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AN ABEL-TAUBER THEOREM FOR HANKEL TRANSFORMS

N. H. BINGHAM and A. INOUE

§1. Abel-Tauber theorems

We write R_0 for the class of slowly varying functions at infinity, that is, the class of positive measurable ℓ, defined on some neighbourhood of infinity, satisfying

$$\ell(\lambda x)/\ell(x) \to 1 \qquad (x \to \infty) \qquad \forall \lambda > 0.$$

For $\ell \in R_0$, the class Π_ℓ is the class of measurable f satisfying

$$\{f(\lambda x) - f(x)\}/\ell(x) \to c \log \lambda \qquad (x \to \infty) \qquad \forall \lambda > 0$$

for some constant c, called the *ℓ-index* of f. For details, we refer to [BGT].

Let $\nu \geq -1/2$, $t^{\nu+\frac{1}{2}} f(t) \in L^1_{loc}[0, \infty)$, and let f be ultimately decreasing to zero at infinity (this plays the role of our Tauberian condition; see [BI1], §2 for discussion of the possibility of weakening it). We consider the *Hankel transform*

$$F_\nu(x) := \int_0^{\infty-} f(t)(xt)^{\frac{1}{2}} J_\nu(xt)dt \qquad (x > 0), \tag{1.1}$$

where $\int_0^{\infty-}$ denotes an improper integral $\lim_{M\to\infty} \int_0^M$ and J_ν is the Bessel function

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu + n + 1)} \left(\frac{x}{2}\right)^{2n} \qquad (0 \leq x < \infty).$$

Since the improper integral converges uniformly on each (a, ∞) with $a > 0$, F_ν is finite and continuous on $(0, \infty)$.

Here is our main theorem:

THEOREM 1.1. *Let $\ell \in R_0$, and let ν, f and F_ν be as above. We set $\bar{F}_\nu(x) := x^{\nu+\frac{1}{2}} F_\nu(1/x)$ for $x > 0$. Then*

$$f(t) \sim t^{-(\nu+\frac{3}{2})} \ell(t) \qquad (t \to \infty) \tag{1.2}$$

if and only if

$$\bar{F}_\nu \in \Pi_\ell \quad \text{with} \quad \ell-\text{index} \quad \frac{1}{2^\nu \Gamma(\nu + 1)}. \tag{1.3}$$

Both imply

$$\{\bar{F}_\nu(x) - \frac{1}{2^\nu\Gamma(\nu+1)}\int_0^x t^{\nu+\frac{1}{2}} f(t)dt\}/(x^{\nu+\frac{3}{2}} f(x))$$

$$\to \frac{1}{2^\nu\Gamma(\nu+1)}\{\log 2 - \frac{\gamma}{2} + \frac{1}{2}\psi(\nu+1)\} \qquad (x\to\infty), \tag{1.4}$$

where γ is Euler's constant and ψ is the digamma function: $\psi(x) := \Gamma'(x)/\Gamma(x)$ for $x > 0$.

The theorem above treats the boundary case of the following known Abel-Tauber theorem for Hankel transforms:

THEOREM 1.2 ([B1], [RS]). *Let ν, f, F_ν and ℓ be as in Theorem 1.1. Then for $\rho \in (-\nu - \frac{3}{2}, 0)$,*

$$f(t) \sim t^\rho \ell(t) \qquad (t\to\infty) \tag{1.5}$$

if and only if

$$F_\nu(x) \sim x^{-\rho-1}\ell(1/x)\cdot 2^{\rho+\frac{1}{2}}\frac{\Gamma(\frac{3}{4}+\frac{\nu}{2}+\frac{\rho}{2})}{\Gamma(\frac{1}{4}+\frac{\nu}{2}-\frac{\rho}{2})} \qquad (x\to 0+).$$

See also [SS, pp. 617-619].

The point of Theorem 1.1 is that one obtains Π-variation rather than regular variation - or in the terminology of [BGT], one uses de Haan theory rather than Karamata theory - in the boundary case $\rho = -\nu - \frac{3}{2}$. The cosine case $\nu = -1/2$ and sine case $\nu = 1/2$ of Theorem 1.1 were recently proved in [I1] and [I2] respectively. However, the proofs of [I1] and [I2] do not apply to general Hankel transforms directly. Our proof of Theorem 1.1 in §2 is closer to those of [B2, Th. 3b] and [BGT, Th. 4.10.1] than those of [I1], [I2].

§2. Proof of Theorem 1.1.

Step 1. Choose X so large that f is positive and non-increasing on $[X,\infty)$. We first show that we lose no generality by supposing that f vanishes on $[0, X)$.

