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Mills' ratio: Monotonicity patterns and functional inequalities ☆

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Dedicated to Professor Péter T. Nagy on the occasion of his sixtieth birthday

Abstract

In this paper we study the monotonicity properties of some functions involving the Mills' ratio of the standard normal law. From these we deduce some new functional inequalities involving the Mills' ratio, and we show that the Mills' ratio is strictly completely monotonic. At the end of this paper we present some Turán-type inequalities for Mills' ratio.

Keywords: Normal distribution; Mills' ratio; Monotone form of l'Hospital's rule; Functional inequality; Complete monotonicity; Schwarz's inequality; Turán-type inequality

1. Introduction

Let us consider the probability density function $\varphi : \mathbb{R} \to \mathbb{R}$ and the reliability function $\overline{\Phi} : \mathbb{R} \to \mathbb{R}$ of the standard normal law, defined by

$$\varphi(x) := \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad \overline{\Phi}(x) := \frac{1}{\sqrt{2\pi}} \int_x^\infty \varphi(t)\, \mathrm{d}t.$$

The function $r : \mathbb{R} \to (0, \infty)$, defined by

$$r(x) := \frac{\overline{\Phi}(x)}{\varphi(x)} = e^{x^2/2} \int_x^\infty e^{-t^2/2}\, \mathrm{d}t,$$

is known in literature as Mills' ratio [19, Section 2.26] of the standard normal law, while its reciprocal $1/r$, defined by $1/r(x) := \varphi(x)/\overline{\Phi}(x)$, is the so-called failure (hazard) rate. It is well known that Mills' ratio is convex and strictly

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decreasing on $\mathbb{R}$, at the origin takes on the value $r(0) = \sqrt{\pi/2}$, and has tails described by the asymptotic expansion as $x \to \infty$

$$r(x) \sim \frac{1}{x} - \frac{1}{x^3} + \frac{1\cdot 3}{x^5} - \frac{1\cdot 3\cdot 5}{x^7} + \cdots.$$

This ratio is frequently used in mathematical statistics, please see for example the paper of A. Feuerverger, M. Menzinger, H.L. Atwood and R.L. Cooper [12]. We note that various lower and upper bounds are known for this ratio, see [19, Section 2.26] and the references therein for results which refer to the problem of finding some simple functions which approximate r. The most known inequalities proved by R.D. Gordon [13] in 1941 are

$$\frac{x}{x^2+1} < r(x) < \frac{1}{x}, \tag{1.1}$$

for all $x > 0$. Recently I. Pinelis [22] using a version of the monotone form of l'Hospital rule, i.e. Lemma 2.1 improved the inequality $r(x) < 1/x$, showing that in fact the function $x \mapsto xr(x)$ is strictly increasing on $(0, \infty)$. We note that in view of the derivative formula $r'(x) = xr(x) - 1$ this implies that the first inequality in (1.1) holds. Motivated by Pinelis' work in Section 2 we show that using Pinelis' idea we can deduce some other known bounds for Mills' ratio, and in Lemma 2.2 we present a simple method to find other new functions which approximate r. Moreover, using the Pinelis version of the monotone l'Hospital's rule we study the monotonicity of some functions involving the Mills' ratio. As an application, at the end of Section 2 we present an interesting chain of inequalities for Mills' ratio. Finally, in Section 3 we prove that the Mills' ratio is strictly completely monotonic on $\mathbb{R}$ and using Schwarz's inequality for integrals we deduce some Turán-type inequalities for nth derivative of Mills' ratio.

2. Functional inequalities involving the Mills' ratio

Before we state our main results of this section let us enounce the following version of the monotone form of l'Hospital rule due to I. Pinelis [24].

Lemma 2.1. *Let $-\infty \leqslant a < b \leqslant \infty$ and let f and g be differentiable functions on (a, b). Assume that either $g' > 0$ everywhere on (a, b) or $g' < 0$ on (a, b). Furthermore, suppose that $f(a^+) = g(a^+) = 0$ or $f(b^-) = g(b^-) = 0$ and f'/g' is (strictly) increasing (decreasing) on (a, b). Then the ratio f/g is (strictly) increasing (decreasing) too on (a, b).*

We note that another version of monotone form of l'Hospital rule was proved by G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [3,4]. Various versions of this monotone form of l'Hospital rule was used since 1982 in different areas of mathematics, for example in

1. differential geometry [10,14],
2. quasiconformal analysis [3,4],
3. statistics and probability [22,23,25],
4. analytic inequalities [2,21].

