L^p–BOUNDS FOR THE RIESZ TRANSFORMS ASSOCIATED WITH THE HODGE LAPLACIAN IN LIPSCHITZ SUBDOMAINS OF RIEMANNIAN MANIFOLDS

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ABSTRACT. We prove L^p-bounds for the Riesz transforms $d/\sqrt{-\Delta}$, $\delta/\sqrt{-\Delta}$ associated with the Hodge-Laplacian $\Delta = -\delta d - d\delta$ equipped with absolute and relative boundary conditions in a Lipschitz subdomain Ω of a (smooth) Riemannian manifold $\mathcal{M}$, for p in a certain interval depending on the Lipschitz character of the domain.

1. INTRODUCTION

Let $\mathcal{M}$ be a smooth, compact, boundaryless manifold, of real dimension n. Assume that this is equipped with a sufficiently smooth Riemannian metric tensor, so that, in particular, a geodesic ball B has volume $|B|$ comparable the $n-$th power of its radius r_B, i.e., we have the "Ahlfors-David" type regularity condition

$$|B| \approx r_B^n, \tag{1.1}$$

where the implicit constants depend only on intrinsic properties of $\mathcal{M}$ and not on B or r_B. Let Ω be a Lipschitz subdomain of $\mathcal{M}$ and denote by ν its outward unit normal (canonically identified with a 1-form). In this paper, $L^p(\Omega;\Lambda^\ell)$ stands for the space of $\ell-$differential forms with $p-$th power integrable coefficients in Ω. Following [19], we define B_ℓ and C_ℓ as unbounded operators in $L^2(\Omega;\Lambda^\ell)$, $\ell = 0, 1, ..., n$, by setting

$$\begin{aligned} D(B_\ell) &:= \Big\{u \in L^2(\Omega;\Lambda^\ell) : \delta u \in L^2(\Omega;\Lambda^{\ell-1}),\, du \in L^2(\Omega;\Lambda^{\ell+1}),\, \nu \vee u = 0 \text{ on } \partial\Omega \\ &\qquad d\delta u \in L^2(\Omega;\Lambda^\ell),\, \delta du \in L^2(\Omega;\Lambda^\ell),\, \nu \vee du = 0 \text{ on } \partial\Omega\Big\}, \end{aligned} \tag{1.2}$$

$$B_\ell u := d\delta u + \delta du = -\Delta u, \qquad \forall\, u \in D(B_\ell), \tag{1.3}$$

and

$$\begin{aligned} D(C_\ell) &:= \Big\{u \in L^2(\Omega;\Lambda^\ell) : du \in L^2(\Omega;\Lambda^{\ell+1}),\, \delta u \in L^2(\Omega;\Lambda^{\ell-1}),\, \nu \wedge u = 0 \text{ on } \partial\Omega, \\ &\qquad \delta du \in L^2(\Omega;\Lambda^\ell),\, d\delta u \in L^2(\Omega;\Lambda^\ell),\, \nu \wedge \delta u = 0 \text{ on } \partial\Omega\Big\}, \end{aligned} \tag{1.4}$$

$$C_\ell u := d\delta u + \delta du = -\Delta u. \qquad \forall\, u \in D(C_\ell). \tag{1.5}$$

Above, d is the exterior derivative operator on $\mathcal{M}$, δ denote its formal adjoint, and Δ is the Hodge-Laplacian on $\mathcal{M}$. Also, $\wedge$ and $\vee$ stand, respectively, for the exterior and interior product of forms. In effect, B_ℓ and C_ℓ are the L^2-realizations of the Hodge-Laplacian with absolute and relative boundary conditions in Ω (cf., e.g., the discussion in [23]). In the language of the theory of unbounded linear operators on Hilbert spaces, we have

$$B_\ell = d_\ell^* \circ d_\ell + d_{\ell-1} \circ d_{\ell-1}^*, \qquad C_\ell = \delta_\ell^* \circ \delta_\ell + \delta_{\ell+1} \circ \delta_{\ell+1}^*, \tag{1.6}$$

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where d_ℓ, δ_ℓ are the L^2 realizations of of d, δ acting on ℓ-forms in Ω, and star denotes adjunction, in the operator theoretic sense.

It follows that the operators B_ℓ and C_ℓ are self-adjoint, non-negative generators of analytic semigroups in $L^2(\Omega;\Lambda^\ell)$. The extent to which the latter property continues to hold with L^2 replaced by L^p, $p\neq 2$, has been recently addressed in in [19]. To explain the nature of the main result in [19], we need to consider the following inhomogeneous problem for the Hodge-Laplacian:

$$\begin{cases} -\Delta u = f\in L^p(\Omega;\Lambda^\ell),\\ u,\, d\delta u,\, \delta du\in L^p(\Omega,\Lambda^\ell),\\ du\in L^p(\Omega,\Lambda^{\ell+1}),\ \delta u\in L^p(\Omega,\Lambda^{\ell-1}),\\ \nu\vee u=0 \text{ on } \partial\Omega,\\ \nu\vee du=0 \text{ on } \partial\Omega.\end{cases} \tag{1.7}$$

In order to ensure uniqueness, it is necessary to assume (cf. [16]) that

$$b_\ell(\Omega),\text{ the } \ell\text{-th Betti number of } \Omega,\text{ vanishes.} \tag{1.8}$$

Assuming that this is the case, let p_Ω, $q_\Omega\in[1,\infty]$ be the critical indices for which (1.7) is well-posed whenever $p\in(p_\Omega,q_\Omega)$. From the work in [16] it is known that

$$1\leq p_\Omega<2<q_\Omega\leq\infty,\qquad 1/p_\Omega+1/q_\Omega=1, \tag{1.9}$$

and, in the case when $n=3$, this further improves, as shown in [18], to

$$1\leq p_\Omega<\tfrac{3}{2}<3<q_\Omega\leq\infty,\qquad 1/p_\Omega+1/q_\Omega=1. \tag{1.10}$$

In general, p_Ω, q_Ω depend only on the Lipschitz character of Ω and the fact that $p_\Omega<3/2$ in the three-dimensional setting is sharp (though the situation in the higher-dimensional setting is less clear). It has been proved in [19] that for every $p\in]p_\Omega,q_\Omega[$ the semigroups in $L^2(\Omega;\Lambda^\ell)$ generated by $-B_\ell$ and $-C_\ell$ extend to analytic semigroups in $L^p(\Omega;\Lambda^\ell)$. In particular (see, e.g., [21]), the fractional powers $B_\ell^{-\alpha}$ and $C_\ell^{-\alpha}$ for $\alpha\in[0,1]$ are bounded in $L^p(\Omega;\Lambda^\ell)$ with p as before, provided

$$b_\ell(\Omega)=b_{n-\ell}(\Omega)=0. \tag{1.11}$$

Corresponding to $\alpha=1/2$, K.O. Friedrichs' theorem gives that

$$D((B_\ell)^{1/2}) = \Big\{u\in L^2(\Omega;\Lambda^\ell):\ du\in L^2(\Omega;\Lambda^{\ell+1}),\ \delta u\in L^2(\Omega;\Lambda^{\ell-1}),\ \nu\vee u=0\Big\}, \tag{1.12}$$

$$D((C_\ell)^{1/2}) = \Big\{u\in L^2(\Omega;\Lambda^\ell):\ du\in L^2(\Omega;\Lambda^{\ell+1}),\ \delta u\in L^2(\Omega;\Lambda^{\ell-1}),\ \nu\wedge u=0\Big\}, \tag{1.13}$$

and, under the assumption (1.11),

$$(B_\ell)^{-1/2}: L^2(\Omega;\Lambda^\ell)\longrightarrow D((B_\ell)^{1/2}), \tag{1.14}$$

$$(C_\ell)^{-1/2}: L^2(\Omega;\Lambda^\ell)\longrightarrow D((C_\ell)^{1/2}), \tag{1.15}$$

are isomorphisms. As a result,

$$d(B_\ell)^{-1/2},\quad \delta(B_\ell)^{-1/2},\quad d(C_\ell)^{-1/2},\quad \delta(C_\ell)^{-1/2} \tag{1.16}$$

are all bounded operators on $L^2(\Omega;\Lambda^\ell)$. Note that since for every $u\in L^2(\Omega;\Lambda^\ell)$ we have

$$\|u\|^2_{L^2(\Omega;\Lambda^\ell)} = \|d(B_\ell)^{-1/2}u\|^2_{L^2(\Omega;\Lambda^{\ell+1})} + \|\delta(B_\ell)^{-1/2}u\|^2_{L^2(\Omega;\Lambda^{\ell-1})}, \tag{1.17}$$

the estimate

$$\|d(B_\ell)^{-1/2}u\|_{L^p(\Omega;\Lambda^\ell)} + \|\delta(B_\ell)^{-1/2}u\|_{L^p(\Omega;\Lambda^{\ell-1})} \leq C\|u\|_{L^p(\Omega;\Lambda^\ell)} \tag{1.18}$$

entails (by polarization and duality) the opposite inequality for the conjugate exponent, i.e.,

$$\|u\|_{L^{p'}(\Omega;\Lambda^\ell)} \leq C\|d(B_\ell)^{-1/2}u\|_{L^{p'}(\Omega;\Lambda^\ell)} + C\|\delta(B_\ell)^{-1/2}u\|_{L^{p'}(\Omega;\Lambda^{\ell-1})}, \tag{1.19}$$

for $1/p+1/p'=1$. It is therefore natural to seek to determine the range of p's for which the equivalence

$$\|\sqrt{B_\ell}u\|_{L^p(\Omega;\Lambda^\ell)} \approx \|du\|_{L^p(\Omega;\Lambda^{\ell+1})} + \|\delta u\|_{L^p(\Omega;\Lambda^{\ell-1})} \tag{1.20}$$

is valid for differential forms u belonging to the space

$$V_p(\Omega,\Lambda^\ell) := \Big\{u \in L^p(\Omega;\Lambda^\ell) :\ du \in L^p(\Omega;\Lambda^{\ell+1}),\ \delta u \in L^p(\Omega;\Lambda^{\ell-1}),\ \nu \vee u = 0\Big\}. \tag{1.21}$$

For a smooth domain $\Omega \subset \mathcal{M}$, all operators in (1.16) are classical pseudodifferential operators of order zero, so in this case one can take $1 < p < \infty$ but the case of irregular domains is considerably more subtle.

