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ON THE SUMMATION FORMULA OF VORONOI([1])

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Abstract. A formula involving sums of the form $\sum d(n)f(n)$ and $\sum d(n)g(n)$ is derived, where $d(n)$ is the number of divisors of n, and $f(x)$, $g(x)$ are Hankel transforms of each other. Many forms of such a formula, generally known as Voronoi's summation formula, are known, but we give a more symmetrical formula. Also, the reciprocal relation between $f(x)$ and $g(x)$ is expressed in terms of an elementary kernel, the cosine kernel, by introducing a function of the class $L^2(0, \infty)$. We use L^2-theory of Mellin and Fourier-Watson transformations.

Introduction. In 1904 Voronoi [10] published the following general formula: If $\tau(n)$ is an arithmetic function and $f(x)$ is continuous and has a finite number of maxima and minima in $a<x<b$, then analytic functions $\alpha(x)$ and $\delta(x)$, dependent on $\tau(n)$, can be determined such that

$$\frac{1}{2}\sum_{n>a}^{n\leqq b} \tau(n)f(n)+\frac{1}{2}\sum_{n\geqq a}^{n<b} \tau(n)f(n) = \int_a^b f(x)\,\delta(x)\,dx+2\pi\sum_{n=1}^{\infty}\tau(n)\int_a^b f(x)\alpha(nx)\,dx.$$

One of the better known special cases of this formula is when $\tau(n)=d(n)$, the number of divisors of n, and

$$\alpha(x) = (2/\pi)K_0(4\pi x^{1/2})-Y_0(4\pi x^{1/2}), \qquad \delta(x) = \log x+2\gamma,$$

γ being Euler's constant and Y_0, K_0 denote Bessel functions of second and third kinds respectively, of order zero. This special case is generally known as Voronoi's summation formula. Later, this formula received considerable attention as a result of which many modifications were put forth by A. L. Dixon and W. L. Ferrar [2], J. R. Wilton [13], A. P. Guinand [3] and others. Most of the authors used complex analysis and in all the new forms of the Voronoi formula, the kernel used was a combination of the Bessel functions $Y_0(x)$ and $K_0(x)$.

Our object in this paper is to obtain a more symmetric and simplified form of Voronoi's formula, which holds under simple conditions. We state below the main result. First, a definition, due to Miller [6] and Guinand [4].

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DEFINITION. A function $f(x) \in G_\lambda^p(0, \infty)$ if and only if, for a fixed $\lambda > 1/p$ and $p > 1$, there exists almost everywhere a function $f^{(\lambda)}(x)$, such that

(i) $$f(x) = \frac{1}{\Gamma(\lambda)} \int_x^\infty (t-x)^{\lambda-1} f^{(\lambda)}(t)\, dt, \qquad x > 0,$$

and

(ii) $$x^\lambda f^{(\lambda)}(x) \in L^p(0, \infty).$$

The function $f^{(\lambda)}(x)$ is the λth derivative (apart from a factor $(-1)^\lambda$) of $f(x)$ when λ is an integer. It can be shown that if $f(x) \in G_\lambda^2(0, \infty)$, then

(1.1) $$x^{r+1/2} f^{(r)}(x) \to 0 \quad \text{as } x \to 0 \text{ or } \infty, \quad 0 \leqq r < \lambda,$$

and that G_λ^2 is a subclass of L^2. In this paper we shall use the class $G_1^2(0, \infty)$. The properties (i) and (ii), in this case, simply mean that (i) $f(x)$ is the integral of its derivative $f'(x)$ (apart from the factor -1) and (ii) $xf'(x) \in L^2(0, \infty)$.

