 2^j u\left(1 + \frac{q-1}{Q(i,j)}\right) < y \leqq 2^j u\left(1 + \frac{q}{Q(i,j)}\right)\right\},$$

and, if $i < j$, then correspondingly by

$$I^0(i, j, q) = \left\{x\colon 2^i u\left(1 + \frac{q-1}{Q(i,j)}\right) < x \leqq 2^i u\left(1 + \frac{q}{Q(i,j)}\right)\right\},$$

$$J^0(i, j, q) = \{y\colon 2^j u < y \leqq 2^{j+1} u\}.$$

Then, with these notations

$$\Sigma_3(X, u) = \sum_{i,j} \sum_{q=1}^{Q(i,j)} \sum_{\substack{N(a) \in I^0(i,j,q) \\ N(d) \in J^0(i,j,q) \\ X < N(ad) \leqq 2X}} A(a)\, D_u(d)\, (a/d)_3\, (d/a)_3 .$$

We shall now remove the condition $X < N(ad) \leqq 2X$. To do this we define intervals $I(i,j,q)$, $J(i,j,q)$; if $i \geqq j$

$$I(i,j,q) = \left\{ x \colon \operatorname{Max}\left(2^i u, \frac{X}{2^j u\left(1+\frac{q-1}{Q(i,j)}\right)}\right) < x \leqq \operatorname{Min}\left(2^{i+1} u, \frac{2X}{2^j u\left(1+\frac{q}{Q(i,j)}\right)}\right)\right\},$$

$$J(i,j,q) = J^0(i,j,q),$$

and correspondingly if $i < j$. We next define

$$M(X) = \operatorname*{Max}_{z \leqq 2X} \operatorname{Card}\{b : N(b) = z\};$$

it is well-known that, if $\varepsilon > 0$,

$$M(X) \ll X^{\varepsilon}.$$

Finally we note that

$$\frac{2X}{2^j u\left(1+\frac{q-1}{Q(i,j)}\right)} - \frac{2X}{2^j u\left(1+\frac{q}{Q(i,j)}\right)} \ll 2^i u Q(i,j)^{-1},$$

which shows that if we allow for the rounding error we have

$$\begin{aligned} |\Sigma_3(X,u)| &\leqq \sum_{i,j,q} \left| \sum_{\substack{N(a) \in I(i,j,q) \\ N(d) \in J(i,j,q)}} A(a)\, D_u(d)\, (a/d)_3\, (d/a)_3 \right| \\ &\quad + O\Big(M_1(X)\, M_2(X,u)\, M(X) \sum_{i,j,q} 2^i u\, 2^j u\, Q(i,j)^{-2}\Big) \\ &= \sum_{i,j,q} \left| \sum_{\substack{N(a) \in I(i,j,q) \\ N(d) \in J(i,j,q)}} A(a)\, D_u(d)\, (a/d)_3\, (d/a)_3 \right| + O\Big(X^{1+3\varepsilon} \sum_{i,j} Q(i,j)^{-1}\Big). \end{aligned}$$

Now, if $I$ is any bounded subinterval of $(0, \infty)$ let

$$\begin{aligned} &t(I) = \operatorname{Sup}(I) \\ &C(I) = \{a \in \mathbf{Z}[\omega] \colon N(a) \in I,\ a \equiv 1\,(3),\ a \text{ square-free}\}, \\ &n(I) = \operatorname{Card}(C(I)). \end{aligned}$$

We shall prove, using methods of Vinogradov and Jutila [6], the following lemma.

**Lemma 1.** *Let $I$, $J$ be bounded subintervals of $(1, \infty)$ and let $X$ (resp. $Y$) be a complex-valued function on $C(I)$ (resp. $C(J)$) so that $|X(a)| \leqq 1$ (resp. $|Y(b)| \leqq 1$). Then, for $\varepsilon > 0$*

$$\left| \sum_{\substack{a \in C(I) \\ b \in C(J)}} X(a)\, Y(b)\, (a/b)_3\, (b/a)_3 \right| \ll (t(I)\, t(J))^{\varepsilon} \big(n(I)^{1/2}\, t(I)^{1/2}\, n(J)^{7/8} + n(I)^{1/2}\, n(J)\, t(J)^{1/4}\big),$$

*where the implied constant depends only on $\varepsilon$.*

Before we prove this we show how it leads to Proposition 2. If $i \geqq j$ we apply Lemma 1 with $I = I(i, j, q)$, $J = J(i, j, q)$, $X(a) = M_1(X)^{-1} A(a)$, $Y(b) = M_2(X, u)^{-1} D_u(b)$. Note that

$$n(I) \ll 2^i u, \qquad t(I) \ll 2^i u,$$
$$n(J) \ll 2^j u\, M(X)\, Q(i,j)^{-1}, \qquad t(J) \ll 2^j u.$$