The question we study in this paper is whether

$$\frac{1}{\sqrt{B_\ell}} := B_\ell^{-\frac{1}{2}} \text{ maps } L^p(\Omega;\Lambda^\ell) \text{ boundedly into } V_p(\Omega,\Lambda^\ell), \tag{1.22}$$

plus a similar issue for the operator C_ℓ. This question can be equivalently reformulated in terms of the Riesz transforms associated to B_ℓ and C_ℓ in $L^p(\Omega;\Lambda^\ell)$, and this is how we choose to state the theorem below, which constitutes the main result of this paper.

Theorem 1.1. *Let $\mathcal{M}$ be a smooth, compact, oriented manifold of real dimension n, equipped with a smooth Riemannian metric tensor. Assume that $\Omega \subset \mathcal{M}$ is a Lipschitz domain and consider the Hodge-Laplacians B_ℓ, C_ℓ, equipped with absolute and relative boundary conditions as in* (1.2)-(1.3) *and* (1.4)-(1.5)*, respectively. Finally, let the critical indices p_Ω, q_Ω retain the same significance as above and introduce*

$$q_\Omega^* := \frac{nq_\Omega}{n-1}, \qquad (q_\Omega^*)' := \left(1 - \frac{1}{q_\Omega^*}\right)^{-1}. \tag{1.23}$$

Then, for every $p \in](q_\Omega^*)', q_\Omega^*[$, the Riesz transforms associated to B_ℓ, that is

$$\frac{d}{\sqrt{B_\ell}} := dB_\ell^{-\frac{1}{2}} : L^p(\Omega;\Lambda^\ell) \longrightarrow L^p(\Omega;\Lambda^{\ell+1}), \tag{1.24}$$

$$\frac{\delta}{\sqrt{B_\ell}} := \delta B_\ell^{-\frac{1}{2}} : L^p(\Omega;\Lambda^\ell) \longrightarrow L^p(\Omega;\Lambda^{\ell-1}), \tag{1.25}$$

are well-defined and bounded provided $b_\ell(\Omega) = 0$.

Likewise, the Riesz transforms associated to C_ℓ, i.e.,

$$\frac{d}{\sqrt{C_\ell}} := dC_\ell^{-\frac{1}{2}} : L^p(\Omega;\Lambda^\ell) \longrightarrow L^p(\Omega;\Lambda^{\ell+1}), \tag{1.26}$$

$$\frac{\delta}{\sqrt{C_\ell}} := \delta C_\ell^{-\frac{1}{2}} : L^p(\Omega;\Lambda^\ell) \longrightarrow L^p(\Omega;\Lambda^{\ell-1}), \tag{1.27}$$

are well-defined and bounded provided $b_{n-\ell}(\Omega) = 0$.

Theorem 1.1 can be viewed as a higher-degree generalization of results corresponding to the Riesz transforms $\nabla(-\Delta_D)^{-\frac{1}{2}}$, $\nabla(-\Delta_N)^{-\frac{1}{2}}$, associated with the scalar Beltrami-Laplacian equipped with (homogeneous) Dirichlet and Neumann boundary conditions from [20]. These, in turn, extend results in the flat, Euclidean setting from [13], [9] and [15].

A significant consequence of Theorem 1.1 is as follows.

Corollary 1.2. *Retain the same background hypotheses as in Theorem 1.1 and assume that $b_\ell(\Omega) = 0$. Then there exist two constants $C_0, C_1 > 0$ with the property that*

$$C_0\|\sqrt{B_\ell}w\|_{L^p(\Omega;\Lambda^\ell)} \leq \|dw\|_{L^p(\Omega;\Lambda^{\ell+1})} + \|\delta w\|_{L^p(\Omega;\Lambda^{\ell-1})} \leq C_1\|\sqrt{B_\ell}w\|_{L^p(\Omega;\Lambda^\ell)} \tag{1.28}$$

for every form $w \in V_p(\Omega;\Lambda^\ell)$. As a corollary, in the above context, the operator

$$\sqrt{B_\ell} : V_p(\Omega;\Lambda^\ell) \longrightarrow L^p(\Omega;\Lambda^\ell) \tag{1.29}$$

is an isomorphism.

This is obtain by taking $u := (B_\ell)^{1/2}w$ (initially for $w \in V_2(\Omega;\Lambda^\ell) \cap V_p(\Omega;\Lambda^\ell)$, then extended by density to the entire space $V_p(\Omega;\Lambda^\ell)$) in (1.18) and (1.19). The heuristic interpretation of (1.28) is that $\sqrt{B_\ell}$, i.e., the square-root of the Laplacian with absolute boundary conditions, behaves the same way in $V_p(\Omega;\Lambda^\ell)$ (with p as before) as the Dirac operator $D := d + \delta$. Of course, a similar result is valid for the operator $\sqrt{C_\ell}$.

The proof of Theorem 1.1 is based on a strategy introduced by X. T. Duong to prove weak-type (1,1) bounds using a modified Hörmander condition adapted to an operator L satisfying pointwise Gaussian heat kernel bounds, in which, for example, the resolvent operator $R(t) = (1 + t^2L)^{-1}$ or the heat kernel e^{-t^2L} replaced the usual dyadic averaging operator, and which appeared in [7], [8] and [4]. See also [10, 11], where some similar ideas had been introduced previously. More recently, Duong's approach was extended by Blunck and Kuntsmann [1], [2], and independently by the first named author and Martell [12], to settings in which pointwise kernel bounds may be lacking, and in which therefore one cannot expect to obtain weak L^1 estimates, but only (p,p) bounds when p is greater than some $p_0 > 1$. Our approach here is based on the techniques of these extensions.

The layout of the paper is as follows. In Section 2 we review a number of basic differential geometric results, further augmented by a discussion of traces of differential forms and commutation identities for the resolvents of B_ℓ, C_ℓ in Section 3. In Section 4 we recall a version of certain off-diagonal estimates from [19] and, following the work in [12] prove that such estimates are stable under composition. Finally, the proof of Theorem 1.1 is presented in Section 5.

Throughout the paper, we will use the standard convention that generic constants C and c may vary from one instance to the next, but will always depend only upon harmless parameters such as the intrinsic properties of our manifold $\mathcal{M}$, the Lipschitz character of our domain Ω, and the particular exponent(s) p for which we are proving L^p norm inequalities.

2. Geometrical preliminaries

Throughout the paper, $\mathcal{M}$ will denote a smooth, compact, oriented manifold of real dimension n, equipped with a smooth Riemannian metric tensor, $\sum_{j,k} g_{jk} dx_j \otimes dx_k$. We denote by $T\mathcal{M}$ and $T^*\mathcal{M}$ the tangent and cotangent bundles to $\mathcal{M}$, respectively. Occasionally, we shall identify $T^*\mathcal{M} \equiv \Lambda^1$ canonically, via the metric. Set Λ^ℓ for the ℓ-th exterior power of $T\mathcal{M}$. Sections in this latter vector bundle are ℓ-differential forms. The Hermitian structure on $T\mathcal{M}$ extends naturally to $T^*\mathcal{M} := \Lambda^1$ and, further, to Λ^ℓ. We denote by $\langle \cdot, \cdot \rangle$ the corresponding (pointwise) inner product. The volume form on $\mathcal{M}$, $\mathcal{V}_M$, is the unique unitary, positively oriented differential form of maximal degree on $\mathcal{M}$. In local coordinates, $\mathcal{V}_\mathcal{M} := [\det(g_{jk})]^{1/2} dx_1 \wedge dx_2 \wedge \dots \wedge dx_n$. In the sequel, we denote by $d\lambda_\mathcal{M}$ the Borelian measure induced by the volume form $\mathcal{V}_\mathcal{M}$ on $\mathcal{M}$, i.e., $d\lambda_\mathcal{M} = [\det(g_{jk})]^{1/2} dx_1 dx_2 \dots dx_n$ in local coordinates.

Going further, we introduce the Hodge star operator as the unique vector bundle morphism $* : \Lambda^\ell \to \Lambda^{n-\ell}$ such that $u \wedge (*u) = |u|^2 \mathcal{V}_\mathcal{M}$ for each $u \in \Lambda^\ell$. In particular, $\mathcal{V}_\mathcal{M} = *1$ and

$$u \wedge (*v) = \langle u, v \rangle \mathcal{V}_\mathcal{M}, \qquad \forall\, u \in \Lambda^\ell,\ \forall\, v \in \Lambda^\ell. \tag{2.1}$$

The interior product between a 1-form ν and a ℓ-form u is then defined by

$$\nu \vee u := (-1)^{\ell(n+1)} * (\nu \wedge *u). \tag{2.2}$$

Let d stand for the (exterior) derivative operator and denote by δ its formal adjoint (with respect to the metric introduced above). For further reference some basic properties of these objects are summarized below.

Proposition 2.1. *For arbitrary 1-form ν, ℓ-forms u, ω, $(n-\ell)$-form v, and $(\ell+1)$-form w, the following are true:*

1. $\langle u, *v \rangle = (-1)^{\ell(n-\ell)} \langle *u, v \rangle$ *and* $\langle *u, *\omega \rangle = \langle u, \omega \rangle$. *Also,* $**u = (-1)^{\ell(n-\ell)} u$;
2. $\langle \nu \wedge u, w \rangle = \langle u, \nu \vee w \rangle$;
3. $*(\nu \wedge u) = (-1)^\ell \nu \vee (*u)$ *and* $*(\nu \vee u) = (-1)^{\ell+1} \nu \wedge (*u)$;
4. $*\delta = (-1)^\ell d*$, $\delta * = (-1)^{\ell+1} * d$, *and* $\delta = (-1)^{n(\ell+1)+1} * d *$ *on ℓ-forms.*

Thus, if $\Delta := -(d\delta + \delta d)$, it follows that $d\Delta = \Delta d$, $\delta\Delta = \Delta\delta$ and $*\Delta = \Delta *$.

