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On the Distributional Mellin Transformation and its Extension to Boehmian Spaces

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Abstract. A classical theory of the Mellin transform is extended to a generalized function space which is a dual of a testing function space developed by A.H. Zamanian. Support of the generalized Mellin transform is obtained on behalf of two characterization theorems. Adopting the concept of Boehmian space, basic general definitions and theorems for the transform of a tempered Boehmian are obtained.

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1. Introduction

The conventional Mellin transform of a suitably restricted $f(x)$,$x \in I(0,\infty)$, *is defined by* [19]

$$(\mu f)(s) = \int_0^\infty f(x)\, x^{s-1} dx, \qquad (1)$$

where s is *a certain strip in the complex s-plane* .

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The Mellin transform is essentially bilateral Laplace transform and, further, it can be expressed as an exponential Fourier transform as well . Let $a, b \in \mathbb{R}$ and

$$\zeta_{a,b}(x) := \begin{cases} x^{-a}, 0 < x \underline{<} 1 \\ x^{-b}, 1 < x < \infty \end{cases} . \tag{2}$$

Then, the space $\mu_{a,b}$ is the linear space of all infinitely smooth complex valued functions ϕ on $I(0, \infty)$ for which

$$y_k(\phi) := \sup_{x \in I} \left| \zeta_{a,b}(x) x^{k+1} D_x^k \phi(x) \right| \tag{3}$$

is finite, $k = 0, 1, 2, ...$

The topology equipped with $\mu_{a,b}$ is generated by the sequence of multinorms $\{y_k\}_{k \in \mathbb{N}_0}$.The space $\mu_{a,b}$ is ,in turn, a complete multinormed space. The dual space of $\mu_{a,b}$ is denoted by $\acute{\mu}_{a,b}$ which assigned the weak topology.

Denoting by D the Schwartz' space of test functions of finite support then D is a subspace of $\mu_{a,b}$ and , hence, the topology of D is stronger than that induced on D by $\mu_{a,b}$. Further, $\mu_{a,b}$ is dense in E (the space of C^∞-functions) and hence, $\acute{E}$ (the space of distributions of bounded support) is a subspace of $\acute{\mu}_{a,b}$.

We say f is a Mellin transformable generalized function if it is a member of $\acute{\mu}_{a,b}$ for some real numbers a, b . Clearly, f is a member of $\acute{\mu}_{a,b}$ for every $c \underline{>} a$ and $d \underline{<} b$.Consequently, there exist real members σ_1 and σ_2(possibily $\sigma_1 = -\infty$ and $\sigma_2 = \infty$) such that $f \in \acute{\mu}_{a,b}$ for every $a < \sigma_1$ and $b > \sigma_2$ and $f \notin \acute{\mu}_{a,b}$ for $a > \sigma_1$ and $b < \sigma_2$. The function x^{s-1} is a member of $\mu_{a,b}$ for all $s, a \underline{<} \operatorname{Re} s \underline{<} b$ .Therefore, the generalized Mellin transform of $f \in \acute{\mu}_{a,b}$ is defined by [19,p.107]

$$(\mu f)(s) := \left\langle f(x), x^{s-1} \right\rangle . \tag{4}$$

where $s \in \mathbb{C}$ and $a \underline{<} \operatorname{Re} s \underline{<} b$(we refer the reader to [19] for further properties) .The integral in (1) converges for $s = c \in \mathbb{R}$,which it converges for $s = c + it, t \in \mathbb{R}$.This suggests to assign real values for s in Theorem 2.1 and Theorem 2.2.

If ϕ is a function on $\mathbb{R}$, the closure of the set $\{x : \phi(x) \neq 0\}$ is called the support of ϕ and denoted by suppϕ. Let Φ be a testing function space and $\acute{\Phi}$ be the dual space of Φ. A generalized function $f \in \acute{\Phi}$ is said to have a compact support in a set u if $\langle f, \phi \rangle = 0$ whenever $\phi \in \Phi$ has a support in $\mathbb{R} \backslash u$. The smallest closed set having this property is called the support of f which is denoted by supp$f$.