MAIN THEOREM. *Let $\phi(x) \in G_1^2(0, \infty)$. Then there exist functions $f(x)$ and $g(x)$, both $\in G_1^2(0, \infty)$, defined by*

$$f(x) = 2 \int_0^\infty \phi(t) \cos 2\pi xt\, dt, \qquad x > 0,$$

and

$$g(x) = 2 \int_0^\infty \frac{1}{t} \phi\left(\frac{1}{t}\right) \cos 2\pi xt\, dt, \qquad x > 0,$$

such that

$$\lim_{N\to\infty} \left\{ \sum_{n=1}^N d(n) f(n) - \int_0^N (\log t + 2\gamma) f(t)\, dt \right\}$$
$$= \lim_{N\to\infty} \left\{ \sum_{n=1}^N d(n) g(n) - \int_0^N (\log t + 2\gamma) g(t)\, dt \right\},$$

where γ is Euler's constant.

This symmetric form of Voronoi's formula could be derived from a general formula [3] of A. P. Guinand, if we had used the kernel $-Y_0(4\pi x^{1/2}) + (2/\pi) K_0(4\pi x^{1/2})$ and employed sophisticated order results. In our proof we make use of easily derived and elementary results, using the theory of mean convergence of functions of $L^2(0, \infty)$.

DEFINITION 2. A kernel $k(x) \in D^2$ if and only if

(i) there is defined a.e. in $(-\infty, \infty)$ a function $K(\frac{1}{2}+it)$, such that $|K(\frac{1}{2}+it)| = 1$, $K(\frac{1}{2}+it)K(\frac{1}{2}-it) = 1$;

(ii) the function $k_1(x)$, defined a.e. by

$$\frac{k_1(x)}{x} = \frac{1}{2\pi i} \underset{T\to\infty}{\text{l.i.m.}} \int_{1/2-iT}^{1/2+iT} \frac{K(s)}{1-s} x^{-s}\, ds,$$

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may be chosen, so that

(a) $k_1(x)$ is differentiable, $k_1(x)=\int_0^x k(t)\,dt$,

(b) $k_1(x)$ is $O(x^{1/2})$, $x\to\infty$, and $O(x^{1/2})$, $x\to 0$,

(c) $k(x)\in L(1/n, n)$, for all finite $n>0$.

Such a class of kernels is due to J. B. Miller [7].

The following results can be deduced from the functional relations and expansions of Bessel functions $Y_n(x)$ and $K_n(x)$ [12, pp. 62–80]. If $L_n(x)=-Y_n(x)-(2/\pi)K_n(x)$ and $M_n(x)=-Y_n(x)+(2/\pi)K_n(x)$, then

$$(d/dx)\{xL_1(x)\} = xM_0(x). \tag{1.2}$$

$$L_1(x) = O(x^{-1/2}), \qquad \text{as } x\to\infty, \tag{1.3}$$

and $=O(x\log x)$, as $x\to 0$.

2. **Preliminary results.** Consider the function

$$h(x) = \Big\{\sum_{n\leq x} d(n)-x(\log x+2\gamma-1)\Big\}x^{-1}. \tag{2.1}$$

Since [8, p. 262]

$$\sum_{n\leq x} d(n)-x(\log x+2\gamma-1) = O(x^{1/2}), \qquad x\to\infty,$$

therefore

$$\begin{aligned} h(x) &= O(x^{-1/2}), \qquad x\to\infty,\\ &= O(\log x), \qquad x\to 0. \end{aligned} \tag{2.2}$$

Then its Mellin transform

$$H(s) = \int_0^\infty h(x)x^{s-1}\,dx \qquad (s=\sigma+it)$$

exists for $0<\sigma<\frac{1}{2}$. Or

$$\begin{aligned} H(s) &= \int_0^1 h(x)x^{s-1}\,dx+\int_1^\infty h(x)x^{s-1}\,dx, \qquad 0<\sigma<\tfrac{1}{2},\\ &= \frac{1}{s^2}-\frac{2\gamma-1}{s}+\int_1^\infty h(x)x^{s-1}\,dx, \qquad \sigma<\tfrac{1}{2}. \end{aligned}$$

This gives the analytic continuation into $\sigma<0$. Now

$$\int_1^\infty h(x)x^{s-1}\,dx = \int_1^\infty \sum_{n\leq x} d(n)x^{s-2}\,dx-\int_1^\infty (\log x+2\gamma-1)x^{s-1}\,dx.$$

By splitting the range of integration $(1,\infty)$ into $(1,2)$, $(2,3)$, ... and solving, we get

$$\int_1^\infty \sum_{n\leq x} d(n)x^{s-2}\,dx = \frac{1}{1-s}\sum_{n=1}^\infty d(n)n^{s-1} = \frac{\zeta^2(1-s)}{1-s},$$

where $\zeta(z)$ is the Riemann-zeta function.