Moving on, let Ω be a Lipschitz subdomain of $\mathcal{M}$. That is, $\partial\Omega$ can be described in appropriate local coordinates by means of graphs of Lipschitz functions. Then the outward unit conormal $\nu \in T^*\mathcal{M}$ of Ω is defined a.e., with respect to the surface measure $d\sigma$ induced by the ambient Riemannian metric on $\partial\Omega$. For any two sufficiently well-behaved differential forms (of compatible degrees) u, w we then have the integration by parts formula

$$\begin{aligned}\int_\Omega \langle du, w\rangle\, d\lambda_{\mathcal{M}} &= \int_\Omega \langle u, \delta w\rangle\, d\lambda_{\mathcal{M}} + \int_{\partial\Omega} \langle \nu\wedge u, w\rangle\, d\sigma \\ &= \int_\Omega \langle u, \delta w\rangle\, d\lambda_{\mathcal{M}} + \int_{\partial\Omega} \langle u, \nu\vee w\rangle\, d\sigma. \end{aligned} \tag{2.3}$$

We continue with a brief discussion of a number of notational conventions used throughout the paper. We denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{N} = \{1, 2, ...\}$ the subset of $\mathbb{Z}$ consisting of positive numbers. Also, we set $\mathbb{N}_o := \mathbb{N} \cup \{0\}$. By $C^k(\Omega)$, $k \in \mathbb{N}_o \cup \{\infty\}$, we shall denote the space of functions of class C^k in Ω, and by $C^\infty_c(\Omega)$ the subspace of $C^\infty(\Omega)$ consisting of compactly supported functions. When viewed as a topological vector space, the latter is equipped with the usual inductive limit topology, and its dual, i.e. the space of distributions in Ω, is denoted by $D'(\Omega) := \Big(C^\infty_c(\Omega)\Big)'$. Also, we set $C^k(\Omega, \Lambda^\ell) := C^k(\Omega) \otimes \Lambda^\ell$, etc. Finally, we would like to alert the reader that, besides denoting the pointwise inner product of forms, $\langle\cdot,\cdot\rangle$ is also used as a duality bracket between a topological vector space and its dual (in each case, the spaces in question should be clear from the context).

3. Traces of differential forms

The Sobolev (or potential) class $L^p_\alpha(\mathcal{M})$, $1 < p < \infty$, $\alpha \in \mathbb{R}$, is obtained by lifting the Euclidean scale $L^p_\alpha(\mathbb{R}^n) := \{(I - \Delta)^{-\alpha/2} f : f \in L^p(\mathbb{R}^n)\}$ to $\mathcal{M}$ (via a C^∞ partition of unity and pull-back). For a Lipschitz subdomain Ω of $\mathcal{M}$, we denote by $L^p_\alpha(\Omega)$ the restriction of elements in $L^p_\alpha(\mathcal{M})$ to Ω, and set $L^p_\alpha(\Omega; \Lambda^\ell) = L^p_\alpha(\Omega)\otimes\Lambda^\ell T\mathcal{M}$, i.e. the collection of ℓ-forms with coefficients in $L^p_\alpha(\Omega)$. In particular, $L^p(\Omega; \Lambda^\ell)$ stands for the space of ℓ-differential forms with p-th power integrable coefficients in Ω. For the sake of simplicity of notation, we will sometimes write $\|f\|_p$ in place of $\|f\|_{L^p(\Omega;\Lambda^\ell)}$ when there is no chance of confusion.

Let us also note here that if $p, p' \in (1, \infty)$ are such that $1/p + 1/p' = 1$, then

$$\Big(L^p_s(\Omega; \Lambda^\ell)\Big)^* = L^{p'}_{-s}(\Omega; \Lambda^\ell), \qquad \forall\, s \in (-1 + 1/p, 1/p). \tag{3.1}$$

The Besov spaces $B^{p,q}_s(\Omega; \Lambda^\ell)$, $1 < p, q < \infty$, $s \in \mathbb{R}$, can be introduced in a similar manner; alternatively, this may be obtained from the Sobolev scale via real-interpolation.

Next, denote by $L^p_1(\partial\Omega)$ the Sobolev space of functions in $L^p(\partial\Omega)$ with tangential gradients in $L^p(\partial\Omega)$, $1 < p < \infty$. Besov spaces on $\partial\Omega$ can then be introduced via real interpolation, i.e.

$$B^{p,q}_s(\partial\Omega) := \left(L^p(\partial\Omega), L^p_1(\partial\Omega)\right)_{s,q}, \quad \text{with } 0 < s < 1,\ 1 < p, q < \infty. \tag{3.2}$$

Finally, if $1 < p, q < \infty$ and $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, we define

$$B^{p,q}_{-s}(\partial\Omega) := \Big(B^{p',q'}_s(\partial\Omega)\Big)^*, \quad 0 < s < 1, \tag{3.3}$$

and, much as before, set $B^{p,q}_s(\partial\Omega; \Lambda^\ell) := B^{p,q}_s(\partial\Omega) \otimes \Lambda^\ell T\mathcal{M}$.

Recall (cf. [14], [13]) that the trace operator

$$\mathrm{Tr} : L^p_s(\Omega; \Lambda^\ell) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega; \Lambda^\ell) \tag{3.4}$$

is well-defined, bounded and onto if $1 < p < \infty$ and $\frac{1}{p} < s < 1 + \frac{1}{p}$. Furthermore, the trace operator has a bounded right inverse

$$\mathrm{Ex} : B^{p,p}_{s-\frac{1}{p}}(\partial\Omega; \Lambda^\ell) \longrightarrow L^p_s(\Omega; \Lambda^\ell), \tag{3.5}$$

and if $1 < p < \infty$, $\frac{1}{p} < \alpha < 1 + \frac{1}{p}$, then

$$\text{Ker}\,(\text{Tr}) = \text{ the closure of } C_c^\infty(\Omega;\Lambda^\ell) \text{ in } L_s^p(\Omega;\Lambda^\ell). \tag{3.6}$$

For $1 < p < \infty$, $s \in \mathbb{R}$, and $\ell \in \{0,1,...,n\}$ we next introduce

$$\mathcal{D}_\ell^p(\Omega;d) := \{u \in L^p(\Omega;\Lambda^\ell) :\, du \in L^p(\Omega;\Lambda^{\ell+1}\}, \tag{3.7}$$

$$\mathcal{D}_\ell^p(\Omega;\delta) := \{u \in L^p(\Omega;\Lambda^\ell) :\, \delta u \in L^p(\Omega;\Lambda^{\ell-1})\}, \tag{3.8}$$

equipped with the natural graph norms. Throughout the paper, all derivatives are taken in the sense of distributions.

Inspired by the identity (2.3), whenever $1 < p < \infty$ and $u \in \mathcal{D}_\ell^p(\Omega;\delta)$ we then define $\nu \vee u$ as a functional in $\big(B^{p',p'}_{1-\frac{1}{p'}}(\partial\Omega;\Lambda^{\ell-1})\big)^* = B^{p,p}_{-\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1})$ (where, as before, $1/p + 1/p' = 1$) by setting

$$\langle \nu \vee u, \varphi\rangle := -\langle \delta u, \Phi\rangle + \langle u, d\Phi\rangle \tag{3.9}$$

for any $\varphi \in B^{p',p'}_{\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1})$ and any $\Phi \in L_1^{p'}(\Omega;\Lambda^{\ell-1})$ with $\text{Tr}\,\Phi = \varphi$. Note that (3.1), (3.6) imply that the operator

$$\nu \vee \cdot\, : \mathcal{D}_\ell^p(\Omega;\delta) \longrightarrow B^{p,p}_{-\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1}) \tag{3.10}$$

is well-defined, linear and bounded for each $p \in (1,\infty)$, i.e.

$$\|\nu \vee u\|_{B^{p,p}_{-\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1})} \leq C\Big(\|u\|_{L_s^p(\Omega;\Lambda^\ell)} + \|\delta u\|_{L_s^p(\Omega;\Lambda^{\ell-1})}\Big). \tag{3.11}$$

Other spaces of interest for us here are defined as follows. For $1 < p < \infty$, $s \in \mathbb{R}$, and $\ell \in \{0,1,...,n\}$, consider

$$\mathcal{D}_\ell^p(\Omega;\delta_\vee) := \{u \in L^p(\Omega;\Lambda^\ell) :\, \delta u \in L^p(\Omega;\Lambda^{\ell-1}),\ \nu \vee u = 0\}, \tag{3.12}$$

once again equipped with the natural graph norm. Based on definitions, it can be readily checked that

$$u \in \mathcal{D}_\ell^p(\Omega;d_\wedge) \Longrightarrow \nu \wedge du = 0 \text{ in } B^{p,p}_{-\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1}), \tag{3.13}$$

$$u \in \mathcal{D}_\ell^p(\Omega;\delta_\vee) \Longrightarrow \nu \vee \delta u = 0 \text{ in } B^{p,p}_{-\frac{1}{p}}(\partial\Omega;\Lambda^{\ell-1}). \tag{3.14}$$

For further use, we record here a useful variation on the integration by parts formula (2.3), namely that if $1 < p, p' < \infty$ satisfy $1/p + 1/p' = 1$ then

$$\langle du, v\rangle = \langle u, \delta v\rangle, \quad \forall\, u \in \mathcal{D}_\ell^{p'}(\Omega;d),\ \forall\, v \in \mathcal{D}_\ell^p(\Omega;\delta_\vee). \tag{3.15}$$

Consider the unbounded operators B_ℓ, C_ℓ on $L^2(\Omega;\Lambda^\ell)$ defined as in (1.2)-(1.3) and (1.4)-(1.5), respectively. In the last part of this section we establish some useful commutation identities between d and δ on the one hand, and the resolvents of the operators B_ℓ and C_ℓ on the other hand. Specifically, we have the following result (compare with [16]).