If $i < j$ then we reverse $I, J$ etc.; we assume also that $Q(i,j) = Q(j,i)$. Then, from Lemma 1, we have

$$|\Sigma_3(X, u)| \ll \sum_{\substack{i,j \\ i \geqq j}} \left((2^i u 2^j u)^{4\varepsilon} \left((2^i u)(2^j u)^{7/8} Q(i,j)^{1/8} + (2^i u)^{1/2} (2^j u)^{5/4}\right) + X^{1+3\varepsilon} Q(i,j)^{-1}\right).$$

Next, from (1) we note that $X \ll 2^i u\, 2^j u \ll X$, and that, to each allowable value of $j$ there correspond at most 5 values of $i$. We now choose

$$Q(i,j) = \mathrm{Min}\left([(2^i u)^{1/9}], [(2^j u)^{1/9}]\right),$$

and sum over the range $2^j \leqq 2X^{1/2}/u$. Since

$$\sum (2^i u)(2^j u)^{7/8} Q(i,j)^{1/8} \ll X \sum (2^j u)^{-1/9} \ll X u^{-1/9},$$
$$\sum (2^i u)^{1/2} (2^j u)^{5/4} \ll X^{1/2} \sum (2^j u)^{3/4} \ll X^{7/8},$$
$$\sum X Q(i,j)^{-1} \ll X u^{-1/9},$$

it now follows that

$$|\Sigma_3(X, u)| \ll X^{4\varepsilon} (X^{7/8} + X u^{-1/9}).$$

This proves Proposition 2.

*Proof of Lemma* 1. We shall make use of the following inequality which is easily deduced from the Cauchy-Schwarz inequality and which embodies Vinogradov's method of handling double sums

$$\left|\sum_{i=1}^{m} \sum_{j=1}^{n} x_i y_j c_{ij}\right| \ll \left(\sum_{i=1}^{m} |x_i|^2\right)^{1/2} \left(\sum_{j=1}^{n} |y_j|^2\right)^{1/2} \left(\sum_{i,i'=1}^{m} \left|\sum_{j=1}^{n} c_{ij} \bar{c}_{i'j}\right|^2\right)^{1/4},$$

where $x_i (1 \leqq i \leqq m)$, $y_j (1 \leqq j \leqq n)$, $c_{ij} (1 \leqq i \leqq m, 1 \leqq j \leqq n)$ are complex numbers. In our case this yields

$$\left|\sum_{\substack{a \in C(I) \\ b \in C(J)}} X(a)\, Y(b)\, (a/b)_3\, (b/a)_3\right| \leqq n(I)^{1/2}\, n(J)^{1/2} \left(\sum_{b_1, b_2} \left|\sum_a \chi_{b_1 b_2}(a)\right|^2\right)^{1/4},$$

where

$$\chi_{b_1 b_2}(a) = (a/b_1)_3\, (b_1/a)_3\, \overline{(a/b_2)_3}\, \overline{(b_2/a)_3} \qquad (\text{if } (a, b_1 b_2) = 1),$$
$$= 0 \qquad (\text{otherwise}).$$

Note that this does *not* only depend on the product $b_1 b_2$. To the sum over $b_1, b_2$ we apply the Cauchy-Schwarz inequality which shows that

$$\left|\sum_{\substack{a \in C(I) \\ b \in C(J)}} X(a)\, Y(b)\, (a/b)_3\, (b/a)_3\right| \leqq n(I)^{1/2}\, n(J)^{3/4} \left(\sum_{b_1, b_2} \left|\sum_a \chi_{b_1 b_2}(a)\right|^4\right)^{1/8}.$$

The remainder of the proof is devoted to the character sum appearing here; in this we essentially follow Jutila [6].

Since, on expanding one has

$$\left|\sum_a \chi_{b_1b_2}(a)\right|^4 = \left|\sum_{a^*} T(a^*)\,\chi_{b_1b_2}(a^*)\right|^2,$$

where $T(a^*)$ is the number of ways of representing $a^*$ as $a_1a_2$ $(a_1, a_2 \in C(I))$, it follows that

$$\sum_{b_1,b_2}\left|\sum_a \chi_{b_1b_2}(a)\right|^4 = \sum_{a_1^*, a_2^*} T(a_1^*)\,T(a_2^*)\left|\sum_b \chi_{a_1^*a_2^*}(b)\right|^2.$$

However $T(a^*) \ll t(I)^\varepsilon$ whence

$$\sum_{b_1,b_2}\left|\sum_a \chi_{b_1b_2}(a)\right|^4 \ll t(I)^{2\varepsilon} \sum_{a_1^*, a_2^*} e^{-2\pi(N(a_1^*)+N(a_2^*))/t(I)^2}\left|\sum_b \chi_{a_1^*a_2^*}(b)\right|^2,$$

where we now let $a_1^*, a_2^*$ run through all integers, $\equiv 1(3)$. On reversing the order of summations on the right-hand side we obtain

$$\sum_{b_1,b_2}\left|\sum_a \chi_{b_1b_2}(a)\right|^4 \ll t(I)^{2\varepsilon} \sum_{b_1,b_2} |\theta(t(I)^{-2}, \chi_{b_1b_2})|^2,$$

where, for any congruence character $\chi$ of modulus $f$ we define

$$\theta(w,\chi) = \sum_{\substack{a\equiv 1(3)\\(a,f)=1}} \chi(a)e^{-2\pi N(a)w}.$$

We now need the following lemma.