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Now, for $\sigma<0$,

$$\int_1^\infty (\log x+2\gamma-1)x^{s-1}\,dx = \frac{1}{s^2}-\frac{2\gamma-1}{s}.$$

Hence, by analytic continuation, we obtain

$$H(s) = \zeta^2(1-s)/(1-s) \qquad (0<\sigma<\tfrac{1}{2}). \tag{2.3}$$

Since $x^{\sigma-1}h(x)\in L^2(0,\infty)$, $0<\sigma<\frac{1}{2}$, by Mellin's inversion formula

$$\tfrac{1}{2}\{h(x+0)+h(x-0)\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}\frac{\zeta^2(1-s)}{1-s}x^{-s}\,ds.$$

Next we shall show that $H(s)\in L^2(-\infty,\infty)$ on $s=\frac{1}{2}+it$ and deduce that

$$h(x)\in L^2(0,\infty).$$

Now [8, p. 92]

$$\zeta(\tfrac{1}{2}+it) = O(t^{1/6}\log t), \qquad t\to\infty.$$

Therefore $\zeta^2(1-s)/(1-s)\in L^2(\frac{1}{2}-i\infty,\frac{1}{2}+i\infty)$ and has a Mellin transform $h_1(x)$, say, belonging to $L^2(0,\infty)$, defined by

$$h_1(x) = \frac{1}{2\pi i}\operatorname*{l.i.m.}_{T\to\infty}\int_{1/2-iT}^{1/2+iT}\frac{\zeta^2(1-s)}{1-s}x^{-s}\,ds$$

a.e. for $x>0$. Let C be the contour $(\sigma-iT,\frac{1}{2}-iT,\frac{1}{2}+iT,\sigma+iT,\sigma-iT)$. By Cauchy's Theorem

$$\int_C\frac{\zeta^2(1-s)}{1-s}x^{-s}\,ds = 0, \qquad 0<\sigma<\tfrac{1}{2},$$

the integrals along the lines $(\sigma-iT,\frac{1}{2}-iT)$ and $(\frac{1}{2}+iT,\sigma+iT)$ vanish as $T\to\infty$, since [8, p. 82] $\zeta(\sigma+it)=O(t^{1/2-\sigma/2})$, $0<\sigma<1$.

We have then

$$\operatorname*{l.i.m.}_{T\to\infty}\int_{1/2-iT}^{1/2+iT}\frac{\zeta^2(1-s)}{1-s}x^{-s}\,ds = \lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT}\frac{\zeta^2(1-s)}{1-s}x^{-s}\,ds$$

a.e. Or, $h_1(x)=h(x)$ a.e. and hence $h(x)\in L^2(0,\infty)$.

Let us define a function

$$A(x) = \operatorname*{l.i.m.}_{T\to\infty}\frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{\mathscr{K}(s)}{1-s}x^{1-s}\,ds, \tag{2.4}$$

where $\mathscr{K}(s)=\psi(1-s)/\psi(s)$ and $\psi(s)=\sum_{n=1}^\infty d(n)n^{-s}$.

Thus $\psi(s)=\zeta^2(s)$ and using the functional equation

$$\zeta(s) = 2^s\pi^{s-1}(\sin\tfrac{1}{2}s\pi)\Gamma(1-s)\zeta(1-s)$$

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we obtain

$$\mathscr{K}(s) = 4(2\pi)^{-2s}\Gamma^2(s)\cos^2 \tfrac{1}{2}s\pi. \tag{2.5}$$

Now

$$|\mathscr{K}(\tfrac{1}{2}+it)| = 1, \qquad \mathscr{K}(\tfrac{1}{2}+it)\mathscr{K}(\tfrac{1}{2}-it) = 1 \tag{2.6}$$

and consequently, on the line $s=\frac{1}{2}+it$, $|\mathscr{K}(s)/(1-s)|=O(t^{-1})$ and thus belongs to $L^2(-\infty,\infty)$ when integrated with respect to t. Hence the integral (2.4) converges in mean square. Also, $x^{-1}A(x)\in L^2(0,\infty)$ and $A(x)$ is a Fourier kernel in Watson's sense [11].