The following result is an immediate application of Lemma 2.1 and provides a generalization of Proposition 1.2 due to I. Pinelis [22].

Lemma 2.2. *Let us consider the differentiable function $h : [a, \infty) \to (0, \infty)$, where $a \in \mathbb{R}$. Assume that for all $x \geqslant a$ the product $h(x)\varphi(x)$ is not constant and $[h\varphi]'$ does not change sign, further $\lim_{x\to\infty} h(x)\varphi(x) = 0$. If the function $g : [a, \infty) \to \mathbb{R}$, defined by*

$$g(x) := \left[xh(x) - h'(x)\right]^{-1},$$

has the limit 1 at infinity and is (strictly) increasing on $[a, \infty)$, then for all $x \geqslant a$ we have $r(x) < h(x)$. Moreover, when g is (strictly) decreasing the above inequality is reversed, i.e. for all $x \geqslant a$, we have $r(x) > h(x)$.

Proof. Since the product $h(x)\varphi(x)$ is not constant on $[a,\infty)$ it follows that $xh(x)-h'(x)\neq 0$ for all $x\in[a,\infty)$, thus the function g makes sense. Suppose that g is (strictly) increasing. But $\lim_{x\to\infty}h(x)\varphi(x)=\lim_{x\to\infty}\overline{\Phi}(x)=0$ and $\varphi'(x)=-x\varphi(x)$, thus from the hypothesis it follows that the function

$$x\mapsto\frac{\overline{\Phi}'(x)}{[h(x)\varphi(x)]'}=\frac{-\varphi(x)}{h'(x)\varphi(x)+h(x)\varphi'(x)}=g(x)$$

is (strictly) increasing. Then applying the monotone form of l'Hospital rule, i.e. Lemma 2.1 we conclude that the ratio

$$\frac{r(x)}{h(x)}=\frac{\overline{\Phi}(x)}{h(x)\varphi(x)}$$

is (strictly) increasing too on $[a,\infty)$. Now using the l'Hospital rule for limits we obtain that

$$\lim_{x\to\infty}\frac{r(x)}{h(x)}=\lim_{x\to\infty}\frac{\overline{\Phi}(x)}{h(x)\varphi(x)}=\lim_{x\to\infty}\frac{\overline{\Phi}'(x)}{[h(x)\varphi(x)]'}=\lim_{x\to\infty}g(x)=1,$$

which implies that for all $x\geqslant a$ the inequality $r(x)<h(x)$ holds. Analogously, when the function g is (strictly) decreasing, we have that the ratio r/h is (strictly) decreasing too, thus for all $x\geqslant a$ we have the inequality $r(x)>h(x)$. □

An interesting application of Lemma 2.2 is the following result.

Theorem 2.3. *Let us consider the Mills' ratio r of the standard normal law. Then for all $x\geqslant 0$ the following inequalities hold*:

$$\frac{2}{\sqrt{x^2+4}+x}<r(x)<\frac{4}{\sqrt{x^2+8}+3x}. \tag{2.4}$$

Proof. We note that the above lower and upper bounds improve the bounds from (1.1) of R.D. Gordon [13]. The first inequality in (2.4) was proved by Z.W. Birnbaum [8] and Y. Komatu [16], while the second inequality in (2.4) is due to M.R. Sampford [27]. However, we give here a different proof for these bounds. For this let us consider the functions $h_1,h_2:[0,\infty)\to(0,\infty)$, defined by

$$h_1(x)=\frac{2}{\sqrt{x^2+4}+x}\quad\text{and}\quad h_2(x)=\frac{4}{\sqrt{x^2+8}+3x}.$$

Clearly we have that the functions $x\mapsto h_1(x)\varphi(x)$ and $x\mapsto h_2(x)\varphi(x)$ are strictly decreasing on $[0,\infty)$. Moreover, it is easy to verify that

$$\lim_{x\to\infty}h_1(x)\varphi(x)=\lim_{x\to\infty}h_2(x)\varphi(x)=0.$$

Now consider the functions $g_1,g_2:[0,\infty)\to\mathbb{R}$, defined by $g_i(x):=[xh_i(x)-h_i'(x)]^{-1}$, where $i=1,2$. Easy computations show that

$$g_1(x)=\frac{x\sqrt{x^2+4}+x^2+4}{2(x\sqrt{x^2+4}+1)}$$

and

$$g_2(x)=\frac{\sqrt{x^2+8}(\sqrt{x^2+8}+3x)^2}{4[3(x^2+1)\sqrt{x^2+8}+x^3+9x]}.$$