**Lemma 2.** *Let $\chi$ be a character of modulus $f(\neq 1)$, not necessarily primitive. Then, if $w \leqq 1$, $\varepsilon > 0$,*

$$\theta(w,\chi) \ll E(\chi)w^{-1} + N(f)^{(1+\varepsilon)/2},$$

*where $E(\chi)=1$ if $\chi$ is principal, $=0$ otherwise. The implied constant depends only on $\varepsilon$.*

In our case, as $b_1, b_2$ are square-free, $\chi_{b_1b_2}$ is principal if and only if $b_1 = b_2$. Moreover, the modulus $f_{b_1b_2}$ of $\chi_{b_1b_2}$ satisfies

$$N(f_{b_1b_2}) \leqq 9t(J)^2.$$

Thus we obtain

$$\sum_{b_1,b_2}\left|\sum_a \chi_{b_1b_2}(a)\right|^4 \ll t(I)^{2\varepsilon}\left(n(J)\,t(I)^4 + n(J)^2\,t(J)^{2(1+\varepsilon)}\right)$$

and from this it follows that

$$\left|\sum_{\substack{a\in C(I)\\ b\in C(J)}} X(a)\,Y(b)\,(a/b)_3\,(b/a)_3\right| \ll t(I)^{\varepsilon/4}\left(n(J)^{7/8}\,n(I)^{1/2}\,t(I)^{1/2} + n(I)^{1/2}\,n(J)\,t(J)^{(1+\varepsilon)/4}\right)$$

which leads immediately to the asserted inequality.

*Proof of Lemma* 2. Let $\chi_1$ be the primitive character associated with $\chi$, and let $f_1$ be its conductor. Define

$$R(s,\chi)=\prod_{\substack{\pi\mid f\\ \pi\nmid f_1}}(1-\chi_1(\pi)\,N(\pi)^{-s}),$$

where $\pi$ is understood to be a prime of $\mathbf{Z}[\omega]$. Let $L(s,\chi_1)$ be the usual $L$-series over $\mathbf{Q}(\omega)$ with character $\chi_1$, and write

$$\Lambda(s,\chi_1)=(2\pi)^{-s}\,\Gamma(s)\,L(s,\chi_1).$$

This satisfies the functional equation

$$\Lambda(s,\chi_1)=W_\chi(3N(f_1))^{1/2-s}\,\Lambda(1-s,\bar{\chi}_1),$$

where $|W_\chi|=1$. The inverse Mellin transform now yields

$$\theta(w,\chi)=\frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} w^{-s}\,\Lambda(s,\chi_1)\,R(s,\chi)\,ds$$

where $\sigma>1$. Since, if $f\neq 1$, $\Lambda(s,\chi_1)\,R(s,\chi)$ is holomorphic in $\mathrm{Re}(s)<1$ even when $\chi$ is principal, we may move the line of integration to $\mathrm{Re}(s)=0$. There is a pole of the integrand at $s=1$ with residue $(6\sqrt{3})^{-1}\,R(1,\chi)w^{-1}$ if $\chi$ is principal, and none otherwise; thus

$$\theta(w,\chi)=\frac{R(1,\chi)}{6\sqrt{3}}\,E(\chi)w^{-1}+O\Big(\int_{\mathrm{Re}(s)=0}|\Lambda(s,\chi_1)\,R(s,\chi)|\,|ds|\Big).$$

If $\chi$ is principal then $|R(1,\chi)|\leqq 1$. Moreover, by applying the Phragmen-Lindelöf theorem to $L(s,\chi_1)\,R(s,\chi)$ in the strip $-\varepsilon/2\leqq\mathrm{Re}(s)\leqq 1+\varepsilon/2$ we see that, on $\mathrm{Re}(s)=0$

$$|\Lambda(s,\chi_1)\,R(s,\chi)|\ll N(f_1)^{(1+\varepsilon)/2}\,N(f)^{\varepsilon/2}e^{-\pi|s|/4}$$

and hence

$$\int_{\mathrm{Re}(s)=0}|\Lambda(s,\chi_1)\,R(s,\chi)|\,|ds|\ll N(f)^{1/2+\varepsilon}.$$

From this Lemma 2 follows immediately.