Substituting the value of $\mathscr{K}(s)$, obtained above, in (2.4), we have

$$A(x) = \underset{T\to\infty}{\text{l.i.m.}}\ \frac{-1}{2\pi i}\int_{1/2-iT}^{1/2+iT} 4(2\pi)^{-2s}\Gamma(s)\Gamma(s-1)\cos^2\tfrac{1}{2}s\pi\cdot x^{1-s}\,ds. \tag{2.7}$$

We shall now evaluate the above integral. It is known [9, p. 195] that for $1<\sigma<\frac{5}{4}$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(\pi)^{-1}(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)\cos\pi s\cdot x^{-s}\,ds = x^{-1/2}Y_1(4\pi x^{1/2}).$$

Moving the line of integration to $\sigma=\frac{1}{2}$ and by applying the theory of residues we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}(\pi)^{-1}(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)\cos\pi s\cdot x^{-s}\,ds \\ = x^{-1/2}Y_1(4\pi x^{1/2})+(2\pi^2x)^{-1}.\end{aligned} \tag{2.8}$$

Also, [9, p. 197] for $\sigma>1$

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}(\pi)^{-1}(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)x^{-s}\,ds = \frac{2}{\pi}x^{-1/2}K_1(4\pi x^{1/2}).$$

Moving the line of integration to $\sigma=\frac{1}{2}$, we have

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}(\pi)^{-1}(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)x^{-s}\,ds = \frac{2x^{-1/2}}{\pi}K_1(4\pi x^{1/2})-(2\pi^2x)^{-1}. \tag{2.9}$$

Now from (2.8) and (2.9),

$$\begin{aligned}-\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{2}{\pi}(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)\cos^2\pi s\cdot x^{-s}\,ds \\ = -x^{-1/2}\{Y_1(4\pi x^{1/2})+(2/\pi)K_1(4\pi x^{1/2})\}.\end{aligned}$$

Thus (2.7) yields $A(x)=x^{+1/2}L_1(4\pi x^{1/2})$.

Note that $A(x)$ is differentiable, and let $A(x)=\int_0^x \chi(t)\,dt$, from whence $\chi(x)=2\pi M_0(4\pi x^{1/2})$ by (1.2). From (1.3) and (2.6) we see that all relevant conditions are satisfied and therefore $\chi(x)$ belongs to the kernel class D^2.

Further, let

$$F(x) = \underset{T\to\infty}{\text{l.i.m.}}\ \frac{1}{2\pi i}\int_{1/2-iT}^{1/2+iT}\frac{s\mathscr{K}(s)}{(1-s)(2-s)}x^{1-s}\,ds. \tag{2.10}$$

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From (2.6), $|s\mathscr{K}(s)/(1-s)(2-s)| = O(t^{-1})$, therefore the integral (2.10) converges in mean square and $x^{-1}F(x) \in L^2(0, \infty)$. Thus $F(x)$ is a generalized Hankel kernel [11].

LEMMA 2.1. *Let $h(x)$ be defined by* (2.1). *Then*

$$\int_0^x th(t)\,dt = x\int_0^\infty h(t)\frac{F(xt)}{t}\,dt,$$

where $F(x)$ is the generalized Hankel kernel defined by (2.10).

Proof. Applying Parseval's theorem to L^2-functions $h(x)$ and $x^{-1}F(x)$, we have

$$x\int_0^\infty h(t)\frac{F(xt)}{t}\,dt = \frac{x}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{H(1-s)s\mathscr{K}(s)}{(1-s)(2-s)}\,x^{1-s}\,ds,$$

which, by (2.3) and (2.5), is

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\frac{\zeta^2(1-s)}{(1-s)(2-s)}\,x^{2-s}\,ds = \int_0^x th(t)\,dt,$$

as required.