2. The Support of the Distributional Mellin Transform

In this section we include two theorems investigating the support of distributions in the space $\mu'_{a,b}$. In the first theorem , we show that the support of the distributional Mellin transform can be obtained in view of a sequence of an infinite number of derivatives in the space $\mu'_{a,b}$. In the second theorem, we show that the support of the distributional Mellin transform, can be predicted from a convergent sequence of certain antiderivatives . This, in short, has been shown as follows :

Theorem 2.1. *Let* $f \in \mu'_{a,b}(0,\infty), \beta > 0$. *The following are equivalent*

(i) *supp* $(\mu f)(s) \subseteq [-\beta,\beta], a < s < b, a, b \in \mathbb{R}$.

(ii) $\lim_{k\to\infty} \eta^{-k}\mu\left((xD_x)^k f(x)\right)(s) = 0$,*for any real number* $\eta > \beta$.

Proof. $(i) \Rightarrow (ii)$. Assume (i) is satisfied.We prove (ii). Choose ρ such that $\beta < \rho < \eta$. Let $\psi \in C^\infty(\mathbb{R})$ be such that

$$\begin{aligned} \psi &\equiv 1 \text{ in anbhd of } [-\beta,\beta] \\ &\equiv 0 \text{ in anbhd of } \{s; |s| > \rho\}. \end{aligned}$$

For $\phi \in D$, properties of the differentiation of the Mellin transform yields

$$\begin{aligned} \left\langle \mu\left(\eta^{-k}(xD_x)^k f(x)\right)(s), \phi(s)\right\rangle &= \left\langle (-1)^k \eta^{-k} s^k (\mu f)(s), \phi(s)\right\rangle \\ &= \left\langle (\mu f)(s), (-1)^k \eta^{-k} s^k (\psi\phi)(s)\right\rangle. \end{aligned}$$

Therefore, to proof this part of the theorem we merely need to show that the sequence

$$(-1)^k \eta^{-k} s^k (\psi\phi)(s) \to 0,$$

in the sense of D ,as $k \to \infty$. For, let $k > j$ .Leibnitz' rule,being employed together with simple calculations imply

$$\begin{aligned} \left|D_s^j (-1)^k \eta^{-k} s^k (\psi\phi)(s)\right| &\leq \eta^{-k}\sum_{r=0}^{j}\begin{pmatrix} j \\ r \end{pmatrix}\left|D_s^j (2k)^r s^{2k-r}\right| \left\|D_s^{j-r}(\psi(s)\phi(s))\right\|_\infty \\ &\leq M\sum_{r=0}^{j}\begin{pmatrix} j \\ r \end{pmatrix}\left|\frac{s}{\eta}\right|^{k-r}\eta^{-k} \\ &= M\sum_{r=0}^{j}\begin{pmatrix} j \\ r \end{pmatrix}\left(\frac{\rho}{\eta}\right)^{k-r}\eta^{-k} \\ &\to 0 \ as \ k \to \infty, \text{since } \frac{\rho}{\eta} < 1. \end{aligned}$$

$(ii) \Rightarrow (i)$. Assume $\lim_{k\to\infty} \eta^{-k}\mu\left((xD_x)^k f(x)\right)(s) = 0$.Then to prove (i) we show

$$\langle(\mu f)(s), \phi(s)\rangle = 0 \text{ , for all } \phi \in D, \tag{5}$$

with the property that, $supp\phi \subset \{s \in \mathbb{R} : |s| > \beta\}$.

Compactness of ϕ leads to the existence of $\rho > \beta\,(\rho > 0)$ where

$$supp\phi \subset \{s \in \mathbb{R} : |s| > \rho\},$$

and $\beta < \eta < \rho$. Properties of differentiation of the Mellin transform [13, P.216] imply

$$\left(\mu(\eta^{-k}(xD_x)^k f(x))(s)\right)_{k\geq 0} = \left((-1)^k \eta^{-k} s^k (\mu f)(s)\right)_{k\geq 0},$$

which is, indeed, bounded.

