KOSHLIAKOV'S FORMULA AND GUINAND'S FORMULA IN RAMANUJAN'S LOST NOTEBOOK

BRUCE C. BERNDT[1], YOONBOK LEE, and JAEBUM SOHN[2]

Abstract. On two pages in his lost notebook, Ramanujan recorded several theorems involving the modified Bessel function $K_\nu(z)$. These include Koshliakov's formula and Guinand's formula, both connected with the functional equation of nonanalytic Eisenstein series, and both discovered by these authors several years after Ramanujan's death. Other formulas, including one by K. Soni and two particularly elegant new results, are stated without proof by Ramanujan. In this paper, we prove all the formulas claimed by Ramanujan on these two pages.

Key Words: Koshliakov's formula, Guinand's formula, modified Bessel functions, nonanalytic Eisenstein series, Maass wave forms, divisor functions

2000 Mathematics Classification Numbers: Primary 33C10; Secondary 11F12, 11F20

1. Introduction

At a conference held on June 1–5, 1987 at the University of Illinois to commemorate the centenary of Ramanujan's birth, R. William Gosper remarked in his lecture that Ramanujan frequently reaches his hand from his grave to snatch your theorems from you. In less colorful language, Gosper asserted that it frequently happens that one proves an important theorem, which later is found to be ensconced somewhere in Ramanujan's writings. In other instances, we learn that Ramanujan had anticipated an important recent development in his own inimitable way.

In this paper, we examine two pages in Ramanujan's lost notebook [22, pp. 253–254] on which Gosper's observation is demonstrated once again. On page 253, Ramanujan states a version of Guinand's formula from which N.S. Koshliakov's formula follows as a corollary. Although there is no indication that Ramanujan made any connections with Epstein zeta functions or nonholomorphic Eisenstein series, it is remarkable that with these two important formulas, Ramanujan anticipated later developments which are now central in the theory of Maass wave forms on $SL(2,\mathbb{Z})$. On page 254, Ramanujan gives applications of Guinand's formula; these results are mostly new.

Koshliakov is chiefly remembered for one theorem, namely, Koshliakov's formula [16], which we now see was proved by Ramanujan about ten years earlier. To state his formula, let $K_\nu(z)$ denote the modified Bessel function of order ν [31, p. 78], and let $d(n)$ denote the number of positive divisors of the positive integer n. Then, if γ denotes Euler's constant and $a>0$,

$$\gamma - \log\left(\frac{4\pi}{a}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi a n) = \frac{1}{a}\left(\gamma - \log(4\pi a) + 4\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi n}{a}\right)\right). \tag{1.1}$$

[1]Research partly supported by NSA grant MSPF-03IG-124.

[2]Research supported by grant No. R01-2005-000-10339-0 from the Basic Research Program of the Korea Science & Engineering Foundation.

Koshliakov's proof, as well as most subsequent proofs, depends upon Voronoï's summation formula [29]

$$\sum_{a\le x\le b}{}' d(n)f(n) = \int_a^b (\log x + 2\gamma)f(x)dx \\ + \sum_{n=1}^{\infty} d(n)\int_a^b f(x)\left(4K_0(4\pi\sqrt{nx}) - 2\pi Y_0(4\pi\sqrt{nx})\right)dx, \quad (1.2)$$

where $Y_\nu(z)$ denotes the Weber-Bessel function of order ν, usually so denoted [31, p. 64]. The prime $'$ on the summation sign on the left-hand side indicates that if a or b is an integer, then only $\frac{1}{2}f(a)$ or $\frac{1}{2}f(b)$, respectively, is counted. For conditions on $f(x)$ that ensure the validity of (1.2), see, for example, Berndt's paper [3].

A. L. Dixon and W. L. Ferrar [9] also proved (1.1) by using the Voronoï summation formula. F. Oberhettinger and K. L. Soni [20] established a generalization of (1.1) using Voronoï's formula (1.2). Soni [26] derived further identities from Koshliakov's formula. In contrast to the work of these authors, Ramanujan evidently did not appeal to Voronoï's formula.

Koshliakov's formula can be considered as an analogue of the transformation formula for the classical theta function, namely,

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2/\tau} = \sqrt{\tau}\sum_{n=-\infty}^{\infty} e^{-\pi n^2\tau}, \qquad \text{Re}\,\tau > 0, \quad (1.3)$$

which, as is well known, is equivalent to the functional equation of the Riemann zeta function $\zeta(s)$ given by [28, p. 22]

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s). \quad (1.4)$$

Ferrar [10] was evidently the first mathematician to prove indeed that (1.1) can be derived from the functional equation of $\zeta^2(s)$. Oberhettinger and Soni [20] showed that this functional equation and Koshliakov's formula are equivalent.

On page 253 in his lost notebook [22], Ramanujan states (1.1) as a corollary of a more general and especially beautiful formula at the top of the same page. This more general formula is stated in an equivalent formulation in Theorem 1.1 below.