Thus we can say that $h(x)$ is the F-transform of itself.

LEMMA 2.2. *Let $f(x) \in G_1^2(0, \infty)$. Then there exists $g(x) \in G_1^2(0, \infty)$, such that*

$$g(x) = 2\pi\int_0^\infty f(t)\chi(xt)\,dt, \qquad x > 0,$$

and

$$f(x) = 2\pi\int_0^\infty g(t)\chi(xt)\,dt, \qquad x > 0.$$

Further $xf'(x)$ and $xg'(x)$ are F-transforms of each other. Here $\chi(x) = 2\pi M_0(4\pi x^{1/2})$.

Proof. The first part is immediate by a result due to J. B. Miller [6], since the kernel $\chi(x) \in D^2$. The second part can be proved by the same method as used in the proof of Lemma 2.1.

LEMMA 2.3. *Let $\phi(x) \in G_1^2(0, \infty)$ and define $f(x)$ by the equation*

$$f(x) = 2\int_0^\infty \phi(t)\cos 2\pi xt\,dt, \qquad x > 0. \tag{2.11}$$

Then $f(x) \in G_1^2(0, \infty)$. Further, if a function $g(x)$ is defined by

$$g(x) = 2\int_0^\infty \frac{1}{t}\phi\left(\frac{1}{t}\right)\cos 2\pi xt\,dt, \qquad x > 0, \tag{2.12}$$

then $g(x) \in G_1^2(0, \infty)$.

Proof. It can be seen that $2\cos 2\pi x \in D^2$. Thus by Theorem I of J. B. Miller [6], $f(x) \in G_1^2(0, \infty)$, since $\phi(x) \in G_1^2(0, \infty)$. Similarly $g(x) \in G_1^2(0, \infty)$, provided we

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can show that $(1/x)\phi(1/x) \in G_1^2(0, \infty)$ when $\phi(x)$ does. Now,

$$x \frac{d}{dx}\left\{\frac{1}{x}\phi\left(\frac{1}{x}\right)\right\} = -\frac{1}{x}\phi\left(\frac{1}{x}\right) - \frac{1}{x^2}\phi'\left(\frac{1}{x}\right).$$

Since $\phi(x) \in G_1^2$, by property (ii), $(1/x)\phi(1/x)$ and $(1/x^2)\phi'(1/x)$ belong to $L^2(0, \infty)$, and using Minkowski's inequality, we can show that $x(d/dx)\{(1/x)\phi(1/x)\}$ also belongs to $L^2(0, \infty)$.

Also,

$$\phi(x) = \frac{1}{x}\int_0^x \frac{d}{dt}\{t\phi(t)\}\, dt.$$

Or,

$$\begin{aligned}\frac{1}{x}\phi\left(\frac{1}{x}\right) &= \int_0^{1/x} \{\phi(t) + t\phi'(t)\}\, dt \\ &= \int_x^\infty \left\{\frac{1}{u^2}\phi\left(\frac{1}{u}\right) + \frac{1}{u^3}\phi'\left(\frac{1}{u}\right)\right\} du = -\int_x^\infty \frac{d}{du}\left\{\frac{1}{u}\phi\left(\frac{1}{u}\right) du\right\}.\end{aligned}$$

Thus $(1/x)\phi(1/x)$ is the integral of its derivative, and hence $(1/x)\phi(1/x) \in G_1^2(0, \infty)$. This proves the lemma.