Set $\tilde{f}(t) := I_{[X,\infty)}(t)f(t)$, and let $\tilde{F}_\nu$ be its Hankel transform:

$$\tilde{F}_\nu(x) := \int_0^{\infty-} \tilde{f}(t)(xt)^{1/2}J_\nu(xt)dt \qquad (x>0).$$

By the mean-value theorem, there exists $c_1 \in (0,\infty)$ such that

$$|x^{-\nu}J_\nu(x) - y^{-\nu}J_\nu(y)| \le c_1|x-y| \qquad (0\le x,y\le 1).$$

So for $\lambda > 1$,

$$\frac{1}{\ell(x)}|\{\bar{F}_\nu(\lambda x) - \bar{F}_\nu(x)\} - \{(\lambda x)^{\nu+\frac{1}{2}}\tilde{F}_\nu(1/\lambda x) - x^{\nu+\frac{1}{2}}\tilde{F}_\nu(1/x)\}|$$

$$= \frac{1}{\ell(x)} \left| \int_0^X t^{\nu+\frac{1}{2}} f(t)\{(t/\lambda x)^{-\nu} J_\nu(t/\lambda x) \; - (t/x)^{-\nu} J_\nu(t/x)\} dt \right|$$

$$\leq c_1 \frac{(1-\lambda^{-1})}{x\ell(x)} \cdot X \cdot \int_0^X t^{\nu+\frac{1}{2}} |f(t)| dt \to 0 \qquad (x \to \infty).$$

So (1.3) holds if and only if $x^{\nu+\frac{1}{2}} \tilde{F}_\nu(1/x) \in \Pi_\ell$ with ℓ-index $1/\{2^\nu \Gamma(\nu+1)\}$.

On the other hand, there exists $c_2 \in (0,,\infty)$ such that

$$|x^{-\nu} J_\nu(x) - \frac{1}{2^\nu \Gamma(\nu+1)}| \leq c_2 x^2 \qquad (0 \leq x \leq 1). \tag{2.1}$$

So for $x > X$,

$$\frac{1}{x^{\nu+\frac{3}{2}} f(x)} |x^{\nu+\frac{3}{2}} \int_0^X f(t)(t/x)^{1/2} J_\nu(t/x) dt - \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^X f(t) t^{\nu+\frac{1}{2}} dt|$$

$$\leq \frac{1}{x^{\nu+\frac{3}{2}} f(x)} \int_0^X t^{\nu+\frac{1}{2}} |f(t)| \cdot |(t/x)^{-\nu} J_\nu(t/x) - \frac{1}{2^\nu \Gamma(\nu+1)}| dt$$

$$\leq \frac{c_2 X^2}{x^{\nu+\frac{3}{2}+2} f(x)} \int_0^X t^{\nu+\frac{1}{2}} |f(t)| dt \to 0 \qquad (x \to \infty),$$

under (1.2). Thus when proving that (1.2) is equivalent to (1.3) and implies (1.4), we may replace f by $\tilde{f}$ - that is, we may assume that f vanishes on $[0, X)$.

Step 2: Abelian part. First we assume (1.2) and show (1.4) and (1.3). For $x > 0$,

$$\frac{1}{x^{\nu+\frac{3}{2}} f(x)} \{\bar{F}_\nu(x) - \frac{1}{2^\nu \Gamma(\nu+1)} \int_0^x f(t) t^{\nu+\frac{1}{2}} dt\} = I(x) + II(x), \tag{2.2}$$

where

$$I(x) := \int_0^1 \frac{(xu)^{\nu+2} f(xu)}{x^{\nu+2} f(x)} \{u^{1/2} J_\nu(u) - \frac{u^{\nu+\frac{1}{2}}}{2^\nu \Gamma(\nu+1)}\} \frac{du}{u^{\nu+2}},$$

$$II(x) := \int_1^{\infty-} \frac{f(xu)}{f(x)} u^{1/2} J_\nu(u) du.$$

By the uniform convergence theorem for regularly varying functions ([BGT, Th. 1.5.2]), $(xu)^{\nu+2} f(xu)/x^{\nu+2} f(x)$ converges to $u^{1/2}$ as $x \to \infty$ uniformly in $u \in (0, 1]$, whence using (2.1) we find that

$$I(x) \to \int_0^1 \frac{1}{u^{\nu+\frac{3}{2}}} \{u^{1/2} J_\nu(u) - \frac{u^{\nu+\frac{1}{2}}}{2^\nu \Gamma(\nu+1)}\} du \qquad (x \to \infty).$$