Write $\langle(\mu f)(s), \phi(s)\rangle = \left\langle(-1)^k \eta^{-k} s^k (\mu f)(s), (-1)^k \eta^{-k} s^{-k}\phi(s)\right\rangle$ .Then due to the boundedness of the sequence $(-1)^k \eta^{-k} s^k (\mu f)(s), k = 0, 1, 2, ...,$ our objective in (5) can be fulfilled when the sequence $(-1)^k \eta^k s^{-k}\phi(s), k = 0, 1, 2, ...,$converges to zero in the sense of the topology of D .Once again, Leibnitz' rule implies

(6)

$$\left|D_s^j (-1)^k \eta^k s^{-k}\phi(s)\right| \leq \eta^k \sum_{r=0}^{j} \begin{pmatrix} j \\ r \end{pmatrix} D_s^r \left|(-1)^k (2k+j-1)^j s^{-2k-r}\right| \left\|D^{j-r}\phi(s)\right\|_\infty .$$

Calculations on (6) then yield

$$\left|D_s^j (-1)^k \eta^{-k} s^k \phi(s)\right| \leq M \sum_{r=0}^{j} r \left(\frac{\eta}{\rho}\right)^k \eta^{-k-r}. \tag{7}$$

Since $\frac{\eta}{\rho} < 1$,Expression (7) approaches 0 as $k \to \infty$ .This completes the proof of the theorem .

Theorem 2.2. *Let* $f \in \mu'_{a,b}(0,\infty)$ *and* $\beta > 0$.*The following are equivalent*

(i)*supp* $(\mu f)(s) \cap (-\beta, \beta) = \emptyset$

(ii)*There is a sequence* (θ_k)*in* $\mu'_{a,b}(0,\infty)$ *with properties:*

(a) $\theta_0(x) = f(x)$,

(b) $(xD_x)\,\theta_{k+1}(x) = \theta_k(x), k \geq 1$,

(c) $\eta^{-k}\mu(\theta_k(x))(s) \to 0$, *as* $k \to \infty$ *for all* $\eta > \beta^{-1}$.

Proof .$(ii) \Rightarrow (i)$ **.** It is sufficient to show that

$$\langle(\mu f)(s), \phi(s)\rangle = 0 \text{ for all } \phi \in D, \tag{8}$$

such that *supp* $\phi \subset [-\alpha, \alpha], 0 < \alpha < \beta$.

From (ii), we have

(9) $$f = \theta_0(x) = (xD_x)\theta_1(x) = (xD_x)^2\theta_2(x) = ... = (xD_x)^k\theta_k(x).$$

Select η such that, $\beta^{-1} < \eta < \alpha^{-1}$. Equation(9) and (ii) suggest that

$$\begin{aligned}\langle(\mu f)(s), \phi(s)\rangle &= \left\langle \mu\left((xD_x)^k\theta_k(x)\right)(s), \phi(s)\right\rangle \\ &= \left\langle (-1)^k s^k (\mu\theta_k)(s), \phi(s)\right\rangle \\ &= \left\langle \eta^{-k}(\mu\theta_k)(s), (-1)^k \eta^k s^k \phi(s)\right\rangle. \qquad (10)\end{aligned}$$

Since $\eta^{-k}(\mu\theta_k)(s)$ is bounded, (8) holds when the sequence

$$\left((-1)^k \eta^k s^k \phi(s)\right)_{k\geq 0} \to 0 \text{ in the space } D.$$

Twice again ,calculations lead to

$$\left|D_s^j (-1)^k \eta^k s^k \phi(s)\right| \leq \sum_{r=0}^{j} \binom{j}{r} \left|(-1)^k \eta^k (2k)^r |s|^{2k-r}\right| \left\|D_s^{j-r}\phi(s)\right\|_\infty$$

(11) $$< M \sum_{r=0}^{j} \binom{j}{r} |\eta\alpha|^{2k} \eta^r \alpha^{-r}.$$

Since $|s| < \alpha$,we have$|s\eta| < |\eta\alpha| < 1$.Hence, Expression (11) approaches zero as $k \to \infty$.