Proposition 3.1. *Let Ω be a Lipschitz subdomain of $\mathcal{M}$. Then for every $\ell \in \{0,1,...,n\}$ and every nonzero number $t \in \mathbb{R}$, the following properties hold.*

(1) *For $f \in L^2(\Omega;\Lambda^\ell)$ such that $df \in L^2(\Omega;\Lambda^{\ell+1})$, we have*

$$d(1 + t^2 B_\ell)^{-1} f = (1 + t^2 B_{\ell+1})^{-1} df. \tag{3.16}$$

If, in addition, $\nu \wedge f = 0$ on $\partial\Omega$, we then also have

$$d(1 + t^2 C_\ell)^{-1} f = (1 + t^2 C_{\ell+1})^{-1} df. \tag{3.17}$$

(2) *For $f \in L^2(\Omega;\Lambda^\ell)$ such that $\delta f \in L^2(\Omega;\Lambda^{\ell-1})$, we have*

$$\delta(1+t^2C_\ell)^{-1}f = (1+t^2C_{\ell-1})^{-1}\delta f. \tag{3.18}$$

If, in addition, $\nu \vee f = 0$ on $\partial\Omega$, we then also have

$$\delta(1+t^2B_\ell)^{-1}f = (1+t^2B_{\ell-1})^{-1}\delta f. \tag{3.19}$$

Proof. To prove (3.16), fix $f \in L^2(\Omega;\Lambda^\ell)$ with the property that $df \in L^2(\Omega;\Lambda^{\ell+1})$, and let $u \in D(B_\ell)$ be the solution of the partial differential equation $u - t^2\Delta u = f$ in Ω. Then the differential form $v := du \in L^2(\Omega;\Lambda^{\ell+1})$ satisfies

$$v - t^2\Delta v = df \in L^2(\Omega;\Lambda^{\ell+1}), \tag{3.20}$$

since $d\Delta = \Delta d$. In addition, we have $\nu \vee v = \nu \vee du = 0$ since $u \in D(B_\ell)$, and $dv = 0$ in Ω since $d^2 = 0$. In particular, $\nu \vee dv = 0$ on $\partial\Omega$. On the other hand, the differential form $w := (1+t^2B_{\ell+1})^{-1}(df) \in L^2(\Omega;\Lambda^{\ell+1})$ is the unique solution of the boundary-value problem

$$w - t^2\Delta w = df \text{ in } \Omega, \quad \nu \vee w = 0 \text{ and } \nu \vee dw = 0 \text{ on } \partial\Omega. \tag{3.21}$$

We therefore necessarily have $v = w$, which amounts to (3.16).

As far as (3.17) is concerned, pick $f \in L^2(\Omega;\Lambda^\ell)$ with the property that $df \in L^2(\Omega;\Lambda^{\ell+1})$ and $\nu \wedge f = 0$ on $\partial\Omega$. Let $u \in D(C_\ell)$ be the unique solution of $u - t^2\Delta u = f$ in Ω, and consider $v := du \in L^2(\Omega;\Lambda^{\ell+1})$. Thanks to (3.13), it follows that $\nu \wedge v = \nu \wedge du = 0$. Let us also note that $\nu \wedge (d\delta u) = 0$ by (3.13) and the fact that $u \in D(C_\ell)$. Consequently,

$$\begin{aligned} \nu \wedge \delta v = \nu \wedge \delta du &= \nu \wedge [(-\Delta - d\delta)u] \\ &= \nu \wedge (t^{-2}f - t^{-2}u - d\delta u) = 0. \end{aligned} \tag{3.22}$$

On the other hand, $w := (1+t^2C_{\ell+1})^{-1}(df)$ is the unique solution of

$$w - t^2\Delta w = df \text{ in } \Omega, \ \nu \wedge w = 0 \text{ and } \nu \wedge \delta w = 0 \text{ on } \partial\Omega. \tag{3.23}$$

Since $v = du$ is a solution of the same boundary-value problem, we may conclude that $v = w$, which proves (3.17). The claims in part (2) of the statement of the proposition then follow from what we have proved so far and Hodge duality. □

4. L^p−OFF-DIAGONAL ESTIMATES

The aim of this section is two fold. On the one hand we record some useful off-diagonal estimates for the resolvents of the Hodge-Laplacian which are akin to (though slightly more general than) those proved in [19]. On the other hand, we will prove here estimates in the same spirit as those established in [12, Section 2]. Throughout the present section, we retain the hypotheses on $\mathcal{M}$ from Section 2 above, and assume that $\Omega \subset \mathcal{M}$ is a Lipschitz domain. We let p_Ω, q_Ω denote the endpoints of the largest interval (stable under Hölder conjugation) where the boundary-value problem (1.7) is well-posed. Finally, we remind the reader that balls on $\mathcal{M}$ are considered with respect to the geodesic distance induced by the Riemannian metric.

The following lemma is stated in [19, Sections 5-6] in the case $t = t_0$. The current version is proved in a similar fashion, by keeping careful track on the constants involved.

Lemma 4.1. *Let $x_0 \in \Omega$, let $t_0, t \in]0,+\infty[$, let $p \in [2, q_\Omega[$, and let $j \in \mathbb{N}$, $j \geq 3$. For each j, assume that $f_j \in L^p(\Omega;\Lambda^\ell)$ is supported in the annulus $B(x_0, 2^{j+1}t_0) \setminus B(x_0, 2^{j-1}t_0)$, and define $u_j := (1+t^2B_\ell)^{-1}f_j \in D(B_\ell)$. Set $p^* := \frac{np}{n-1}$, and suppose that $r \in [p, p^*]$. Then*

$$\begin{aligned} &\|u_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} + \|t\,\delta u_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^{\ell-1})} + \|t\,du_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^{\ell+1})} \\ &\quad \leq C\, t_0^{n(\frac{1}{r}-\frac{1}{p})} \exp\Big(-c\,2^j\,\tfrac{t_0}{t}\Big)\,\|f_j\|_p. \end{aligned} \tag{4.1}$$

Moreover, if $f \in L^p(\Omega;\Lambda^\ell)$, and $u(t) := (1+t^2B_\ell)^{-1}f$, then

$$\|u(t)\|_{L^r(\Omega;\Lambda^\ell)} + \|t\delta u(t)\|_{L^r(\Omega;\Lambda^{\ell-1})} + \|tdu(t)\|_{L^r(\Omega;\Lambda^{\ell+1})} \leq Ct^{n(\frac{1}{r}-\frac{1}{p})}\|f\|_p. \tag{4.2}$$

Our next result is similar to Lemma 2.3 in [12], and essentially states that the class of operators satisfying off-diagonal estimates is stable under composition.

Lemma 4.2. *Let $x_0 \in \Omega$, let $t_0 \in]0,+\infty[$ and let $p \in [2, q_\Omega[$. Let $\{T_t; t>0\}$ and $\{S_s; s>0\}$ be two families of operators, uniformly bounded in $L^q(\Omega;\Lambda^\ell)$ for $p \le q \le r \le p^* = \frac{np}{n-1}$ and satisfying, for $f_j \in L^p(\Omega;\Lambda^\ell)$ supported in $B(x_0, 2^{j+1}t_0) \setminus B(x_0, 2^{j-1}t_0)$, $j \ge 3$, the estimates*

$$\|T_t f_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} \le C_1 \exp\Big(-c_1 2^j \frac{t_0}{t}\Big)\, t_0^{n(\frac{1}{r}-\frac{1}{p})}\|f_j\|_p, \quad t>0, \tag{4.3}$$

and

$$\|S_s f_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} \le C_2 \exp\Big(-c_2 2^j \frac{t_0}{s}\Big)\, t_0^{n(\frac{1}{r}-\frac{1}{p})}\|f_j\|_p, \quad s>0. \tag{4.4}$$

Then for each for $j \ge 6$ we have

$$\|T_t S_s f_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} \le C \exp\Big(-c 2^j \frac{t_0}{\max\{t,s\}}\Big)\, t_0^{n(\frac{1}{r}-\frac{1}{p})}\|f_j\|_p, \quad t,s>0. \tag{4.5}$$

Proof. Let $j \ge 6$ and $G = B(x_0, 2^{j-3}t_0)$. Fix $r \in [p,p^*]$. We decompose $S_s f_j$ into two terms $g_j = S_s f_j \chi_{\Omega\cap G}$ and $h_j = S_s f_j \chi_{\Omega\setminus G}$. We have

$$\begin{aligned} \|T_t g_j\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} &\overset{(1)}{\le} c\|g_j\|_r \\ &\overset{(2)}{\le} c\|S_s f_j\|_{L^r(\Omega\cap B(x_0,2^{j-3}t_0);\Lambda^\ell)} \\ &\overset{(3)}{\le} c\,C_2 e^{-c_2 2^3 \frac{t_1}{s}} t_1^{n(\frac{1}{r}-\frac{1}{p})}\|f_j\|_p \\ &\overset{(4)}{\le} C e^{-c_2 2^j \frac{t_0}{s}} t_0^{n(\frac{1}{r}-\frac{1}{p})}\|f_j\|_p. \end{aligned} \tag{4.6}$$

The first inequality is obtained thanks to the uniform boundedness of $\{T_t; t>0\}$ in L^r. The second inequality is an equality, using the definition of g_j. The third inequality is obtained by applying (4.4) written for t_1 (in place of t_0) and $j=3$. The last inequality is obtained by choosing $t_1 = 2^{j-3}t_0$ and the fact that $r>p$, so that $2^{n(j-3)(\frac{1}{r}-\frac{1}{p})} \le 1$. Next, we decompose h_j the following way

$$\begin{aligned} h_j &= \sum_{k=j-2}^{\infty} h_j^{(k)} \\ &= \sum_{k=j-2}^{\infty} S_s f_j \chi_{\Omega\cap(B(x_0,2^k t_0)\setminus B(x_0,2^{k-1}t_0))}. \end{aligned} \tag{4.7}$$