## 4. Proof of Theorem 4

The proof of Theorem 4 is based on the ideas developed in [15], [16]. We shall first define a generalised cubic Gauss sum; for $c, r\in\mathbf{Z}[\omega]$, $c\equiv 1(3)$ we let

$$g(r,c)=\sum_{d(\mathrm{mod}\,c)}(d/c)_3\,e(rd/c),$$

so that $g(c)=g(1,c)$. We shall need some algebraic properties of these sums. First of all, if $s\in\mathbf{Z}[\omega]$, $(s,c)=1$ then

$$g(rs,c)=\overline{(s/c)_3}\,g(r,c).$$

Next, if $c_1, c_2 \equiv 1\,(3)$, $(c_1, c_2)=1$ then

$$\begin{aligned} g(r, c_1 c_2) &= (c_1/c_2)_3\,(c_2/c_1)_3\, g(r, c_1)\, g(r, c_2) \\ &= \overline{(c_1/c_2)_3}\, g(r, c_1)\, g(r, c_2) \\ &= g(c_2 r, c_1)\, g(r, c_2). \end{aligned}$$

Now let $\pi$ be a prime of $\mathbf{Z}[\omega]$, $\pi \equiv 1\,(3)$. Then, if $(\pi, r)=1$ one has the following

$$\begin{aligned} &g(r, c)=0 \quad \text{if} \quad \pi^2 | c, \\ &g(r\pi, c)=0 \quad \text{unless} \quad (c, \pi)=1 \quad \text{or} \quad \pi^2 \| c, \\ &g(r\pi, \pi^2)=N(\pi)\,\overline{g(r, \pi)}. \end{aligned}$$

We shall consider three families of Dirichlet series. The first is

$$\psi(r, s, l)=\sum_{c \equiv 1\,(3)} g(r, c)\, N(c)^{-s}\,(\bar{c}/|c|)^l \qquad (l \in \mathbf{Z})$$

which converges absolutely if $\operatorname{Re}(s)>3/2$. Next, let $\alpha \in \mathbf{Z}[\omega]$, $\alpha \equiv 1\,(3)$, $\alpha$ square-free and define

$$\psi_\alpha(r, s, l)=\sum_{\substack{c \equiv 1\,(3) \\ c \equiv 0\,(\alpha)}} g(r, c)\, N(c)^{-s}\,(\bar{c}/|c|)^l \qquad (l \in \mathbf{Z})$$

and

$$\tilde{\psi}_\alpha(r, s, l)=\sum_{\substack{c \equiv 1\,(3) \\ (c, \alpha)=1}} g(r, c)\, N(c)^{-s}\,(\bar{c}/|c|)^l. \qquad (l \in \mathbf{Z})$$

We shall also need, with notations as above,

$$\begin{aligned} \zeta(s, l) &= \sum_{c \equiv 1\,(3)} N(c)^{-s}\,(\bar{c}/|c|)^{3l}, \qquad (l \in \mathbf{Z}) \\ \Delta_\alpha(s, l) &= \prod \left(1-N(\pi)^{2-3s}\,(\bar{\pi}/|\pi|)^{3l}\right), \end{aligned}$$

where $\pi\,(\equiv 1\,(3))$ runs through the prime divisors of $\alpha$. Our first problem is to express the $\psi_\alpha(r, s, l)$ in terms of the $\psi(r, s, l)$. In order to make the formulae more compact we shall define

$$\Omega(c)=N(c)^{-s}\,(\bar{c}/|c|)^l$$

and we shall write $\psi(r, \Omega)$, etc., in place of $\psi(r, s, l)$, etc. Clearly $\Omega$ is a Größencharakter of conductor dividing 3.

**Lemma 3.** *Let* $(\alpha, r)=1$ *and let* $\alpha$ *be square-free. Then*

(i) $\tilde{\psi}_\alpha(\alpha r, \Omega)\, \Delta_\alpha(\Omega)=\sum_{\substack{d | \alpha \\ d \equiv 1\,(3)}} \mu(d)\, N(d)\, \Omega(d^2)\, \overline{g(r\alpha/d, d)}\, \psi(r\alpha/d, \Omega)$,

(ii) $\psi_\alpha(r, \Omega)=\Omega(\alpha)\, g(r, \alpha)\, \tilde{\psi}_\alpha(r\alpha, \Omega)$,

(iii) $\psi_\alpha(r, \Omega)\, \Delta_\alpha(\Omega)=\sum_{\substack{d | \alpha \\ d \equiv 1\,(3)}} \mu(d)\, N(d^2)\, \Omega(d^2)\, \Omega(\alpha)\, g(r, \alpha/d)\, \psi(r\alpha/d, \Omega)$.