$(i) \Rightarrow (ii)$ Assume that (i) holds. Define a sequence $(\theta_k)_{k\geq 0}$ by

$$(\mu\theta_0)(s) = (\mu f)(s) \text{ and } (\mu\theta_k)(s) = (-1)^k s^k (\mu f)(s), k\geq 1.$$

Since μf vanishes in a nbhd of zero ,the sequence (θ_k) is, indeed, well-defined.

Let $\eta > \beta^{-1}$.Choose α such that

(12) $$\eta^{-1} < \alpha < \beta.$$

Then if ψ is a C^∞-function satisfying

$$\begin{aligned}\psi(s) &\equiv 1, |s| \geq \beta - \frac{\beta - \alpha}{2} \\ &\equiv 0, |s| \leq \alpha.\end{aligned}$$

Then $\psi(\mu f)(s) = (\mu f)(s)$.Therefore,

$$\begin{aligned}\left\langle \eta^{-k}(\mu\theta_k)(s), \phi(s)\right\rangle &= \left\langle (-1)^k \eta^{-k} s^k \psi(\mu f)(s), \phi(s)\right\rangle \\ &= \left\langle (\mu f)(s), (-1)^k \eta^{-k} s^k (\phi\psi)(s)\right\rangle.\end{aligned}$$

From above it is sufficient to show that

$$(-1)^k \eta^{-k} s^k (\phi\psi)(s) \to 0, \quad \text{in the sense of } D.$$

The proof of this relation is already obtained by the first part of Theorem 2.1. and, thus avoided.The theorem is completely proved.

3. **Tempered Bohemian**

One of the most recent generalizations of Schwartz' theory of distributions is the theory of Boehmians. The concept of Boehmians is motivated by regular operators. The space of Boehmians is obtained by an algebraic construction similar to that of the field of quotient [8]. Applying this construction to various function spaces yields various spaces of all regular operators, distributions and further, the Roumieu and Beurling types ultradistributions [2],[3],[4],[5],[6],[10][15],[16] and [17] are Boehmians.

As Boehmians are represented by convolution quotients, integral transforms have natural extensions onto defined various spaces of Beohmians [1,11,12].

In [11, Lemma1] Mikusinski shows that the Fourier transform of an integrable Beohmian is a continuous function .On the other hand, he shows that every distribution is the Fourier transform of a tempered Boehmian [12, Theorem 3.4].Therefore, the Fourier transform of a tempered Boehmian is a subspace of the Schwartz space of distributions.

Denote by κ *the space of infinitely smooth functions possessing the property that for some polynomial* $p(x)$ *we have* $|f(x)| \underline{\leq} p(x), \forall x \in \mathbb{R}$.

Recalling [12,p.p.813] , *a sequence* (ϕ_n)*of elements in* D *is said to be a delta sequence if and only if*

(i) $\int \phi_n = 1$ *for all* $n \in \mathbb{N}$,

(ii)$\int |\phi_n| \underline{\leq} C$ *for some constant* C *and all* $n \in \mathbb{N}$,

(iii)$supp\phi_n \to 0$ *as* $n \to \infty$.