We have then

$$\begin{aligned} \|T_t h_j\|_{L^r(B(x_0,t_0);\Lambda^\ell)} &\overset{(1)}{\le} \sum_{k=j-2}^{\infty} \|T_t h_j^{(k)}\|_{L^r(\Omega\cap B(x_0,t_0);\Lambda^\ell)} \\ &\overset{(2)}{\le} C_1 t_0^{n(\frac{1}{r}-\frac{1}{p})} \sum_{k=j-2}^{\infty} \exp\Big(-c_1 2^k \frac{t_0}{t}\Big)\|h_j^{(k)}\|_p \\ &\overset{(3)}{\le} C_1 t_0^{n(\frac{1}{r}-\frac{1}{p})} \exp\Big(-c_1 2^{j-3}\tfrac{t_0}{t}\Big)\Big(\textstyle\sum_{k=1}^{\infty} \exp\Big(-c_1 2^{j-3}(2^k-1)\tfrac{t_0}{t}\Big)\Big) \\ &\qquad \times \ \|S_s f_j\|_{L^p((\Omega\cap B(x_0,2^{k+j-3}t_0))\setminus B(x_0,2^{k+j-4}t_0));\Lambda^\ell)} \\ &\overset{(4)}{\le} C_1 t_0^{n(\frac{1}{r}-\frac{1}{p})} \exp\Big(-c_1 2^{j-3}\tfrac{t_0}{t}\Big)\|S_s f_j\|_{L^p(\Omega\setminus G;\Lambda^\ell)} \\ &\overset{(5)}{\le} C\, t_0^{n(\frac{1}{r}-\frac{1}{p})} \exp\Big(-c 2^j \tfrac{t_0}{t}\Big)\|f_j\|_{L^p(\Omega;\Lambda^\ell)}. \end{aligned} \tag{4.8}$$

The first inequality comes from the decomposition (4.7). The second inequality is obtained by applying (4.3) for each $h_j^{(k)}$. The third inequality is an equality (replacing k by $k-j+3$), the fourth inequality comes from the fact that $e^{-c_1 2^{j-3}(2^k-1)\frac{t_0}{t}} \le 1$ for all $k \ge 1$, and that the complement of G in $\mathcal{M}$ is

$$\mathcal{M} \setminus G = \bigcup_{k=1}^{\infty} B(x_0, 2^{k+j-3}t_0) \setminus B(x_0, 2^{k+j-4}t_0).$$

Finally, the fifth inequality comes from the uniform boundedness of the operators S_s, $s > 0$ and by denoting $c = 2^{-3}c_1$. Putting (4.6) and (4.8) together, we get (4.5). □

We now state a corollary of the previous two lemmata, that will be useful in the sequel. Set

$$R(t) := (1 + t^2 B_\ell)^{-1}.$$

Corollary 4.3. *There exists $q < 2$ such that if $f \in L^q(\Omega; \Lambda^\ell)$, and if $f_j \in L^q(\Omega; \Lambda^\ell)$ is supported in the annulus $B(x_0, 2^{j+1}t) \setminus B(x_0, 2^{j-1}t)$, then for every $i \in \mathbb{N}$,*

$$\|(R(t))^i f\|_{L^2(\Omega;\Lambda^\ell)} \le C_i\, t^{n(\frac{1}{2}-\frac{1}{q})}\, \|f\|_q. \tag{4.9}$$

and

$$\|(R(t))^i f_j\|_{L^2(\Omega\cap B(x_0,t);\Lambda^\ell)} \le C_i\, t^{n(\frac{1}{2}-\frac{1}{q})} \exp\left(-c\, 2^j\right)\, \|f_j\|_q. \tag{4.10}$$

Proof. Let $2 < p < q_\Omega$ and set $p^* := \frac{np}{n-1}$ as above. A simple iteration argument shows that (4.2) holds for each $r \in [p, p^*]$, with $R(t)^i$ in place of $R(t)$. Since the resolvent is self-adjoint, dualizing the latter estimate with $r = p^*$ yields

$$\|(R(t)^i f\|_{L^{p'}(\Omega;\Lambda^\ell)} \le C t^{n\left(\frac{1}{p'}-\frac{1}{(p^*)'}\right)} \|f\|_{(p^*)'}. \tag{4.11}$$

The case $p = 2 = p'$ is (4.9). Moreover, by Lemma 4.2, we have that (4.1) also holds with $(R(t))^i$ in place of $R(t)$. This fact, plus (4.11) with $p' < 2$, yield the second conclusion of the corollary by a straightforward interpolation argument. We omit the details. □

We conclude this section with two more corollaries that will be useful in the sequel. To set the stage, we introduce some notation. Given a measurable set $E \subset \mathcal{M}$, we denote by $|E|$ its Riemannian volume, i.e., $|E| := \int_{\mathcal{M}} \chi_E \, d\lambda_{\mathcal{M}}$, where χ_E denotes the characteristic function of E, and where $d\lambda_{\mathcal{M}}$ is the Borel measure induced by the volume form as described in Section 2. When equipped with the geodesic distance and the measure $d\lambda_{\mathcal{M}}$, the Lipschitz domain $\Omega \subset \mathcal{M}$ becomes a space of homogeneous type. In fact, we have the Ahlfors-David condition

$$|B_\Omega(x, r)| \approx r^n, \qquad 0 < r < 4\,\mathrm{diam}(\Omega), \tag{4.12}$$

where we have set

$$B_\Omega(x, r) := \{y \in \Omega : \mathrm{dist}\,(x, y) < r\} = B(x, r) \cap \Omega, \tag{4.13}$$

and where the implicit constants depend only on intrinsic properties of $\mathcal{M}$, and on the Lipschitz character of Ω. In particular, (4.12) implies the doubling property

$$|B_\Omega(x, 2r)| \le C|B_\Omega(x, r)|, \quad \text{for every } x \in \Omega,\ 0 < r < \mathrm{diam}\,(\Omega). \tag{4.14}$$

We now define the non-centered Hardy-Littlewood maximal function (relative to Ω) by

$$M_\Omega f(x) := \sup_{x \in B_\Omega} \left(\frac{1}{|B_\Omega|} \int_{B_\Omega} |f(y)|\, d\lambda_{\mathcal{M}} \right), \qquad x \in \Omega, \tag{4.15}$$

where the supremum runs over all "Ω balls" $B_\Omega := B_\Omega(z, r)$ containing x, such that $z \in \Omega$ and $0 < r < 2\mathrm{diam}(\Omega)$. We note that by the doubling property (4.14), M_Ω is bounded on $L^p(\Omega)$ whenever $1 < p \le \infty$ and, corresponding to $p = 1$, is of weak-type $(1, 1)$.

Recall that $R(t) := (1 + t^2 B_\ell)^{-1}$. We have the following

Corollary 4.4. *Let $x \in \Omega$, $t > 0$, and set $B := B(x, t)$, $B_\Omega := B \cap \Omega$. Suppose that $p \in [2, q_\Omega[$, and that $r \in [p, p^*]$, with $p^* := \frac{np}{n-1}$. Then for $f \in L^p(\Omega; \Lambda^\ell)$, and for each $i \in \mathbb{N}$, we have*

$$\left(\frac{1}{|B_\Omega|} \int_{B_\Omega} \left| \left(R(t)\right)^i f \right|^r d\lambda_{\mathcal{M}} \right)^{1/r} \le C_i \,\mathrm{ess\,inf}_{B_\Omega} \left(M_\Omega(|f|^p) \right)^{1/p}.$$

Proof. Let $f \in L^p(\Omega;\Lambda^\ell)$. Given a ball $B_\Omega = B_\Omega(x,t) = B(x,t)\cap\Omega$, set

$$S_0(B_\Omega) := B_\Omega\,, \qquad S_j(B_\Omega) := B_\Omega(x,2^j t)\setminus B_\Omega(x,2^{j-1}t)\,, \quad j\in\mathbb{N}. \tag{4.16}$$

Let J_{max} denote the largest j such that $2^j t \le 2\operatorname{diam}\Omega$ (so that $S_j(B_\Omega)$ is non-empty). Let $\chi_{S_j(B_\Omega)}$ denote the characteristic function of $S_j(B_\Omega)$, and set $f_j := f\chi_{S_j(B_\Omega)}$. Then, using (4.12), Lemma 4.1 and Lemma 4.2, we obtain

$$\begin{aligned}\left(\frac{1}{|B_\Omega|}\int_{B_\Omega}\left|\big(R(t)\big)^i f\right|^r d\lambda_{\mathcal{M}}\right)^{1/r} &\le \sum_{j=0}^{J_{max}} t^{-n/r}\left(\int_{B_\Omega}\left|\big(R(t)\big)^i f_j\right|^r d\lambda_{\mathcal{M}}\right)^{1/r}\\ &\le C\sum_{j=0}^{J_{max}}\exp(-c2^j)t^{-n/p}\left(\int_{S_j(B_\Omega)}|f_j|^p\,d\lambda_{\mathcal{M}}\right)^{1/p}\\ &\le C\sum_{j=0}^{J_{max}}2^{jn/p}\exp(-c2^j)\left(\frac{1}{|B_\Omega(x,2^jt)|}\int_{B_\Omega(x,2^jt)}|f|^p\,d\lambda_{\mathcal{M}}\right)^{1/p}.\end{aligned} \tag{4.17}$$

Since $B_\Omega(x,t)$ is contained in each of the balls $B_\Omega(x,2^jt)$, $j\ge 0$, the desired bound in terms of the non-centered maximal function follows readily. □

Finally, we have

Corollary 4.5. *Let $x\in\Omega$, $t>0$, and set $B := B(x,t)$, $B_\Omega := B\cap\Omega$. Let $q<2$ be as in Corollary 4.3. Then for $f\in L^q(\Omega;\Lambda^\ell)$, and for each $i\in\mathbb{N}$, we have*

$$\left(\frac{1}{|B_\Omega|}\int_{B_\Omega}\left|\big(R(t)\big)^i f\right|^2 d\lambda_{\mathcal{M}}\right)^{1/2} \le C_i\operatorname{ess\,inf}_{B_\Omega}\Big(M_\Omega(|f|^q)\Big)^{1/q}.$$

Sketch of proof. The proof follows that of the previous corollary *mutatis mutandi*, using Corollary 4.3 in place of the two lemmata. We omit the details. □

5. The boundedness of the Riesz transform

We retain the same assumptions that we imposed on $\mathcal{M}$, Ω, p_Ω, q_Ω in § 4. Here we shall present the proof of Theorem 1.1. In fact, we only discuss the case of the Riesz transform (1.24), since (1.25) is handled similarly, while (1.26)-(1.27) are the Hodge duals of (1.24)-(1.25). For the convenience of the reader we state the targeted case in more precise form.