**Remark** Assertion (i) is a reflection of the Hecke theory of the metaplectic group applied to Eisenstein series.

*Proof.* (i) This we prove by induction on the number of primes dividing $\alpha$; the assertion is trivial if $\alpha=1$. Suppose now that $\alpha$ is square-free and that $\pi\equiv 1(3)$ is a prime of $\mathbf{Z}[\omega]$, $\pi\nmid\alpha r$. We consider the Dirichlet series defining $\tilde{\psi}_\alpha(r,\Omega)$; this can be written as

$$\tilde{\psi}_\alpha(r,\Omega)=\sum_{k=0}^{\infty}\sum_{\substack{c'\equiv 1(3)\\(c',\pi\alpha)=1}} g(r,c'\pi^k)\,\Omega(c'\pi^k).$$

However, $g(r,c'\pi^k)=0$ unless $k=0$, 1 and

$$g(r,c'\pi)=g(r\pi,c')\,g(r,\pi),$$

whence

$$\tilde{\psi}_\alpha(r,\Omega)=\tilde{\psi}_{\alpha\pi}(r,\Omega)+\Omega(\pi)\,g(r,\pi)\,\tilde{\psi}_{\alpha\pi}(r\pi,\Omega).$$

Similarly, one proves that

$$\tilde{\psi}_\alpha(r\pi,\Omega)=\tilde{\psi}_{\alpha\pi}(r\pi,\Omega)+\Omega(\pi^2)\,N(\pi)\,\overline{g(r,\pi)}\,\tilde{\psi}_{\alpha\pi}(r,\Omega).$$

From these two equations one has, on eliminating $\tilde{\psi}_{\alpha\pi}(r,\Omega)$,

$$\tilde{\psi}_{\alpha\pi}(r\pi,\Omega)\,\Delta_\pi(\Omega)=\tilde{\psi}_\alpha(r\pi,\Omega)-\Omega(\pi^2)\,N(\pi)\,\overline{g(r,\pi)}\,\tilde{\psi}_\alpha(r,\Omega).$$

Now we shall assume that (i) is true for $\alpha$ and all $r$; we shall verify that it is true with $\alpha$ replaced by $\alpha\pi$. Indeed, by the above with $\alpha r$ in place of $r$, we find on multiplying through by $\Delta_\alpha(\Omega)$

$$\tilde{\psi}_{\alpha\pi}(\alpha\pi r,\Omega)\,\Delta_{\alpha\pi}(\Omega)=\tilde{\psi}_\alpha(\alpha\pi r,\Omega)\,\Delta_\alpha(\Omega)-\Omega(\pi^2)\,N(\pi)\,\overline{g(\alpha r,\pi)}\,\tilde{\psi}_\alpha(\alpha r,\Omega)\,\Delta_\alpha(\Omega).$$

We now make use of the induction hypothesis which shows that the right-hand side is equal to

$$\sum_{d|\alpha}\mu(d)\,N(d)\,\Omega(d^2)\,\overline{g(r\alpha\pi/d,d)}\,\psi(r\alpha\pi/d,\Omega)$$
$$+\sum_{d|\alpha}\mu(d\pi)\,N(d\pi)\,\Omega(d^2\pi^2)\,\overline{g(r\alpha/d,d)}\,\overline{g(r\alpha,\pi)}\,\psi(r\alpha/d,\Omega).$$

Since $g(r\alpha/d,d)\,g(r\alpha,\pi)=g(r\alpha/d,d\pi)$ we may replace $d\pi$ in the second sum by $d$, which yields for the expression above

$$\sum_{d|\alpha\pi}\mu(d)\,N(d)\,\Omega(d^2)\,\overline{g(r\alpha\pi/d,d)}\,\psi(r\alpha\pi/d,\Omega).$$

This completes the induction step and the proof of (i).

To prove (ii) we merely remark that if $g(r,c)\neq 0$ and $c\equiv 0(\alpha)$ then $c$ is of the form $c_1\alpha$ where $(c_1,\alpha)=1$. Thus

$$\begin{aligned}\psi_\alpha(r,\Omega)&=\sum_{\substack{c_1\equiv 1(3)\\(c_1,\alpha)=1}} g(r,c_1\alpha)\,\Omega(c_1\alpha)\\&=\Bigg(\sum_{\substack{c_1\equiv 1(3)\\(c_1,\alpha)=1}} g(r\alpha,c_1)\,\Omega(c_1)\Bigg)g(r,\alpha)\,\Omega(\alpha)\\&=\Omega(\alpha)\,g(r,\alpha)\,\tilde{\psi}_\alpha(r\alpha,\Omega),\end{aligned}$$

as asserted.