Moreover, $\lim_{x\to\infty}g_1(x)=\lim_{x\to\infty}g_2(x)=1$ and for all $x\geqslant 0$ we have

$$\frac{\mathrm{d}g_1(x)}{\mathrm{d}x}=\frac{x\sqrt{x^2+4}-x^2-6}{\sqrt{x^2+4}(x\sqrt{x^2+4}+1)^2}<0,$$

$$\frac{\mathrm{d}g_2(x)}{\mathrm{d}x}=6\frac{x(x^2+4)\sqrt{x^2+8}-x^4-8x^2+24}{\sqrt{x^2+8}[3(x^2+1)\sqrt{x^2+8}+x^3+9x]^2}>0.$$

Here we used that the function $q:[0,\infty)\to\mathbb{R}$, defined by

$$q(x)=x^4+8x^2-24-x\bigl(x^2+4\bigr)\sqrt{x^2+8},$$

is strictly decreasing, i.e. for all $x\geqslant 0$ we have

$$\frac{\mathrm{d}q(x)}{\mathrm{d}x}=4\frac{x(x^2+4)\sqrt{x^2+8}-x^4-8x^2-8}{\sqrt{x^2+8}}<0.$$

Consequently $q(x)\leqslant q(0)=-24<0$, and hence the required monotonicity follows. Finally, using that g_1 is strictly decreasing and g_2 is strictly increasing from Lemma 2.2 it follows that $h_1(x)<r(x)<h_2(x)$ for all $x\geqslant 0$. Thus the proof is complete. □

Another application of Lemma 2.1 is the following result.

Theorem 2.5. *The following assertions are true*:

(a) *The Mills' ratio r is strictly log-convex on $\mathbb{R}$.*
(b) *The function $x\mapsto xr'(x)/r(x)$ is strictly decreasing on $(0,\infty)$.*
(c) *The function $x\mapsto xr'(x)$ is strictly decreasing on $(0,x_0)$ and is strictly increasing on (x_0,∞), where $x_0\approx 1.161527889\ldots$ is the unique positive root of the transcendent equation $x(x^2+2)\overline{\Phi}(x)=(x^2+1)\varphi(x)$.*
(d) *The function $x\mapsto x^2r'(x)$ is strictly decreasing on $(0,\infty)$.*

Proof. (a) Due to M.R. Sampford [27] we know that for all $x\in\mathbb{R}$ we have $(1/r(x))'<1$. Using the derivative formula $r'(x)=xr(x)-1$ we get $(r'(x)/r(x))'=1-(1/r(x))'>0$, i.e. the Mills' ratio r is strictly log-convex on $\mathbb{R}$. It is worth mentioning here that this above result can be derived too using Lemma 2.1. For this let us write the quotient $r'(x)/r(x)$ as follows

$$\frac{r'(x)}{r(x)}=\frac{x\overline{\Phi}(x)-\varphi(x)}{\overline{\Phi}(x)}.$$

Since $\lim_{x\to\infty}[x\overline{\Phi}(x)-\varphi(x)]=\lim_{x\to\infty}\overline{\Phi}(x)=0$, using Lemma 2.1 it is enough to show that

$$x\mapsto\frac{[x\overline{\Phi}(x)-\varphi(x)]'}{[\overline{\Phi}(x)]'}=-\frac{\overline{\Phi}(x)}{\varphi(x)}=-r(x)$$

is strictly increasing, which is clearly true.