Theorem 5.1. *Assume that $b_\ell(\Omega)=0$. Then for $p\in](q_\Omega^*)',q_\Omega^*[$, where $q_\Omega^* = \frac{nq_\Omega}{n-1}$ and $\frac{1}{q_\Omega^*}+\frac{1}{(q_\Omega^*)'}=1$, the Riesz transform $T := dB_\ell^{-\frac12}$ extends to a bounded operator in $L^p(\Omega;\Lambda^\ell)$.*

Proof. The strategy of the proof is to implement the following bootstrap scheme: Assume that for a given index $p\in[2,q_\Omega[$ the operator T is bounded on $L^q(\Omega;\Lambda^\ell)$ for every $q\in[p',p]$, and set $p^* := \frac{np}{n-1}$ and $\frac{1}{p^*}+\frac{1}{(p^*)'}=1$. Then T is of weak type $((p^*)',(p^*)')$, i.e., there exists a constant $C>0$ independent of f such that

$$\sup_{\alpha>0}\alpha^{(p^*)'}|\{x\in\Omega:\ |Tf(x)|>\alpha\}| \le C\|f\|_{(p^*)'}, \tag{5.1}$$

(where, generally speaking, $\|\cdot\|_p$ denotes the norm in $L^p(\Omega;\Lambda^\ell)$). We will prove this in 5 steps, but first, we discuss certain preliminary matters.

As noted above, our domain Ω is a space of homogeneous type when equipped with the geodesic distance and the measure induced by the volume element. In particular, there exists a family of dyadic "cubes" à la M. Christ [3]. Adapted to our context, Christ's result yields, in particular, the following:

Lemma 5.2. *There exist constants $a_0>0$, $\eta>0$ and $C_1<\infty$ such that for each $j\in\mathbb{Z}$, there is a collection of open sets ("cubes") $\mathbb{D}_j := \{Q_\gamma^j\subset\Omega:\gamma\in I_j\}$, where I_j denotes some (possibly finite) index set depending on j, satisfying* *(i)* $|\Omega \setminus \cup_\gamma Q^j_\gamma| = 0$ *for all* $j \in \mathbb{Z}$

(ii) If $i \geq j$ *then either* $Q^i_\beta \subset Q^j_\gamma$ *or* $Q^i_\beta \cap Q^j_\gamma = \emptyset$.

(iii) For each (j, γ) *and each* $i < j$, *there is a unique* β *such that* $Q^j_\gamma \subset Q^i_\beta$.

(iv) Diameter $(Q^j_\gamma) \leq C_1 2^{-j}$.

(v) Each Q^j_γ *contains some ball* $B_\Omega(z^j_\gamma, a_0 2^{-j})$.

(vi) $|\{x \in Q^j_\gamma : \mathrm{dist}(x, \Omega \setminus Q^j_\gamma) \leq \tau\, 2^{-k}\}| \leq C_1\, \tau^\eta |Q^j_\gamma|$, *for all* k, γ *and for all* $\tau > 0$.

Remark 1. In the setting of a general space of homogeneous type, the dyadic parameter $1/2$ should be replaced by some constant $\delta \in (0,1)$. It is a routine matter to verify that one may take $\delta = 1/2$ in the presence of the Ahlfors-David property (4.12).

Remark 2. For our purposes, we may ignore those $j \in \mathbb{Z}$ such that $2^{-j} \geq 4a_0^{-1}\mathrm{diam}(\Omega)$.

Remark 3. We shall denote by $\mathbb{D}$ the collection of all relevant Q^j_α, i.e., $\mathcal{D} := \cup_{j:2^{-j} \leq 4a_0^{-1}\mathrm{diam}(\Omega)} \mathbb{D}_j$.

Observe that properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_j$, with $2^{-j} \leq 4a_0^{-1}\mathrm{diam}(\Omega)$, there is a ball $B_\Omega(x, r)$ such that $r \approx 2^{-j} \approx \mathrm{diam}(Q)$ and

$$B_\Omega(x, cr) \subset Q \subset B_\Omega(x, r), \tag{5.2}$$

for some uniform (and harmless) constant c. We shall write

$$B_\Omega(x, r) \approx Q \tag{5.3}$$

when $B_\Omega(x, r)$ is the ball corresponding to Q for which (5.2) holds.

With these preliminaries at hand, we now turn to the main steps of the proof of Theorem 5.1. As mentioned in the introduction, the proof is based on the techniques in [1], [2], and [12].

For each $q \in [2, p]$, we denote by K_q the norm of T in $L^q(\Omega; \Lambda^\ell)$. In the sequel, for simplicity of notation, we agree to abbreviate

$$d\lambda_{\mathcal{M}}(x) =: dx.$$

Step 1. We fix an arbitrary number $\alpha > 0$. Let $f \in L^{(p^*)'}(\Omega; \Lambda^\ell) \cap L^2(\Omega; \Lambda^\ell)$, which is dense in $L^{(p^*)'}(\Omega; \Lambda^\ell)$. As in [12], we then use a version of the Calderón-Zygmund decomposition for $|f|^{(p^*)'}$ at height $\alpha^{(p^*)'}$. More precisely, by Whitney's covering lemma (which, as is well known, extends to the present setting by virtue of the existence of Christ's dyadic grid), there exists a collection of pairwise disjoint cubes $Q_k \in \mathbb{D}$, $k \geq 1$ such that, up to a set of measure zero,

$$\big\{x \in \Omega : \, M_\Omega(|f|^{(p^*)'})(x)^{(p^*)'} > \alpha\big\} = \bigcup_{k \geq 1} Q_k, \tag{5.4}$$

and

$$\Big(\frac{1}{|Q_k|}\int_{Q_k} |f(x)|^{(p^*)'}\, dx\Big)^{\frac{1}{(p^*)'}} \leq C\alpha. \tag{5.5}$$

We then write $f = g + \sum_{k \geq 1} b_k$ where

$$g = f\chi_{(\Omega \setminus \bigcup_{k\geq 1} Q_k)} \quad \text{and} \quad b_k = f\chi_{Q_k}, \;\; k \geq 1. \tag{5.6}$$

Then we have $|g(x)| \leq c\alpha$ for almost every $x \in \Omega$ and $\|g\|_{(p^*)'} \leq \|f\|_{(p^*)'}$. Since $b_k = f$ on Q_k, we also have

$$\Big(\frac{1}{|Q_k|}\int_{Q_k} |b_k(x)|^{(p^*)'}\, dx\Big)^{\frac{1}{(p^*)'}} \leq C\alpha. \tag{5.7}$$

Moreover, qualitatively $f \in L^2(\Omega; \Lambda^\ell)$, although of course its L^2 bound will never enter quantitatively into our estimates. Consequently, by construction, $b = \sum b_k$ converges in L^2. We shall use this fact in the sequel to justify certain formal manipulations in the proof.

Step 2. We now decompose Tf as follows. For each $k \geq 1$, let $B_\Omega(x_k, t_k) \approx Q_k$ in the sense of (5.3), so that $t_k \approx \operatorname{diam} Q_k$. We then write

$$
\begin{aligned}
Tf &= Tg + T\left(\sum_{k\geq 1} b_k\right) \\
&= Tg + T\left(\sum_{k\geq 1}(1-(1-R_k)^m)b_k\right) + T\left(\sum_{k\geq 1}(1-R_k)^m b_k\right) =: I + II + III,
\end{aligned}
\tag{5.8}
$$

where

$$R_k := R(t_k) := (1 + t_k^2 B_\ell)^{-1},$$

and where m is a fixed positive integer such that $m > n/(2p)$. For future reference, we note that

$$III = \sum_{k\geq 1} T(1-R_k)^m b_k. \tag{5.9}$$

Indeed, as observed above (cf. the discussion immediately following (5.7)), we have that $\sum b_k$ converges in $L^2(\Omega;\Lambda^\ell)$, by virtue of our qualitative assumption that $f \in L^2(\Omega;\Lambda^\ell)$. Moreover, we claim that for each $i = 1, 2, ..., m$ fixed,

$$\sum_{k\geq 1}(R_k)^i b_k \text{ converges in } L^2(\Omega;\Lambda^\ell). \tag{5.10}$$

Momentarily taking this claim for granted, we have that $\sum_{k\geq 1}(1-R_k)^m b_k$ also converges in L^2, and we may then commute T with the sum to obtain (5.9), since T is bounded on $L^2(\Omega;\Lambda^\ell)$. We defer the proof of (5.10) until the end of the paper.