From (i), (ii) formula (iii) follows at once since

$$g(r,\alpha)\,\overline{g(r\alpha/d,d)}=g(r,\alpha/d)\,g(\alpha r/d,d)\,\overline{g(\alpha r/d,d)}$$
$$=N(d)\,g(r,\alpha/d).$$

Whereas Lemma 3 is essentially formal the next lemma is a consequence of Kubota's theory, as developed in [15], and in this rests the possibility of making the estimates we need.

**Lemma 4.** *The function* $\psi(r,s,l)$ *can be continued as a meromorphic function of* $s$ *to the complex plane; it is entire in* $\mathrm{Re}(s)>1$ *if* $l\neq 0$, *and has at most a simple pole at* $s=4/3$ *if* $l=0$. *If* $r\equiv 1\,(3)$, $r$ *square-free, the residue of* $\psi(r,s,0)$ *at* $s=4/3$ *is*

$$p_0(r)=c_0\overline{g(1,r)}\,N(r)^{-2/3}$$

*where* $c_0$ *is a known constant. Consequently* $\psi_\alpha(r,s,l)$ $((r,\alpha)=1,\ \alpha$ *square-free*$)$ *can be meromorphically continued to the complex plane; it is also entire in* $\mathrm{Re}(s)>1$ *if* $l\neq 0$, *and if* $l=0$ *it has at most a simple pole at* $s=4/3$. *If* $r\equiv 1\,(3)$, $r$ *square-free, the residue* $p(\alpha,r)$ *of* $\psi_\alpha(r,s,0)$ *at* $s=4/3$ *is*

$$p(\alpha,r)=c_0N(\alpha)^{-1}\,\overline{g(1,r)}\,N(r)^{-2/3}\prod_{\pi|\alpha}(1+N(\pi)^{-1})^{-1}.$$

*Let* $\varepsilon>0$ *and* $\sigma_1=3/2+\varepsilon$. *If* $s=\sigma+it$, $\sigma_1\geqq\sigma\geqq\sigma_1-1/2$, $|s-4/3|>1/12$ *then*

$$\psi_\alpha(r,s,l)\ll N(\alpha)^{3/2(\sigma_1-\sigma)-1}\,N(r)^{1/2(\sigma_1-\sigma)}\,(1+l^2+t^2)^{\sigma_1-\sigma},$$

*where the implied constant depends only on* $\varepsilon$.

*Proof.* The meromorphic continuation of $\psi(r,s,l)$ and hence, by Lemma 3, that of $\psi_\alpha(r,s,l)$ follows from [15] Theorem 6. 1 as does the assertion about the poles in $\mathrm{Re}(s)>1$. The residue $p_0(r)$ is given by [15] Theorem 9. 1 and [15] II, from which it follows that

$$c_0=(2\pi)^{5/3}\,(3^{3/2}\,8\,\Gamma(2/3)\,\zeta_{\mathbf{Q}(\omega)}(2))^{-1},$$

where $\zeta_{\mathbf{Q}(\omega)}(s)$ is the Dedekind zeta-function of $\mathbf{Q}(\omega)$. From Lemma 3 (iii) we have

$$p(\alpha,r)=c_0\Delta_\alpha(4/3,0)^{-1}\sum_{d|\alpha}\mu(d)\,N(\alpha)^{-2}\,N(r)^{-2/3}\,g(r,\alpha/d)\,\overline{g(1,\alpha r/d)}$$
$$=c_0N(r)^{-2/3}\,N(\alpha)^{-1}\,\overline{g(1,r)}\,\Delta_\alpha(4/3,0)^{-1}\sum_{\substack{d|\alpha\\ d\equiv 1\,(3)}}\mu(d)\,N(d)^{-1},$$

from which the stated result follows easily.

It therefore remains to prove the estimates. Let

$$G_l(s)=(2\pi)^{-2s}\,\Gamma(s+|l|/2-1/3)\,\Gamma(s+|l|/2-2/3)$$

and

$$F(r,s,l)=G_l(s)\,\psi(r,s,l)\,\zeta(3s-2,l).$$

Then it is shown in [15] Theorem 6. 1 that $F(r,s,l)$ can be continued to the complex plane as an entire function if $l\neq 0$, and as a meromorphic function having at most simple poles at $s=4/3$, $2/3$ if $l=0$. Note that consequently $\psi(r,s,l)\,\zeta(3s-2,l)$ is regular at $s=2/3$. In vertical strips of finite width $F(s,r,l)$ remains bounded as $|\mathrm{Im}(s)|\to\infty$.