In the same way, for any $Y > X$,

$$\int_1^Y \frac{f(xu)}{f(x)} u^{1/2} J_\nu(u) du \to \int_1^Y \frac{1}{u^{\nu+\frac{3}{2}}} \cdot u^{\frac{1}{2}} J_\nu(u) du \qquad (x \to \infty).$$

By the second integral mean-value theorem ([WW, §4.14]), for $x > 1$,

$$\int_Y^{\infty-} \frac{f(xu)}{f(x)} u^{1/2} J_\nu(u) du = \frac{f(xY+)}{f(x)} \int_Y^{Y'} u^{1/2} J_\nu(u) du$$

for some $Y' \in (Y, \infty)$. If we set

$$c_3 := \sup\{| \int_x^y u^{1/2} J_\nu(u) du| : 0 < x < y < \infty\},$$

then

$$\limsup_{x\to\infty} | \int_Y^{\infty-} \frac{f(xu)}{f(x)} u^{1/2} J_\nu(u) du| \le \limsup_{x\to\infty} \frac{f(xY)}{f(x)} . c_3 = c_3 / Y^{\nu+\frac{3}{2}},$$

which can be made arbitrarily small by choosing Y large enough. So

$$II(x) \to \int_1^\infty \frac{1}{u^{\nu+\frac{3}{2}}} \cdot u^{1/2} J_\nu(u) du \qquad (x \to \infty).$$

The sum of the limits of $I(x)$ and $II(x)$ is identified in the Appendix by classical results on special functions, and we obtain (1.4).

We can prove the implication from (1.4) to (1.3) by a standard argument (see e.g. [I1, p. 767]).

Step 3: Tauberian part. We now prove the implication from (1.3) to (1.2). By a formula of Gegenbauer,

$$\int_0^\infty t^{\nu+\frac{1}{2}} e^{-xt} (yt)^{1/2} J_\nu(yt) dt = d_\nu \cdot \frac{xy^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} \qquad (x > 0),$$

where

$$d_\nu := \pi^{-1/2} 2^{\nu+1} \Gamma(\nu + \frac{3}{2})$$

([WW, §13.2 (5)]; cf. [BI1, §2]). So by Parseval's formula for Hankel transforms ([RS]; cf. [BI1, §1]), for $x > 0$

$$\int_0^\infty f(t) t^{\nu+\frac{1}{2}} e^{-xt} dt = d_\nu \int_{0+}^{\infty-} F_\nu(y) \frac{xy^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy \qquad (2.3)$$

(for $\int_{0+}$ on the right, recall the assumption at the end of Step 1). By the second integral mean-value theorem,

$$|F_\nu(y)| \le c_3 f(X)/y \qquad (0 < y < \infty), \qquad (2.4)$$

whence

$$\int_1^\infty |F_\nu(y)| \cdot \frac{y^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy < \infty.$$

On the other hand, by [BGT, Th. 3.7.4], (1.3) implies $|\bar{F}_\nu| \in R_0$, whence

$$\int_0^1 |F_\nu(y)| \cdot \frac{y^{\nu+\frac{1}{2}}}{(x^2+y^2)^{\nu+\frac{3}{2}}} dy = \int_1^\infty \frac{|\bar{F}_\nu(u)|}{(x^2u^2+1)^{\nu+\frac{3}{2}}} du < \infty.$$

Thus the integral on the right of (2.3) converges absolutely - and so the results of [BGT Ch. 4] apply.

We use Laplace transforms. Write

$$U(t) := \int_0^t f(u) u^{\nu+\frac{1}{2}} du \qquad (t \geq 0),$$

$$\hat{U}(x) := \int_{[0,\infty)} e^{-xt} dU(t) = \int_0^\infty f(t) t^{\nu+\frac{1}{2}} e^{-xt} dt \qquad (x > 0).$$

Then by (2.3),

$$\hat{U}(1/x) = (k * \bar{F}_\nu)(x) \qquad (x > 0),$$

where

$$k(x) := d_\nu \frac{x^{2\nu+2}}{(1+x^2)^{\nu+\frac{3}{2}}} \qquad (x > 0)$$

and $k * \bar{F}_\nu$ denotes the Mellin convolution

$$(k * \bar{F}_\nu)(x) = \int_0^\infty k(x/t) \bar{F}_\nu(t) dt/t.$$

The absolute convergence strip of the Mellin transform

$$\check{k}(z) := \int_0^\infty t^{-z} k(t) dt/t$$

is $-1 < \Re z < 2\nu + 2$, and for z in the strip

$$\check{k}(z) = \frac{2^\nu}{\pi^{1/2}} \Gamma(\nu + 1 - \frac{z}{2}) \Gamma(\frac{1}{2} + \frac{z}{2}),$$