If $f_n \in \kappa$ *and* (ϕ_n) *is a delta sequence, then a pair of sequences* (f_n, ϕ_n) *is said to be a quotient of sequences, denoted by* (f_n/ϕ_n) *if* $f_n * \phi_m = f_m * \phi_n, \forall m, n \in \mathbb{N}$,where $*$is the usual Mellin convolution.*Two quotients of sequences* f_n/ϕ_n *and* g_n/ψ_n *are said to be equivalent if* $f_n * \psi_m = g_m * \phi_n$. *The equivalence class* $[f_n/\phi_n]$ *containing* f_n/ϕ_n *is called a tempered Boehmian. The space of equivalence classes of quotients of sequences is denoted by* B_κ*and is so called the space of tempered Boehmians .The space* B_κ *is a vector space with the pointwise addition and scalar multiplication*

$$[f_n/\phi_n] + [g_n/\psi_n] = [(f_n * \psi_n + g_n * \phi_n) / (\phi_n * \psi_n)]$$

and,

$$\alpha [f_n/\phi_n] = [\alpha f_n/\phi_n].$$

The operation $*$ and the differentiation of an arbitrary tempered Boehmian in the space are ,resp.,defined by$[f_n/\phi_n] * [g_n/\psi_n] = [f_n * g_n/\phi_n * \psi_n]$ and $D_x^k [f_n/\phi_n] = \left[D_x^k f_n/\phi_n\right]$.

In B_κ, a sequence (F_n) is said to be a delta convergent, $\delta-$convergent, to a Boehmian F ,denoted by $F_n \xrightarrow{\delta} F$, if there is a delta sequence (δ_n) such that $(F_n * \delta_n), (F * \delta_n) \in B_\kappa$,and $(F_n * \delta_k) \to (F * \delta_k)$,for all $k, n \in \mathbb{N}$.

4. **Mellin Transform of Tempered Boehmians.**

Lemma 4.1. *Let $F = [f_n/\delta_n] \in B_\kappa$. Then $\mu(F)$ converges uniformly on each compact subset of $I(0,\infty)$.*

Proof. As $F * \delta_n = f_n$ we have $\mu(F * \delta_n) = \mu(f_n)$.Thus, $\mu(F)\mu(\delta_n) = \mu(f_n)$.Since δ_n is a delta sequence, $\mu(\delta_n) \to 1$ as $n \to \infty$ on compact subsets of $I(0,\infty)$ [18].Therefore,

$$\mu(F) = \lim_{n\to\infty} \mu(f_n). \tag{13}$$

Hence, the Mellin transform of a tempered Boehmian can be viewed as the limit $\lim_{n\to\infty} \mu(f_n)$ where this limit is obtained by uniform convergence on compact subsets of $I(0,\infty)$.To verify that this definition is well defined , assume $F = G$ in B_κ where ,$F = [f_n/\phi_n]$ and $G = [g_n/\psi_n]$. $F = G$, implies f_n/ϕ_n and g_n/ψ_n are equivalent quotients of sequences in B_κ.That is, $f_n * \psi_m = g_m * \phi_n$ .Then, $\mu(f_n * \psi_m) = \mu(g_m * \phi_n) = \mu(g_n * \phi_m)$,by the properties of delta sequences.Hence,employing the convolution properties we have

$$\lim_{n\to\infty} \mu f_n = \lim_{n\to\infty} \mu g_n.$$

Lemma 4.2. *If $F, G \in B_\kappa$ and $\alpha, \beta \in \mathbb{C}$ then,*

$$\alpha F + \beta G \in B_\kappa \text{ and } \mu(\alpha F + \beta G) = \alpha\mu(F) + \beta\mu(G).$$

Proof. *Let $F = [f_n/\phi_n]$* and $G = [g_n/\psi_n] \in B_\kappa$.In view of Equ.13 we have

$$\mu(F) = \lim_{n\to\infty} \mu f_n \text{ and } \mu(G) = \lim_{n\to\infty} \mu g_n.$$

Since $\alpha f_n * \psi_n + \beta g_n * \phi_n \in \kappa$ and the convolution of delta sequences is a delta sequence we have

$$\alpha F + \beta G = [((\alpha f_n * \psi_n) + (\beta g_n * \phi_n)) / (\phi_n * \psi_n)] \in B_\kappa$$