3. **The Main Theorem.** Applying Parseval's theorem [1] for the two pairs $h(x)$, $h(x)$ and $xf'(x)$, $xg'(x)$ of F-transforms of the class $L^2(0, \infty)$, we have

$$\int_0^\infty xh(x)f'(x)\, dx = \int_0^\infty xh(x)g'(x)\, dx. \tag{3.1}$$

The left-hand side is

$$\begin{aligned}&\int_0^\infty \left\{\sum_{n\leqq x} d(n) - x(\log x + 2\gamma - 1)\right\} f'(x)\, dx \\ &\quad = \lim_{N\to\infty} \left\{\left[\left\{\sum_{n\leqq x} d(n) - x(\log x + 2\gamma - 1)\right\} f(x)\right]_0^N \right. \\ &\qquad\qquad \left. - \int_0^N f(x)\, d\left(\sum_{n\leqq x} d(n)\right) + \int_0^N (\log x + 2\gamma) f(x)\, dx\right\}.\end{aligned}$$

Since $f(x)$ and $h(x)$ satisfy (1.1) and (2.2) respectively, the integrated term vanishes at both limits, and the above expression reduces to

$$\lim_{N\to\infty} \left\{-\sum_{n=1}^N d(n)f(n) + \int_0^N (\log x + 2\gamma) f(x)\, dx\right\}.$$

Treating the right-hand side of (3.1) in the same manner, we obtain

THEOREM 3.1. *Let $f(x) \in G_1^2(0, \infty)$. If $g(x)$ is defined by*

$$g(x) = 2\pi \int_0^\infty f(t)\chi(xt)\, dt$$

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then $g(x)$ *belongs to* $G_1^2(0, \infty)$, *where* $\chi(x)=2\pi M_0(4\pi x^{1/2})$. *Further*

$$\lim_{N\to\infty}\left\{\sum_{n=1}^{N} d(n)f(n)-\int_0^N (\log x+2\gamma)f(x)\,dx\right\}$$
$$=\lim_{N\to\infty}\left\{\sum_{n=1}^{N} d(n)g(n)-\int_0^N (\log x+2\gamma)g(x)\,dx\right\}.$$

THEOREM 3.2. *Let* $\phi(x)\in G_1^2(0,\infty)$. *If there exist functions* $f(x)$ *and* $g(x)$ *defined by the equations* (2.11) *and* (2.12), *then the equations*

$$f(x)=\int_0^\infty g(t)\chi(xt)\,dt, \qquad g(x)=\int_0^\infty f(t)\chi(xt)\,dt$$

hold for $x>0$, *where* $\chi(x)=2\pi M_0(4\pi x^{1/2})$.

Proof. Integrating by parts the integral in (2.11), we get

$$f(x)=\left[\phi(t)\frac{\sin 2\pi xt}{\pi x}\right]_{\to 0}^{\to\infty}-\int_0^\infty \phi'(t)\frac{\sin 2\pi xt}{\pi x}\,dt \tag{3.2}$$
$$=-\int_0^\infty t\phi'(t)\frac{\sin 2\pi xt}{\pi xt}\,dt.$$

The integrated term vanishes by (1.1) since $\phi(x)\in G_1^2(0,\infty)$. If $\Phi(s)$ denotes the Mellin transform of $\phi(x)$, then $-s\Phi(s)$ is the Mellin transform of $x\phi'(x)$. Now, we know that $t\phi'(t)$ and $(\sin 2\pi xt)/\pi xt$ both belong to $L^2(0,\infty)$. Therefore by applying, to the right side of (3.2), the Parseval theorem for Mellin transforms of L^2-functions, we obtain

$$f(x)=-\frac{1}{\pi i}\int_{1/2-i\infty}^{1/2+i\infty} s\Phi(s)(2\pi x)^{s-1}\Gamma(-s)\sin\tfrac{1}{2}s\pi\,ds. \tag{3.3}$$

Now, from (2.12),

$$\int_0^x g(u)\,du=\frac{1}{\pi}\int_0^\infty \phi\left(\frac{1}{t}\right)\frac{\sin 2\pi xt}{t^2}\,dt.$$

Let $G(s)$ be the Mellin transform of $g(x)$. It can be shown easily that $\phi(1-s)$ is the Mellin transform of $(1/x)\phi(1/x)$ and x^s/s is the Mellin transform of the function $1,\ 0<u<x;\ 0,\ u>x$. Applying the Parseval theorem for Mellin transforms to both sides of the last equation, we get