Moreover, there is a functional equation which expresses $F(r, 2-s, -l)$ as a linear combination of functions $N(r)^{s-1} F(r\eta, s, l)$ where $\eta$ runs through the divisors of 9 (in $\mathbf{Z}[\omega]$). The coefficients of this linear combination are quotients of finite Dirichlet series. The precise statement is not relevant at the moment since we use it only to derive an upper bound for $\psi(r, s, l)\,\zeta(3s-2, l)$ on $\mathrm{Re}(s) = 2-\sigma_1$. Indeed, if $r'$ has no squared factors other than powers of $\sqrt{-3}$ then, on $\mathrm{Re}(s) = \sigma_1$,

$$\psi(r', s, l)\,\zeta(3s-2, l) \ll 1,$$

where the implied constant depends only on $\varepsilon$; this follows from the multiplicativity of $|g(r, c)|$ in $c$ whence one obtains, if $\sigma = \mathrm{Re}(s) > 3/2$, the following majorization

$$|\psi(r', s, l)| \leqq \prod_{\substack{\pi \nmid r' \\ \pi \neq \sqrt{-3}}} (1+N(\pi)^{1/2-\sigma}) \prod_{\substack{\pi | r' \\ \pi \neq \sqrt{-3}}} (1+N(\pi)^{3/2-2\sigma}) \leqq \prod_{\pi \neq \sqrt{-3}} (1+N(\pi)^{1/2-\sigma}).$$

From the functional equation one deduces that if $\mathrm{Re}(s) = 2-\sigma_1$

$$\psi(r, s, l)\,\zeta(3s-2, l) \ll N(r)^{\sigma_1 - 1}\,|G_l(2-s)/G_l(s)|.$$

But

$$G_l(2-s)/G_l(s) = \frac{\Gamma(5/3-s+|l|/2)\,\Gamma(4/3-s+|l|/2)}{\Gamma(s-1/3+|l|/2)\,\Gamma(s-2/3+|l|/2)}\,(2\pi)^{4(s-1)},$$

from which one obtains, by Stirling's formula, that on $\mathrm{Re}(s) = 2-\sigma_1$

$$|G_l(2-s)/G_l(s)| \ll (1+l^2+t^2)^{2(\sigma_1-1)},$$

where $t = \mathrm{Im}(s)$. Thus, on $\mathrm{Re}(s) = 2-\sigma_1$

$$\psi(r, s, l)\,\zeta(3s-2, l) \ll N(r)^{\sigma_1-1}\,(1+l^2+t^2)^{2(\sigma_1-1)}.$$

We may now consider the entire function

$$(s-4/3)s^{-1}\,\psi(r, s, l)\,\zeta(3s-2, l)$$

in the strip $2-\sigma_1 \leqq \mathrm{Re}(s) \leqq \sigma_1$; it satisfies the conditions of the precise form of the Phragmen-Lindelöf theorem due to Rademacher [17] p. 66 from which one deduces that if $|s-4/3| > 1/12$, $2-\sigma_1 \leqq \mathrm{Re}(s) = \sigma \leqq \sigma_1$, then

$$\psi(r, s, l)\,\zeta(3s-2, l) \ll N(r)^{1/2(\sigma_1-\sigma)}\,(1+l^2+t^2)^{\sigma_1-\sigma}.$$

On noting that $\Delta_\alpha(s, l)$ is made up of local factors of $\zeta(3s-2, l)$ (those corresponding to primes dividing $\alpha$) we see that

$$(\Delta_\alpha(s, l)\,\zeta(3s-2, l))^{-1} \ll 1, \qquad (\sigma_1 - 1/2 \leqq \sigma \leqq \sigma_1).$$

Hence

$$\psi(r, s, l)\,\Delta_\alpha(s, l)^{-1} \ll N(r)^{1/2(\sigma_1-\sigma)}\,(1+l^2+t^2)^{\sigma_1-\sigma}$$

if $|s-4/3| > 1/12$, $\sigma_1 - 1/2 \leqq \sigma \leqq \sigma_1$. This does not require that $(\alpha, r) = 1$.

We now use Lemma 3 (iii) which yields

$$\psi_\alpha(r, s, l) \ll N(\alpha)^{1/2-\sigma+1/2(\sigma_1-\sigma)} N(r)^{1/2(\sigma_1-\sigma)} (1+l^2+t^2)^{\sigma_1-\sigma} \sum_{d|\alpha} N(d)^{2-2\sigma-1/2-1/2(\sigma_1-\sigma)}$$

in the same region. But as

$$2-2\sigma-1/2-1/2(\sigma_1-\sigma) < -3/4$$

it follows that

$$\sum_{d|\alpha} N(d)^{2-2\sigma-1/2-1/2(\sigma_1-\sigma)} \ll N(\alpha)^{\varepsilon}.$$

Thus, as $\sigma_1 = 3/2+\varepsilon$, we see that

$$\psi_\alpha(r, s, l) \ll N(\alpha)^{3/2(\sigma_1-\sigma)-1} N(r)^{1/2(\sigma_1-\sigma)} (1+l^2+t^2)^{\sigma_1-\sigma},$$

as asserted. Lemma 4 is now proved.