(b) To prove the required result let us write the quotient $xr'(x)/r(x)$ as follows

$$\frac{xr'(x)}{r(x)}=\frac{x^2\overline{\Phi}(x)-x\varphi(x)}{\overline{\Phi}(x)}.$$

Clearly we have $\lim_{x\to\infty}[x^2\overline{\Phi}(x)-x\varphi(x)]=\lim_{x\to\infty}\overline{\Phi}(x)=0$, and the function

$$x\mapsto\frac{[x^2\overline{\Phi}(x)-x\varphi(x)]'}{[\overline{\Phi}(x)]'}=\frac{\varphi(x)-2x\overline{\Phi}(x)}{\varphi(x)}=1-2xr(x)$$

is strictly decreasing on $(0,\infty)$, because from Lemma 2.1 we have that [22, Proposition 1.2] the function $x\mapsto xr(x)$ is strictly increasing on $(0,\infty)$. In view of Lemma 2.1 this shows that the function $x\mapsto xr'(x)/r(x)$ is strictly decreasing on $(0,\infty)$.

(c) To prove the required result we need to use instead of Lemma 2.1 a more general monotone form of l'Hospital's rule due to I. Pinelis [25, Theorem 1.16], please see also the recent work of I. Pinelis [20]. For this let us write the product $xr'(x)$ as follows:

$$s_1(x):=xr'(x)=\frac{xr(x)-1}{1/x}=\frac{\overline{\Phi}(x)-\varphi(x)/x}{\varphi(x)/x^2}.$$

Since for all $x > 0$ we have

$$\frac{\varphi(x)}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\varphi(x)}{x^2}\right] = -\frac{1}{x^3}\left(1+\frac{2}{x^2}\right)\varphi^2(x) < 0,$$

to prove that $x \mapsto xr'(x)$ is strictly decreasing on $(0, x_0)$ and is strictly increasing on (x_0, ∞), in view of [25, Remark 1.17] and [25, Theorem 1.16], it suffices to show that the function

$$x \mapsto s_2(x) := \frac{[\overline{\Phi}(x) - \varphi(x)/x]'}{[\varphi(x)/x^2]'} = -\frac{x}{x^2+2}$$

is strictly decreasing on $(0, \sqrt{2})$ and is strictly increasing on $(\sqrt{2}, \infty)$, which is clearly true. With other words we proved that the waves of s_1 follows the waves of s_2. We note that the behaviour of the roots, i.e. the inequality $x_0 < \sqrt{2}$ is necessary, because in view of [25, Remark 1.15] s_1 may switch from decrease to increase only on intervals of decrease of s_2.

(d) Let us write the product $x^2 r'(x)$ as follows

$$x^2 r'(x) = \frac{xr(x) - 1}{1/x^2} = \frac{\overline{\Phi}(x) - \varphi(x)/x}{\varphi(x)/x^3}.$$

Since $\lim_{x\to\infty}[\overline{\Phi}(x) - \varphi(x)/x] = \lim_{x\to\infty}\varphi(x)/x^3 = 0$, from monotone form of l'Hospital rule, i.e. Lemma 2.1 it is enough to show that

$$x \mapsto \frac{[\overline{\Phi}(x) - \varphi(x)/x]'}{[\varphi(x)/x^3]'} = -\frac{x^4}{x^4+3x^2}$$

is strictly decreasing on $(0, \infty)$, which is true. With this the proof is complete. □

An immediate application of Theorem 2.5 is the following result.

Corollary 2.6. *If $x, y > x_0$, then the following chain of inequalities holds*

$$\frac{2r(x)r(y)}{r(x)+r(y)} \leqslant r\left(\frac{x+y}{2}\right) \leqslant \sqrt{r(x)r(y)} \leqslant r(\sqrt{xy}) \leqslant \frac{r(x)+r(y)}{2} \leqslant r\left(\frac{2xy}{x+y}\right). \tag{2.7}$$

Moreover, the first, second, third and fifth inequalities hold for all x, y strict positive real numbers, while the fourth inequality is reversed if $x, y \in (0, x_0)$. In each of the above inequalities equality holds if and only if $x = y$.