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} G(s)\frac{x^{1-s}}{1-s}\,ds=\frac{-1}{\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\Phi(s)(2\pi)^{-s}\Gamma(s-1)\cos\tfrac{1}{2}s\pi\cdot x^{1-s}\,ds.$$

Or,

$$\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty}\{G(s)-2(2\pi)^{-s}\Phi(s)\Gamma(s)\cos\tfrac{1}{2}s\pi\}\frac{x^{1-s}}{1-s}\,ds=0,$$

and, by Mellin inversion formula,

$$G(s)=2(2\pi)^{-s}\Phi(s)\Gamma(s)\cos\tfrac{1}{2}s\pi \tag{3.4}$$

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a.e. on $R(s)=\frac{1}{2}$. Substituting the value of $\Phi(s)$ in (3.4) and using the functional equation $\Gamma(s)\Gamma(1-s)=\pi \operatorname{cosec} \pi s$, we obtain from (3.3)

$$f(x) = -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{2}{\pi}(2\pi)^{2s-1}\Gamma(-s)\Gamma(1-s)\sin^2 \tfrac{1}{2}s\pi x^{s-1} sG(s)\,ds$$
$$= -\frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} sG(s)\mathscr{L}(1-s)\,ds,$$

say, where

$$\mathscr{L}(s) = (2/\pi)(2\pi)^{1-2s}\Gamma(s)\Gamma(s-1)\cos^2 \tfrac{1}{2}s\pi x^{-s}.$$

It can be easily deduced from the value of the integral in (2.7) that $\mathscr{L}(s)$ is the Mellin transform of $-(xt)^{1/2}L_1(4\pi(xt)^{1/2})$, when considered as a function of t. Now $xg'(x)$ and $x^{-1/2}L_1(4\pi x^{1/2})$ both belong to $L^2(0,\infty)$ due to (1.1), as $g(x)\in G_1^2$, and (1.3). Thus applying Parseval's theorem to the above pair of L^2-functions, we obtain

$$\text{(3.5)} \qquad -\int_0^\infty tg'(t)(xt)^{-1/2}L_1(4\pi(xt)^{1/2})\,dt = \frac{-1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} sG(s)\mathscr{L}(1-s)\,ds = f(x).$$

Integrating the left-hand side by parts, we can write (3.5) as

$$f(x) = -[x^{-1/2}t^{1/2}g(t)L_1(4\pi(xt)^{1/2})]_0^\infty + 2\pi\int_0^\infty g(t)M_0(4\pi(xt)^{1/2})\,dt.$$

The integrated term vanishes at both the limits by (1.1) and (1.3). Hence

$$f(x) = 2\pi\int_0^\infty g(t)M_0(4\pi(xt)^{1/2})\,dt, \qquad x>0,$$
$$= \int_0^\infty g(t)\chi(xt)\,dt,$$

as required. Similarly

$$g(x) = \int_0^\infty f(t)\chi(xt)\,dt, \qquad x>0.$$

Finally, the main theorem stated in the introduction follows by combining the results obtained in Theorems 3.1 and 3.2.

4. **An example.** Let

$$f(x) = K_0(2\pi zx), \qquad R(z)>0.$$

Then

$$\phi(x) = 2\int_0^\infty K_0(2\pi zt)\cos 2\pi xt\,dt$$
$$= \tfrac{1}{2}(z^2+x^2)^{-1/2}, \qquad \text{cf. [12, p. 388].}$$

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Now define a function

$$g(x) = 2\int_0^\infty \frac{1}{t}\,\phi\left(\frac{1}{t}\right)\cos 2\pi xt\,dt$$
$$= \int_0^\infty t^{-1}(z^2+t^{-2})^{-1/2}\cos 2\pi xt\,dt = \int_0^\infty (1+z^2t^2)^{-1/2}\cos 2\pi xt\,dt$$
$$= z^{-1}\int_0^\infty (1+u^2)^{-1/2}\cos\frac{2\pi xu}{z}\,du = z^{-1}K_0\left(\frac{2\pi x}{z}\right), \qquad R(z) > 0,$$

cf. [12, p. 434]. Also,

$$K_0(x) = O(x^{-1/2}e^{-x}), \qquad x \to \infty,$$
$$(4.1) \qquad = O(\log x), \qquad x \to 0.$$