We are now in a position to prove Theorem 4 which is achieved by a Tauberian argument. We let

$$a_n = \sum_{\substack{N(c)=n \\ c\equiv 1(3) \\ c\equiv 0(\alpha)}} \tilde{g}_l(c);$$

note that $a_n = 0$ unless $n \equiv 0(N(\alpha))$. Define the Dirichlet series

$$f(s) = N(\alpha)^s \sum a_n n^{-s} = N(\alpha)^s \psi_\alpha(1, s+1/2, l).$$

We shall apply [18] Lemma 3. 12 to this function. To justify this we note that if $s > 1$

$$\sum |a_n| n^{-s} \leqq N(\alpha)^s \sum_{\substack{c\equiv 1(3) \\ c\equiv 0(\alpha)}} |\tilde{g}_l(c)|\, N(c)^{-s}$$

$$\leqq \sum_{c\equiv 1(3)} N(c)^{-s} \ll (s-1)^{-1}$$

as $s \to 1$. Also, for fixed $\varepsilon > 0$

$$a_{nN(\alpha)} \ll n^{\varepsilon/2},$$

where the implied constant depends only on $\varepsilon$. If also $X/N(\alpha) \equiv 1/2 \pmod 1$ then [18] Lemma 3. 12 shows that

$$\sum_{n\leqq X} a_n = \frac{1}{2\pi i} \int_{\sigma_2-iT}^{\sigma_2+iT} f(s)\, (X/N(\alpha))^s s^{-1}\, ds + O\big(X^{\sigma_2} N(\alpha)^{-1} T^{-1}\big),$$

where $\sigma_2 = 1+\varepsilon$. The left-hand side is $F_l(X, \alpha)$; an application of Cauchy's theorem to the right-hand side yields now, if $\sigma_3 = \sigma_2 - 1/2$,

$$F_l(X, \alpha) = \delta_l\, p(\alpha, 1)\, 6/5\, X^{5/6} + \frac{1}{2\pi i} \left( \int_{\sigma_2-iT}^{\sigma_3-iT} + \int_{\sigma_3-iT}^{\sigma_3+iT} + \int_{\sigma_3+iT}^{\sigma_2+iT} \right) f(s)\, (X/N(\alpha))^s s^{-1}\, ds$$

$$+ O\big(X^{\sigma_2} N(\alpha)^{-1} T^{-1}\big).$$

We shall estimate the three integrals using Lemma 4; we shall assume that $\varepsilon < 1/4$. Since our bound for

$$|f(s)\,(X/N(\alpha))^s|$$

is monotonic on the horizontal lines the first and third integrals are bounded by

$$O\big(X^{\sigma_2} N(\alpha)^{-1}\, T^{-1}\big) + O\big(X^{\sigma_3}(1 + l^2 + T^2)^{1/2}\, N(\alpha)^{-1/4}\, T^{-1}\big).$$

The second integral is bounded by

$$O\left(X^{\sigma_3} N(\alpha)^{-1/4} \int_{-T}^{T} (1 + l^2 + t^2)^{1/2}\,(1 + t^2)^{-1/2} dt\right).$$

However, as

$$(1 + l^2 + t^2)^{1/2} \ll (1 + t^2)^{1/2} + |l|,$$

we have

$$\int_{-T}^{T} (1 + l^2 + t^2)^{1/2}\,(1 + t^2)^{-1/2} dt \ll T + |l| \log T.$$

These results taken together show that

$$F_l(X, \alpha) = \delta_l\, p(\alpha, 1)\; 6/5\; X^{5/6} + O\big(X^{\sigma_2} N(\alpha)^{-1}\, T^{-1}\big) + O\big(X^{\sigma_3} N(\alpha)^{-1/4}\, T\big) + O\big(X^{\sigma_3} N(\alpha)^{-1/4} |l| \log T\big).$$

Assuming that $N(\alpha) \leqq X^{2/3}$ we take $T = N(\alpha)^{-3/8} X^{1/4}$ so that

$$F_l(X, \alpha) = \delta_l\, p(\alpha, 1)\; 6/5\; X^{5/6} + O\big(X^{3/4+\varepsilon} N(\alpha)^{-5/8}\big) + O\big(X^{1/2+\varepsilon} N(\alpha)^{-1/4} |l| \log X\big).$$

This, together with the observation from Lemma 4 that

$$p(\alpha, 1) \ll N(\alpha)^{-1},$$

proves Theorem 4.

**Remark.** Had we not known $p(\alpha, 1)$ it would still have been possible to estimate it by the Phragmen-Lindelöf theorem. This would have shown that

$$p(\alpha, 1) \ll N(\alpha)^{-3/4+\varepsilon},$$

which would have been quite sufficient to prove Theorem 1. Thus, as indicated in Section 1, we could have avoided using [15] Theorem 9. 1.

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