Linearity of the Mellin transform implies

$$\begin{aligned} \mu(\alpha f_n * \psi_n + \beta g_n * \phi_n) &= \mu(\alpha f_n * \psi_n) + \mu(\beta g_n * \phi_n) \\ &= \alpha(\mu f_n)(\mu\psi_n) + \beta(\mu g_n)(\mu\phi_n). \end{aligned} \tag{14}$$

Therefore, according to (14) and due to the algebraic properties of distributions we write

$$\begin{aligned}\mu(\alpha F+\beta G) &= \lim_{n\to\infty}\left[\alpha(\mu f_n)(\mu\psi_n)+\beta(\mu g_n)(\mu\phi_n)\right]\\ &= \alpha\lim_{n\to\infty}\mu f_n+\beta\lim_{n\to\infty}\mu g_n\\ &= \alpha\mu F+\beta\mu G\end{aligned}$$

which represents a Schwartz distribution .The proof of the lemma ,thus ,com-

pletes.

Theorem 4.3. *The Mellin transform of a tempered Boehmian is linear.*

Proof of above theorem is a straight forward consequence of Lemma 4.2 and thus avoided.

5. Comparative Study

Let $F=[f_n/\phi_n]\in B_\kappa$.It worths noting that if *the kth dirivative of the tempered Boehmian is defined by the relation*

$$(xD_x)^k F=\left[\left((xD_x)^k f_n*\phi_n\right)/(\phi_n*\phi_n)\right],k=1,2,...,$$

then the derivative is well-defined ,since every rapidly decreasing function is a tempered function.We have the following assertions:

Theorem 5.1 *Let* $F\in B_\kappa$ *then ,on compact subsets of* $I(0,\infty)$,we have

$$\mu\left((xD_x)^k F\right)=(-1)^k s^k\mu(F),k=1,2,3,....$$

Proof. Properties of the *Mellin* transform and Eq.13 imply

$$\begin{aligned}\mu\left((xD_x)^k F\right) &= \mu\left[\left((xD_x)^k f_n*\phi_n\right)/(\phi_n*\phi_n)\right]\\ &= \lim_{n\to 0}\mu\left((xD_x)^k f_n*\phi_n\right)\\ &= \lim_{n\to 0}\mu\left((xD_x)^k f_n\right)(s)\,\mu(\phi_n)(s)\\ &= \lim_{n\to 0}(-1)^k s^k\mu(f_n)(s)\\ &= (-1)^k s^k\mu(F),\text{ since }\mu(\theta_n)\to 1\text{ ,on compact subsets of }I\end{aligned}$$

,as $n\to\infty$.

Theorem 5.2 Let $\alpha,\beta\in\mathbb{C},F,G\in B_\kappa$,where $F=[f_n/\phi_n]$ and $G=[g_n/\psi_n]$.Then

$$\mu\left((xD_x)^k(\alpha F+\beta G)\right)=(-1)^k\ s^k[\alpha\mu(F)+\beta\mu(G)].$$

Proof . Since the proof of the above theorem is similar to that employed for Theorem 5.1, we avoid the same.

Remark: It is interesting to observe the following possible changes

(a) If Condition (ii) is replaced by $\underset{k\to\infty}{Lim}\eta^{-k}\mu\left(x^k f^{(k)}(x)\right)(s) = 0$, in the statement of Theorem 2.1, then we obtain unbouded value

$\dfrac{\Gamma(s+k)}{\Gamma(s)} = (s+k-1)(s+k-2)\dots(s+1)s$ (by the Stirling's formula).

Hence to ensure the boundedness, in the sense of D ,we forcely need to divide this limit by the large value $(s+k)^k$.

(b) Replacing $(xD_x)\theta_{k+1}(x) = \theta_k(x), k \geq 1$ by $x\theta'_{k+1}(x) = \theta_k(x)\ k \geq 1$, in the statement of Theorem 2.2, implies

$$f = \theta_0 = x\theta'_1(x) = x^2\theta_2^{(2)}(x) = ... = x^k\theta_k^{(k)}(x).$$

Therefore, a corresponding change in the proof considered for Theorem 2.2 the new result can be asserted .

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