Following the standard approach, we may now write

$$|\{x\in\Omega : |Tf| > \alpha\}| \leq |\{x\in\Omega : |I| > \alpha/3\}| + |\{x\in\Omega : |II| > \alpha/3\}| + |\{x\in\Omega : |III| > \alpha/3\}|;$$

the contribution of I is then handled by a variant of the usual argument, using Tchebychev's inequality with exponent p', the known $L^{p'}$ boundedness of T, and the fact that

$$\int_\Omega |g|^{p'} \leq C\alpha^{p'-(p^*)'} \int_\Omega |g|^{(p^*)'} \leq C\alpha^{p'-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'}.$$

We omit the routine details.

It therefore remains to deal with II and III.

Step 3. Next, we consider the contribution of II. Since

$$(1-(1-R_k)^m) = \sum_{i=1}^m C_{m,i}\,(R_k)^i\,, \tag{5.11}$$

it is enough to prove that $\sum_{k\geq 1}(R_k)^i\, b_k \in L^{p'}(\Omega;\Lambda^\ell)$ with a suitable bound, for all $i \in \{1, ..., m\}$. Indeed, the contribution of term II may then be handled exactly like that of I. To this end, fix $h \in L^p(\Omega;\Lambda^\ell) \cap L^2(\Omega;\Lambda^\ell)$ with $\|h\|_p = 1$. By (5.10) and the self-adjointness of the resolvents, we have

$$\int_\Omega h\left(\sum_{k\geq 1}(R_k)^i\, b_k\right) = \sum_{k\geq 1}\int_\Omega h\left((R_k)^i\, b_k\right) = \sum_{k\geq 1}\int_{Q_k}\left((R_k)^i\, h\right)\, b_k. \tag{5.12}$$

At this point, we proceed as in [12, p. 511]. We recall that for each $k \geq 1$, there is an "Ω-ball" $B_\Omega(x_k, t_k) \approx Q_k$ in the sense of (5.2)-(5.3). By (5.12), we have

$$
\begin{aligned}
\Big|\Big\langle h, \sum_{k\geq 1} (R_k)^i\, b_k \Big\rangle\Big| &\overset{(1)}{\leq} \sum_{k\geq 1} |Q_k|^{1/p^*} \|b_k\|_{(p^*)'} \left(\frac{1}{|Q_k|} \int_{Q_k} |\,(R_k)^i\, h|^{p^*} \right)^{1/p^*} \\
&\overset{(2)}{\leq} \sum_{k\geq 1} C\alpha\, |Q_k| \left(\operatorname{ess\,inf}_{Q_k} [M_\Omega(|h|^p)]^{\frac{1}{p}} \right) \\
&\overset{(3)}{\leq} \sum_{k\geq 1} C\alpha \int_{Q_k} [M_\Omega(|h|^p)]^{\frac{1}{p}} \\
&\overset{(4)}{=} C\alpha \int_{\cup_{k\geq 1} Q_k} [M_\Omega(|h|^p)]^{\frac{1}{p}} \\
&\overset{(5)}{\leq} C\alpha \Big| \bigcup_{k\geq 1} Q_k \Big|^{\frac{1}{p'}} \| |h|^p \|_1^{\frac{1}{p}} \\
&\overset{(6)}{\leq} C\alpha \Big| \{ x \in \Omega :\, (M_\Omega(|f|^{(p^*)'})(x))^{\frac{1}{(p^*)'}} > \alpha \} \Big|^{\frac{1}{p'}}, \qquad (5.13)
\end{aligned}
$$

where in the second line we have used (5.7) and Corollary 4.4, taking $B_\Omega = B_\Omega(x_k, t_k) \approx Q_k$; the third and fourth lines are trivial; line 5 is Kolmogorov's characterization of weak-L^1 (see, e.g., [6], p. 102), and the last inequality is just the definition of the cubes Q_k in (5.4) and the fact that $\|h\|_p = 1$. Taking a supremum over all h as above, we obtain that for each $i \in \{1, ..., m\}$,

$$
\Big\| \sum_{k\geq 1} (R_k)^i\, b_k \Big\|_{p'}^{p'} \leq C\alpha^{p'} \Big| \{ x \in \Omega :\, (M_\Omega(|f|^{(p^*)'})(x))^{\frac{1}{(p^*)'}} > \alpha \} \Big|.
$$

Thus, since $T : L^{p'} \to L^{p'}$, we have

$$
\begin{aligned}
\left| \left\{ x \in \Omega : \left| T \left(\sum_{k\geq 1} (R_k)^i\, b_k \right)(x) \right| > c\alpha \right\} \right| &\leq C \Big| \{ x \in \Omega :\, (M_\Omega(|f|^{(p^*)'})(x))^{\frac{1}{(p^*)'}} > \alpha \} \Big| \\
&\leq C\alpha^{-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'}, \qquad (5.14)
\end{aligned}
$$

where in the last inequality we have used the weak-type (1,1) bound for M_Ω. In turn, we obtain

$$
\left| \left\{ x \in \Omega :\, |II| > \frac{\alpha}{3} \right\} \right| \leq C\, \alpha^{-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'} \qquad (5.15)
$$

as desired.

Step 4. It remains to estimate term III. Let $B_\Omega(x_k, t_k) \approx Q_k$ retain the same significance as in Step 3 (cf. (5.2), (5.3)). We set

$$
E_\alpha^* := \Omega \setminus \left(\bigcup_k B_\Omega(x_k, 8t_k) \right),
$$

and by the usual argument involving the doubling property, it is enough to show that

$$
\big| \{ x \in E_\alpha^* : |III| > \alpha/3 \} \big| \leq C\, \alpha^{-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'} \qquad (5.16)
$$

We decompose the operator T as follows: for each $k \geq 1$ we write

$$\begin{aligned} T &= c_{m+1} \int_0^\infty d\, R(t)^{m+1}\, dt \\ &= c_{m+1} \int_0^{2t_k} d\, R(t)^{m+1}\, dt + c_{m+1} \int_{2t_k}^\infty d\, R(t)^{m+1}\, dt \\ &=: T_1^{(k)} + T_2^{(k)} \end{aligned} \tag{5.17}$$

where

$$c_{m+1} := \Big(\int_0^\infty (1+t^2)^{-(m+1)}\, dt \Big)^{-1} \quad \text{and} \quad R(t) := (1+t^2 B_\ell)^{-1}. \tag{5.18}$$

Accordingly, bearing in mind (5.9), we obtain that $III = III_1 + III_2$, where

$$III_1 := \sum_{k\geq 1} T_1^{(k)} (1-R_k)^m b_k \qquad III_2 := \sum_{k\geq 1} T_2^{(k)} (1-R_k)^m b_k$$

and it is enough to show that

$$\big| \{ x \in E_\alpha^* : |III_1| > \alpha/6 \} \big| \leq C\, \alpha^{-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'}, \tag{5.19}$$

and similarly for III_2.

Let us consider first the contribution of the term III_1. By Tchebychev's inequality, it suffices to prove that

$$\|III_1\|_{L^{p'}(E_\alpha^*)}^{p'} \leq C\, \alpha^{p'-(p^*)'} \|f\|_{(p^*)'}^{(p^*)'}. \tag{5.20}$$

In order to prove this estimate, we first establish the following technical fact.

Lemma 5.3. *Define*

$$R(t) := (1+t^2 B_\ell)^{-1}, \quad \widetilde{R}(t) := (1+t^2 B_{\ell+1})^{-1}.$$

Suppose that $f \in L^2(\Omega; \Lambda^\ell)$, $h \in L^2(\Omega; \Lambda^{\ell+1})$, *and that* $t, s > 0$. *Then*

$$\Big\langle d\, R(t) R(s) f, h \Big\rangle = \Big\langle f, \delta\, \widetilde{R}(t) \widetilde{R}(s) h \Big\rangle$$

Proof. By (3.16), we have that $d\, R(t)R(s)f = \widetilde{R}(t) dR(s) f$, so by self-adjointness of the resolvents we obtain

$$\Big\langle d\, R(t) R(s) f, h \Big\rangle = \Big\langle d\, R(s) f, \widetilde{R}(t) h \Big\rangle.$$

Since $\widetilde{R}(t)h \in D(B_{\ell+1})$, we have in particular that $\nu \vee \widetilde{R}(t) h = 0$ on $\partial\Omega$. Consequently, by (2.3), we have

$$\Big\langle d\, R(s) f, \widetilde{R}(t) h \Big\rangle = \Big\langle R(s) f, \delta\, \widetilde{R}(t) h \Big\rangle.$$

We may then obtain the conclusion of the Lemma by using the self-adjointness of $R(s)$, along with (3.19) and the commutativity of the resolvents. We leave the remaining details to the reader. □

We now return to the proof of (5.20). To this end, we write

$$III_1 = c_{m+1} \sum_k \int_0^{2t_k} dR(t)^{m+1} (1-R_k)^m b_k\, dt. \tag{5.21}$$

Let $h \in L^p(E^*_\alpha; \Lambda^{\ell+1}) \cap L^2(E^*_\alpha; \Lambda^{\ell+1})$ with $\|h\|_p = 1$, and consider

$$\begin{aligned}
(5.22)\quad \frac{1}{c_{m+1}}\langle III_1, h\rangle &= \sum_k \int_0^{2t_k} \langle dR(t)^{m+1}(1-R_k)^m b_k, h\rangle\, dt \\
&= \sum_k \int_0^{2t_k} \langle b_k, \delta\widetilde{R}(t)^{m+1}(1-\widetilde{R}_k)^m h\rangle\, dt \\
&= \sum_k \sum_{j\geq 1} \int_0^{2t_k} \langle b_k, \delta\widetilde{R}(t)^{m+1}(1-\widetilde{R}_k)^m \left(h\chi_{A_k^j}\right)\rangle\, dt \\
&= \sum_k \sum_{j\geq 1} \int_0^{2t_k} \langle b_k, t\,\delta\widetilde{R}(t)^{m+1} \left(h\chi_{A_k^j}\right)\rangle \frac{dt}{t} \\
&\quad - \sum_{i=1}^m C_{m,i} \sum_k \sum_{j\geq 1} \frac{1}{t_k}\int_0^{2t_k} \langle b_k, t_k\,\delta(\widetilde{R}_k)^i \widetilde{R}(t)^{m+1}\left(h\chi_{A_k^j}\right)\rangle\, dt =: \mathbf{A} + \mathbf{B},
\end{aligned}$$

where

$$(5.23)\qquad A_k^j := B_\Omega(x_k, 2^j 8t_k) \setminus B_\Omega(x_k, 2^{j-1}8t_k)$$

and where the second line follows by iteration of Lemma 5.3 (recall that we are working with the qualitative a priori assumption that $b \in L^2$). In the third line, we have used that h is supported in E^*_α, and in the last two lines we have used the identity (5.11) and the commutativity of resolvents.