Proof. Observe that the first inequality in (2.7) follows from the strict convexity of the failure rate $1/r$ on $(0, \infty)$, which was proved by M.R. Sampford [27]. For the second inequality in (2.7) we use that from part (a) of Theorem 2.5 the function r is strictly log-convex on $(0, \infty)$. To prove the third inequality recall that from part (b) of Theorem 2.5 we know that the function $x \mapsto xr'(x)/r(x)$ is strictly decreasing on $(0, \infty)$. This implies that the Mills' ratio is strictly multiplicatively concave on $(0, \infty)$, i.e. for all $x, y > 0$ and $\lambda \in (0, 1)$, the inequality $r(x^\lambda y^{1-\lambda}) \geqslant [r(x)]^\lambda [r(y)]^{1-\lambda}$ holds. Here we used part 5 of Corollary 2.5 due to G.D. Anderson, M.K. Vamanamurthy and M. Vuorinen [1]. We use again Corollary 2.5 from [1], namely part 4, in order to prove the fourth inequality in (2.7). More precisely if the function $x \mapsto xr'(x)$ is strictly increasing (decreasing respectively) then we have that the fourth inequality in (2.7) (and its reverse respectively) holds. Now from part (c) of Theorem 2.5 the asserted monotonicity follows. Finally, observe that for the last inequality in (2.7) it is enough to show that $x \mapsto r(1/x)$ is strictly concave, i.e. using part 7 of Corollary 2.5 from [1] the function $x \mapsto x^2 r'(x)$ is strictly decreasing, which is proved in part (d) of Theorem 2.5.

From the strict convexity of $1/r$, strict log-convexity, strict multiplicatively concavity of r, strict monotonicity of $x \mapsto xr'(x)$ and strict concavity of $x \mapsto r(1/x)$ we deduce that equality holds in (2.7) if and only if $x = y$. □

3. Complete monotonicity of Mills' ratio

Let $r_0(x) := 0$ for all $x > 0$. Further for all $n \geqslant 1$ and $x > 0$ let us consider the nth initial segment of the Laplace continuous fraction for Mills' ratio

$$r_n(x) := \cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{\ddots\, x + \cfrac{n-1}{x}}}}}.$$

In [22] I. Pinelis using the monotone form of l'Hospital rule investigated monotonicity properties of the relative error $\delta_n(x) := (r(x) - r_n(x))/r(x)$ and among other things proved that [22, Theorem 1.5] for all $n \geqslant 0$ and all $x > 0$ we have $(-1)^n \delta_n > 0$. Observe that this inequality is actually equivalent with the following result of L.R. Shenton [28, Eq. (19)]

$$r_{2n}(x) < r(x) < r_{2n+1}(x), \tag{3.1}$$

where $n \geqslant 1$ and $x > 0$. In what follows we show that the inequality $(-1)^n \delta_n > 0$, or (3.1) is in fact equivalent with the strict complete monotonicity of Mills' ratio on $(0, \infty)$. Moreover, we show that Mills' ratio is strictly completely monotonic on $\mathbb{R}$, and we obtain some Turán-type inequalities for nth derivative of Mills' ratio. Before we state our last main result of this paper let us recall the following inequality for Legendre polynomials

$$P_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\frac{(x^2-1)^n}{n!2^n}\right],$$

that is

$$\left[P_{n+1}(x)\right]^2 \geqslant P_n(x) P_{n+2}(x)$$

which holds for all $x \in [-1, 1]$ and $n = 0, 1, 2, \ldots$. This inequality is known in literature as Turán's inequality [30] and has been extended in many directions for various orthogonal polynomials and special functions. For more information about this the interested reader is referred to the papers of the author [5,6] as well as to the work of A. Laforgia and P. Natalini [18], where Schwarz inequality (3.7) is used among other things to prove some interesting reversed Turán-type inequalities for generalized elliptic integrals, modified Bessel functions of the first kind, confluent hypergeometric functions, polygamma function and Riemann zeta function.

Theorem 3.2. *Let x be a real number and $n \geqslant 1$ a natural number. Further let $r^{(n)}$ be the nth derivative of Mills' ratio and let us consider the expression $\Delta_n = (-1)^n r^{(n)}(x)$. Then the Mills' ratio is strictly completely monotonic on $\mathbb{R}$, i.e. $\Delta_n > 0$, and satisfies the following reversed Turán-type inequality*

$$\Delta_n \cdot \Delta_{n+2} \geqslant [\Delta_{n+1}]^2. \tag{3.3}$$

Moreover the function $x \mapsto |r^{(n)}(x)|$ is strictly log-convex on $\mathbb{R}$ and consequently in view of (3.3) *the following Turán-type inequalities hold*:

$$\Delta_{2n-1} \cdot \Delta_{2n+1} \geqslant [\Delta_{2n}]^2 \geqslant \frac{2n}{2n+1} \Delta_{2n-1} \cdot \Delta_{2n+1}. \tag{3.4}$$