Thus $K_0(2\pi zx)$ and $z^{-1}K_0(2\pi x/z)$, as function of x, satisfy the conditions of the main theorem, which yields the formula

$$\sum_{n=1}^\infty d(n)K_0(2\pi zn) - \int_0^\infty (\log t + 2\gamma)K_0(2\pi zt)\,dt$$
$$(4.2) \qquad = z^{-1}\sum_{n=1}^\infty d(n)K_0\left(\frac{2\pi n}{z}\right) - z^{-1}\int_0^\infty (\log t+2\gamma)K_0\left(\frac{2\pi t}{z}\right)\,dt.$$

We shall now evaluate the two integrals in (4.2). First consider

$$I_1 = \int_0^\infty (\log t + 2\gamma)K_0(2\pi zt)\,dt$$
$$= \frac{1}{2\pi z}\left\{(2\gamma - \log 2\pi z)\int_0^\infty K_0(u)\,du + \int_0^\infty \log u K_0(u)\,du\right\}.$$

Now [12, p. 388]

$$(4.3) \qquad \int_0^\infty K_0(u)\,du = \frac{\pi}{2}.$$

Let $\int_0^\infty \log u K_0(u)\,du = I$, say.

It is known that [12, p. 172] $K_0(z) = \int_1^\infty e^{-zt}(t^2-1)^{-1/2}\,dt$. Therefore

$$I = \int_0^\infty \log u\,du \int_1^\infty e^{-ut}(t^2-1)^{-1/2}\,dt = \int_1^\infty (t^2-1)^{-1/2}\,dt \int_0^\infty \log u\,e^{-ut}\,du.$$

The inversion of order of integration is justified by absolute convergence.

Now

$$\int_0^\infty \log u\,e^{-ut}\,du = -t^{-1}\log(e^\gamma t),$$

γ being Euler's constant. Thus

$$I = -\int_1^\infty t^{-1}(t^2-1)^{-1/2}\log(e^\gamma t)\,dt$$
$$(4.4) \qquad = -\gamma\int_1^\infty t^{-1}(t^2-1)^{-1/2}\,dt - \int_1^\infty t^{-1}(t^2-1)^{-1/2}\log t\,dt$$
$$= -\gamma\frac{\pi}{2} - \frac{\pi}{2}\log 2.$$

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Hence from (4.3) and (4.4)

$$I_1 = \frac{1}{2\pi z}\left\{(2\gamma - \log 2\pi z)\frac{\pi}{2} - \frac{\pi}{2}(\gamma + \log 2)\right\} = (4z)^{-1}\{\gamma - \log 4\pi z\}.$$

Next consider

$$\begin{aligned} I_2 &= z^{-1}\int_0^\infty (\log t + 2\gamma) K_0\left(\frac{2\pi t}{z}\right) dt \\ &= \frac{1}{2\pi}\left(2\gamma - \log\frac{2\pi}{z}\right)\int_0^\infty K_0(u)\,du + \frac{1}{2\pi}\int_0^\infty \log u K_0(u)\,du \\ &= \tfrac{1}{4}(2\gamma - \log(2\pi/z)) - \tfrac{1}{4}(\gamma + \log 2), \end{aligned}$$

by (4.3) and (4.4). Thus $I_2 = \frac{1}{4}(\gamma - \log(4\pi/z))$. Substituting the values of the integrals I_1 and I_2 in (4.2) and rearranging the terms, we obtain

$$\sum_{n=1}^{\infty} d(n)K(2\pi z n) - z^{-1}\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{z}\right) = \tfrac{1}{4}z^{-1}(\gamma - \log 4\pi z) - \tfrac{1}{4}(\gamma - \log(4\pi/z)),$$

which is a known formula due to N. S. Koshliakov [5].

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