By Hölder's inequality, (4.1) and Lemma 4.2, we have

$$\begin{aligned}
(5.24)\quad |\mathbf{A}| &\leq C \sum_k \sum_{j\geq 1} \int_0^{2t_k} \|b_k\|_{(p^*)'} |Q_k|^{\frac{1}{p^*}-\frac{1}{p}} \exp\left(-c2^j \frac{t_k}{t}\right) \|h\chi_{A_k^j}\|_p \frac{dt}{t} \\
&\leq C\,\alpha \sum_k |Q_k| \sum_{j\geq 1} \int_0^{2t_k} 2^{jn/p} \exp\left(-c2^j\frac{t_k}{t}\right) \left(2^{jn}|Q_k|\right)^{-\frac{1}{p}} \|h\chi_{A_k^j}\|_p \frac{dt}{t} \\
&\leq C\,\alpha \sum_k |Q_k| \operatorname{ess\,inf}_{Q_k} \left(M_\Omega(|h|^p)\right)^{1/p} \sum_{j\geq 1} 2^{jn/p} \int_0^{2t_k} \exp\left(-c2^j\frac{t_k}{t}\right)\frac{dt}{t} \\
&\leq C\,\alpha \sum_k \int_{Q_k} \left(M_\Omega(|h|^p)\right)^{1/p},
\end{aligned}$$

where the second line follows from (5.7), the third line from the definition of M_Ω in (4.15), the fourth line from a routine calculation. At this point, we now have a term identical to that on line 3 of (5.13), so we may continue exactly as above, using Kolmogorov's Lemma, to deduce that

$$(5.25)\qquad |\mathbf{A}|^{p'} \leq C\alpha^{p'} \left|\left\{x \in \Omega :\, \left(M_\Omega(|f|^{(p^*)'})(x)\right)^{\frac{1}{(p^*)'}} > \alpha\right\}\right|.$$

Similarly, since $t \lesssim t_k$ in the integrals in (5.22), we have by (4.1), Lemma 4.2 and our previous argument that

$$\begin{aligned}
(5.26)\quad |\mathbf{B}| &\leq \sum_{i=1}^m C_{m,i}\, C\,\alpha \sum_k \int_{Q_k} \left(M_\Omega(|h|^p)\right)^{1/p} \sum_{j\geq 1} 2^{jn/p} \exp\left(-c2^j\right) \frac{1}{t_k}\int_0^{2t_k} dt \\
&\leq C\alpha \int_{\cup Q_k} \left(M_\Omega(|h|^p)\right)^{1/p} \\
&\leq C\alpha \left|\left\{x \in \Omega :\, \left(M_\Omega(|f|^{(p^*)'})(x)\right)^{\frac{1}{(p^*)'}} > \alpha\right\}\right|^{1/p'},
\end{aligned}$$

and we may proceed as in Step 3, using (5.25) and (5.26), along with the weak-type (1,1) bound for M_Ω, to obtain (5.20).

Step 5. Finally, we consider term III_2. As before, it is enough to establish the analogue of (5.20), but with III_2 in place of III_1. An individual term in the sum defining $c_{m+1}^{-1}\,III_2$ is given by

$$
\begin{aligned}
(5.27)\quad \frac{1}{c_{m+1}} T_2^{(k)}(1-R_k)^m b_k &= \int_{2t_k}^{\infty} dR(t)^{m+1}(1-R_k)^m b_k\,dt \\
&= \int_{2t_k}^{\infty} dR(t)^{m+1}(t_k^2 B_\ell R_k)^m b_k\,dt \\
&= \int_{2t_k}^{\infty} dR(t)\left(\frac{t_k}{t}\right)^{2m}\left(t^2 B_\ell R(t)\right)^m R_k^m\, b_k\,dt \\
&= \int_{2t_k}^{\infty} t\,dR(t)\left(\frac{t_k}{t}\right)^{2m}\left(1-R(t)\right)^m R_k^m\, b_k\,\frac{dt}{t}
\end{aligned}
$$

Once again, we proceed via duality. Let $h \in L^p(E_\alpha^*;\Lambda^{\ell+1})\cap L^2(E_\alpha^*;\Lambda^{\ell+1})$ with $\|h\|_p = 1$. By (5.27) and Lemma 5.3, we have that

$$
\begin{aligned}
(5.28)\quad \frac{1}{c_{m+1}}\langle III_2, h\rangle &= \sum_k \int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}\left\langle b_k, t\,\delta\,\widetilde{R}(t)\left(1-\widetilde{R}(t)\right)^m \widetilde{R}_k^m\, h\right\rangle \frac{dt}{t} \\
&= \sum_k\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}\left\langle b_k, t\,\delta\,\widetilde{R}(t)\left(1-\widetilde{R}(t)\right)^m \widetilde{R}_k^m\,\left(h\chi_{A_k^j}\right)\right\rangle \frac{dt}{t},
\end{aligned}
$$

where again the annulus A_k^j is defined as in (5.23). Then by Lemmas 4.1 and 4.2, we have that

$$
\begin{aligned}
\left|\langle III_2, h\rangle\right| &\leq C\sum_k\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}\|b_k\|_{(p^*)'}\|h\chi_{A_k^j}\|_p \exp\left(-c\,2^j\frac{t_k}{t}\right)|Q_k|^{(1/(p^*)-1/p)}\frac{dt}{t} \\
&\leq C\,\alpha\sum_k\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}|Q_k|\,2^{jn/p}\left(\frac{1}{|B(x_k,82^j t)|}\int_{B(x_k,82^j t)}|h\chi_{A_k^j}|^p\right)^{1/p}\exp\left(-c\,2^j\frac{t_k}{t}\right)\frac{dt}{t} \\
&\leq C\,\alpha\sum_k\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}|Q_k|\,2^{jn/p}\,\mathrm{ess\,inf}_{Q_k}\left(M_\Omega(|h|^p)\right)^{1/p}\exp\left(-c\,2^j\frac{t_k}{t}\right)\frac{dt}{t} \\
&\leq C\,\alpha\sum_k\int_{Q_k}\left(M_\Omega(|h|^p)\right)^{1/p}\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}2^{jn/p}\exp\left(-c\,2^j\frac{t_k}{t}\right)\frac{dt}{t},
\end{aligned}
$$

where we have used (5.7) to obtain the second inequality, and then we have proceeded more or less as in (5.13) and (5.24). Exactly as was the case for those previous estimates, it therefore suffices to observe that

$$
\begin{aligned}
\sum_{j\geq 1}\int_{2t_k}^{\infty}\left(\frac{t_k}{t}\right)^{2m}2^{jn/p}\exp\left(-c\,2^j\frac{t_k}{t}\right)\frac{dt}{t} &= \sum_{j\geq 1}\int_2^{\infty}t^{-2m}\,2^{jn/p}\exp\left(-c\,2^j/t\right)\frac{dt}{t} \\
&= \sum_{j\geq 1}2^{jn/p}\,2^{-2mj}\int_2^{\infty}t^{-2m}\exp\left(-c/t\right)\frac{dt}{t}\leq C,
\end{aligned}
$$

as long as we choose $m > n/(2p)$.

This concludes the proof of Theorem 5.1, modulo the claim (5.10), which we now establish. It is enough to show that the partial sums $S_N := \sum_{k=1}^N (R_k)^i b_k$ form a Cauchy sequence in $L^2(\Omega;\Lambda^\ell)$. To this end, let $N, M \in \mathbb{N}$, with $M > N$, and fix $g_{N,M}\in L^2(\Omega;\Lambda^\ell)$, with $\|g_{N,M}\|_2 = 1$. By self-adjointness of the resolvents, and the definition of b_k (cf. (5.6)), we then have

$$\Big|\int_\Omega (S_M - S_N)\, g_{N,M}\Big| = \Big|\sum_{k=N+1}^{M}\int_{Q_k} f\left((R_k)^i g_{N,M}\right)\Big|$$
$$\leq \left(\sum_{k=N+1}^{M}\int_{Q_k}|f|^2\right)^{1/2}\left(\sum_{k=N+1}^{M}\int_{Q_k}\big|\left((R_k)^i g_{N,M}\right)\big|^2\right)^{1/2} =: I_{N,M} + II_{N,M}.$$

Now, $I_{N,M} \to 0$, as $N, M \to \infty$, by our qualitative assumption that $f \in L^2$. Thus, we need only show that $II_{N,M}$ is uniformly bounded in N and M. In fact, by Corollary 4.5, and (5.2)-(5.3), we have for some $q < 2$ that

$$II_{N,M} \leq C\left(\sum_{k\geq 1}|Q_k|\ \operatorname{ess\,inf}_{Q_k}\left(M_\Omega(|g_{N,M}|^q)\right)^{2/q}\right)^{1/2}$$
$$\leq C\left(\sum_{k\geq 1}\int_{Q_k}\Big(M_\Omega(|g_{N,M}|^q)\Big)^{2/q}\right)^{1/2}$$
$$\leq C\left(\int_\Omega\Big(M_\Omega(|g_{N,M}|^q)\Big)^{2/q}\right)^{1/2} \leq C,$$

where in the last line we have used disjointness of the cubes Q_k, and the fact that $\|g_{N,M}\|_2 = 1$. □

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