Proof. Let us consider the following integral:

$$\gamma_n(x) := \int_x^\infty (x-t)^n \varphi(t)\, \mathrm{d}t = (-1)^n \int_x^\infty (t-x)^n \varphi(t)\, \mathrm{d}t, \quad x \in \mathbb{R},\ n \geqslant 1. \tag{3.5}$$

Then it is known that [28, Eq. (21)] $r^{(n)}(x) = \gamma_n(x)\varphi(x)$, and hence

$$0 < \Delta_n = (-1)^n r^{(n)}(x) = (-1)^n \gamma_n(x)\varphi(x) = \varphi(x) \int_x^\infty (t-x)^n \varphi(t)\,\mathrm{d}t.$$

We note here that actually

$$r(x) = e^{x^2/2} \int_x^\infty e^{-t^2/2}\,\mathrm{d}t = \int_0^\infty e^{-xt} e^{-t^2/2}\,\mathrm{d}t, \tag{3.6}$$

i.e. Mills' ratio r is the Laplace transform of the function $t \mapsto e^{-t^2/2}$. Thus using the classical Bernstein theorem [11], which characterizes completely monotonic functions as Laplace transforms of positive measures whose support is contained in $[0,\infty)$, we conclude that r is indeed strictly completely monotonic.

Now consider the following form of the Schwarz inequality:

$$\int_a^b f(t)\big[g(t)\big]^\alpha\,\mathrm{d}t \cdot \int_a^b f(t)\big[g(t)\big]^\beta\,\mathrm{d}t \geqslant \left[\int_a^b f(t)\big[g(t)\big]^{\frac{\alpha+\beta}{2}}\,\mathrm{d}t\right]^2, \tag{3.7}$$

where f and g are two nonnegative functions of a real variable defined on $[a,b]$, α and β are real numbers such that the integrals exist. Then taking in (3.7) $\alpha = n$, $\beta = n+2$, $a = x$, $b = \infty$, and for all $x \in \mathbb{R}$ fixed, $f(t) = \varphi(t)$ and $g(t) = t - x$, where $t \in [x,\infty]$, we obtain that

$$\int_x^\infty \varphi(t)(t-x)^n\,\mathrm{d}t \cdot \int_x^\infty \varphi(t)(t-x)^{n+2}\,\mathrm{d}t \geqslant \left[\int_x^\infty \varphi(t)(t-x)^{n+1}\,\mathrm{d}t\right]^2$$

holds, which is equivalent with inequality (3.3). Now changing n with $2n-1$ in (3.3) immediately we get the first inequality in (3.4). Observe that to prove that the function $x \mapsto |r^{(n)}(x)|$ is log-convex on $\mathbb{R}$ it is enough to show that $\Delta_n \cdot \Delta_{n+2} \geqslant [\Delta_{n+1}]^2$ holds, which is exactly (3.3). Now for the strict log-convexity of the function $x \mapsto |r^{(n)}(x)|$ observe that using mathematical induction from (3.6) we have

$$\big|r^{(n)}(x)\big| = (-1)^n r^{(n)}(x) = \int_0^\infty e^{-xt} t^n e^{-t^2/2}\,\mathrm{d}t.$$

Let us observe that the integrand in the above integral is log-convex in x. Thus $x \mapsto |r^{(n)}(x)|$ is strictly log-convex by Theorem B-6 in [9, p. 296].

On the other hand from (3.5) easily follows that for all $n \geqslant 1$ we have $\gamma_n' = n\gamma_{n-1}$. Thus from the log-convexity of $x \mapsto r^{(2n)}(x)$ we obtain that

$$\begin{aligned} 0 &\leqslant \frac{\mathrm{d}^2}{\mathrm{d}x^2}\big[\log r^{(n)}(x)\big] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{r^{(2n+1)}(x)}{r^{(2n)}(x)}\right] = \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\gamma_{2n+1}(x)}{\gamma_{2n}(x)}\right] \\ &= \frac{(2n+1)[\gamma_{2n}(x)]^2}{[\gamma_{2n}(x)]^2} - \frac{(2n)\gamma_{2n-1}(x)\gamma_{2n+1}(x)}{[\gamma_{2n}(x)]^2}, \end{aligned}$$

i.e. the second inequality in (3.4) holds. Thus the proof is complete. □

Concluding remarks. 1. Consider the following polynomial expressions f_n and g_n defined by the following formulas:

$$g_0(x) = 1; \quad f_0(x) = 0; \quad g_1(x) = x; \quad f_1(x) = 1;$$
$$f_n(x) = xf_{n-1}(x) + (n-1)f_{n-2}(x) \quad \text{and} \quad g_n(x) = xg_{n-1}(x) + (n-1)g_{n-2}(x),$$

where $x \in \mathbb{R}$, $n \geqslant 2$. It is easy to verify that [22, Lemma 2.2] for all $n \geqslant 1$ and $x \in \mathbb{R}$

$$f_n'(x) = xf_n(x) + nf_{n-1}(x) - g_n(x) \quad \text{and} \quad g_n'(x) = ng_{n-1}(x). \tag{3.8}$$

Due to L.R. Shenton [28, Eq. (17)] it is known that $r_n(x) = f_n(x)/g_n(x)$, moreover from general properties of continuous fractions [28, Eq. (19)] we have that (3.1) holds. As we have seen in Theorem 2.3 some known approximations for Mills' ratio can be deduced easily using the monotone form of l'Hospital rule. We note that with the second inequality in (3.1) the situation is the same. In what follows we would like to present an alternative proof of this inequality using Lemma 2.2. For this let us consider the function $h_u : (0,\infty) \to (0,\infty)$, defined by

$$h_u(x) := r_{2n+1}(x) = \frac{f_{2n+1}(x)}{g_{2n+1}(x)},$$

where $n \geqslant 1$. Since for all $n \geqslant 1$ and $x \in \mathbb{R}$ we have [28, Eq. (18)]

$$f_n(x)g_{n-1}(x) - g_n(x)f_{n-1}(x) = (-1)^{n-1}(n-1)!,$$

in view of (3.8) it is easy to verify that

$$\big[h_u(x)\varphi(x)\big]' = \varphi(x)\big[h_u'(x) - xh_u(x)\big] = -\varphi(x)\left[\frac{(2n+1)!}{[g_{2n+1}(x)]^2} + 1\right] < 0,$$

i.e. for all $x > 0$ the product $h_u(x)\varphi(x)$ is not constant and $[h_u\varphi]'$ does not change sign, further $\lim_{x\to\infty} h_u(x)\varphi(x) = 0$. Now consider the function $g_u : (0,\infty) \to (0,\infty)$, defined by

$$g_u(x) = \frac{1}{xh_u(x) - h_u'(x)} = \frac{[g_{2n+1}(x)]^2}{(2n+1)! + [g_{2n+1}(x)]^2}.$$

Then the function g_u has the limit 1 at infinity and simple computations shows that

$$g_u'(x) = \frac{2(2n+1)(2n+1)!g_{2n}(x)g_{2n+1}(x)}{[(2n+1)! + [g_{2n+1}(x)]^2]^2} > 0,$$

thus from Lemma 2.2 we have that $r(x) < h_u(x) = r_{2n+1}(x)$.

2. A positive function f is called strictly logarithmically completely monotonic [26] on an interval I if f has derivatives of all orders on I and its logarithm $\log f$ satisfies $(-1)^n[\log f(x)]^{(n)} > 0$, for all $x \in I$ and $n \geqslant 1$. Note that a strictly logarithmically completely monotonic function is always strictly completely monotonic, but not conversely. Recently, C. Berg [7] pointed out that the logarithmically complete monotonic functions in fact are the same as those studied by R.A. Horn [15] under the name infinitely divisible completely monotonic functions. It is worth mentioning here that using the results of M.R. Sampford [27], i.e. inequalities $(1/r(x))' < 1$ and $(1/r(x))'' > 0$, it is easy to verify that for all $x \in \mathbb{R}$ we have $(-1)^n[\log r(x)]^{(n)} > 0$, where $n = 1, 2, 3$. All the same, as C. Berg pointed out in private communication, the Mills' ratio is not logarithmically completely monotonic, because in [29, p. 126] it is proved that the "half-normal density" is not infinitely divisible and this means that the Mills' ratio r is not logarithmically completely monotonic.

Finally, we note that after this manuscript had been completed we found on the preprint server www.arxiv.org the paper of O. Kouba [17], where the complete monotonicity of r and the strict log-convexity of $x \mapsto |r^{(n)}(x)|$ were proved among other things using a different approach.

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