in particular,

$$\check{k}(0) = 2^\nu \Gamma(\nu + 1).$$

By (2.4), $\bar{F}_\nu$ is locally bounded on $[0, \infty)$. So by the argument of [BGT, p. 242] (Abelian theorem for differences), we find that (1.3) implies $k * \bar{F}_\nu \in \Pi_\ell$ with ℓ-index 1 - i.e., so is $\hat{U}(1/.)$. So by a Tauberian theorem of de Haan (cf. [BGT, Th. 3.9.1]), we see that $U \in \Pi_\ell$ with ℓ-index 1. Finally, by [BGT, Th. 3.6.10] (with slow decrease replaced by slow increase), we obtain (1.2), completing the proof. •

§3. Mercerian theorems.

For positive f, the *upper order* $\rho(f)$ is defined by

$$\rho(f) := \limsup_{x \to \infty} \frac{\log f(x)}{\log x}.$$

In [BI1, BI2] we recently proved the following Mercerian counterpart to Theorem 1.2:

THEOREM 3.1. *Let ν, f, F_ν be as in Theorem 1.1. We write $\rho := \rho(f)$. If*

$$-\nu - \frac{3}{2} < \rho < 0, \qquad \rho \neq -\frac{1}{2}$$

and

$$x^{-1}F_\nu(1/x)/f(x) \to c > 0 \qquad (x \to \infty),$$

then $c = 2^{\rho+\frac{1}{2}}\Gamma(\frac{3}{4}+\frac{\nu}{2}+\frac{\rho}{2})/\Gamma(\frac{1}{4}+\frac{\nu}{2}-\frac{\rho}{2})$ and $f \in R_\rho$: f varies regularly with index ρ.

The question arises of proving the Mercerian counterpart of Theorem 1.1, that is, the implication from (1.4) to (1.2), under mild assumptions. We leave this question open here, but we think that the techniques of [BI1, BI2] will be helpful in this problem too. For corresponding results for Laplace transforms, see Embrechts [E] and [BGT, Th. 5.4.1].

Appendix

We complete the proof above by showing

$$\int_0^1 \frac{1}{u^{\nu+\frac{3}{2}}}\{u^{1/2}J_\nu(u) - \frac{u^{\nu+\frac{1}{2}}}{2^\nu\Gamma(\nu+1)}\}du + \int_1^\infty \frac{1}{u^{\nu+\frac{3}{2}}}u^{1/2}J_\nu(u)du$$

$$= \frac{1}{2^\nu\Gamma(\nu+1)}\{\log 2 - \frac{\gamma}{2} + \frac{\psi(\nu+1)}{2}\}. \qquad (A1)$$

By Weber's integral ([W, §13.24 (1); cf. [BI1, §1]) for the Mellin transform of the Bessel function, for $x \in (-\nu-\frac{3}{2}, \nu+\frac{1}{2})$,

$$\int_0^1 u^x\{u^{1/2}J_\nu(u) - \frac{u^{\nu+\frac{1}{2}}}{2^\nu\Gamma(\nu+1)}\}du + \int_1^\infty u^x.u^{1/2}J_\nu(u)du$$

$$= \int_0^\infty u^{-x}\cdot u^{-3/2}J_\nu(1/u)du/u - \frac{1}{2^\nu\Gamma(\nu+1)}\int_0^1 u^{x+\nu+\frac{1}{2}}du$$

$$= \frac{1}{2^\nu\Gamma(\nu+1)(x+\nu+\frac{3}{2})}\{g_1(x+\nu+\frac{3}{2}) - 1\}, \qquad (A2)$$

where

$$g_1(x) := \Gamma(\nu+1)\frac{2^x\Gamma(\frac{x}{2}+1)}{\Gamma(-\frac{x}{2}+\nu+1)}.$$

We have $g_1(0) = 1$. For $x \in (-2, 2\nu+2)$, we write

$$g_2(x) := \log g_1(x) = x\log 2 + \log\Gamma(\nu+1) + \log\Gamma(1+\frac{x}{2}) - \log\Gamma(-\frac{x}{2}+\nu+1).$$

Then

$$g_1'(x)/g_1(x) = g_2'(x) = \log 2 + \frac{1}{2}\psi(1+\frac{x}{2}) + \frac{1}{2}\psi(-\frac{x}{2}+\nu+1),$$

and so

$$g_1'(0) = g_2'(0) = \log 2 - \frac{\gamma}{2} + \frac{1}{2}\psi(\nu + 1).$$

Letting $x \downarrow -\nu - \frac{3}{2}$ in (A2), we obtain (A1), as required.

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