Theorem 1.1. *Let $\sigma_k(n) = \sum_{d|n} d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta = \pi^2$, and if s is any complex number, then*

$$\sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\beta) \\ = \frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\} + \frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2} - \alpha^{(1+s)/2}\}. \quad (1.5)$$

The identity (1.5) is equivalent to a formula established by A. P. Guinand [12] in 1955. The series in Theorem 1.1 are remindful of the Fourier expansion of nonanalytic

Eisenstein series on $SL(2,\mathbb{Z})$, or Maass wave forms [18], [19, 230–232], [17, pp. 15–16], [27, pp. 208–209]. This Fourier series was published by H. Maass [18] in the language of Eisenstein series in the same year, 1949, that A. Selberg and S. Chowla [24], [23, pp. 367–378] published it in the similar vein of the Epstein zeta function, but with their proof not published until several years later [25], [23, pp. 521–545]. In the meanwhile, P. T. Bateman and E. Grosswald [2] published a proof. These Eisenstein series were shown by Maass [18] to satisfy a functional equation for automorphic forms. C. J. Moreno kindly informed the author that he was easily able to derive Theorem 1.1 from the aforementioned Fourier series expansion and functional equation. One may then regard (1.5) as an equivalent formulation of the functional equation of these nonholomorphic Eisenstein series or these particular Maass wave forms. S. Zwegers [32] has recently found connections between nonholomorphic modular forms and Ramanujan's mock theta functions. We have been unable to find evidence in Ramanujan's work that he made connections of (1.5) with nonholomorphic Eisenstein series or any other nonholomorphic modular forms. The proof of Theorem 1.1 that we give below is essentially the same as that of Guinand [12] and is completely independent of any considerations with nonanalytic Eisenstein series or their closely associated Epstein zeta functions. As is well known, Ramanujan made a large number of original contributions to *analytic* Eisenstein series, many of which can be found in his lost notebook [1], [8].

On page 254, Ramanujan records formulas similar to Koshliakov's formula (1.1) or to Guinand's formula (1.5). We show that each of the three main results on this page can be deduced from Ramanujan's (and Guinand's) beautiful generalization (1.5) of Koshliakov's formula.

We close this Introduction by mentioning two recent papers by S. Kanemitsu, Y. Tanigawa, H. Tsukada, and M. Yoshimoto [13] and S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto [14], where the formulas of Koshliakov and Guinand are used or generalized.

2. Preliminary Results

The following integral evaluation in terms of the modified Bessel function $K_\nu(z)$ is the key to proving the primary theorems of this paper, and so we state it as a lemma.

Lemma 2.1. *For any complex number* s *and* $\operatorname{Re}\beta>0, \operatorname{Re}\gamma>0$,

$$\int_0^\infty x^{\nu-1}e^{-\beta/x-\gamma x}dx = 2\left(\frac{\beta}{\gamma}\right)^{\nu/2}K_\nu(2\sqrt{\beta\gamma}). \tag{2.1}$$

We use the well-known fact [11, p. 978, formula 8.469, no. 3]

$$K_{1/2}(z)=\sqrt{\frac{\pi}{2z}}e^{-z}. \tag{2.2}$$

More generally, we need the asymptotic behavior [31, p. 202]

$$K_\nu(z)\sim\sqrt{\frac{\pi}{2z}}e^{-z}, \qquad z\to\infty,$$

to ensure the convergence of series and integrals and to also justify the interchange of integration and summation several times in the sequel. We need several integrals of Bessel functions beginning with [11, p. 705, formula 6.544, no. 8]

$$\int_0^\infty K_\nu\left(\frac{a}{x}\right)K_\nu(bx)\frac{dx}{x^2}=\frac{\pi}{a}K_{2\nu}(2\sqrt{ab}),\qquad \text{Re } a>0, \text{Re } b>0. \tag{2.3}$$

We also need another related pair [26, p. 544, Equation (8)]

$$\int_0^\infty xK_0(ax)K_0(bx)dx=\frac{\log(a/b)}{a^2-b^2},\qquad a,b>0, \tag{2.4}$$

and [11, p. 697, formula 6.521, no. 3]

$$\int_0^\infty xK_\nu(ax)K_\nu(bx)dx=\frac{\pi(ab)^{-\nu}(a^{2\nu}-b^{2\nu})}{2\sin(\pi\nu)(a^2-b^2)},\qquad |\text{Re } \nu|<1,\quad \text{Re } (a+b)>0. \tag{2.5}$$

Lastly, we need the evaluation [11, p. 708, formula 6.561, no. 16], for Re $a>0$,

$$\int_0^\infty x^\mu K_\nu(ax)dx=2^{\mu-1}a^{-\mu-1}\Gamma\left(\frac{1+\mu+\nu}{2}\right)\Gamma\left(\frac{1+\mu-\nu}{2}\right),\qquad \text{Re}(\mu+1\pm\nu)>0. \tag{2.6}$$

3. A Generalization of Koshliakov's Formula

We begin by stating Ramanujan's generalization as he recorded it. Theorem 1.1 in the Introduction is a restatement of this result in terms of modified Bessel functions.

Theorem 3.1. *For any complex number s and $a>0$, define*

$$\phi_s(a):=\int_0^\infty x^{s-1}e^{-x^2-a^2/x^2}dx. \tag{3.1}$$

As usual, let $\sigma_k(n)=\sum_{d|n}d^k$, and let $\zeta(s)$ denote the Riemann zeta function. If α and β are positive numbers such that $\alpha\beta=\pi^2$, and if s is any complex number, then

$$\alpha^{(1-s)/2}\sum_{n=1}^\infty\sigma_{-s}(n)\phi_s(n\alpha)-\beta^{(1-s)/2}\sum_{n=1}^\infty\sigma_{-s}(n)\phi_s(n\beta)$$
$$=\frac{1}{4}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2}-\alpha^{(1-s)/2}\}+\frac{1}{4}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\{\beta^{(1+s)/2}-\alpha^{(1+s)/2}\}. \tag{3.2}$$

To prove Theorem 3.1, we need the following lemma.

Lemma 3.2. *Let $K_\nu(z)$ denote the modified Bessel function of order ν. If $x>0$ and* Re $\nu>0$, *then*

$$\frac{1}{4}(\pi x)^{-\nu}\Gamma(\nu)+\sum_{n=1}^\infty n^\nu K_\nu(2\pi nx)$$
$$=\frac{1}{4}\sqrt{\pi}(\pi x)^{-\nu-1}\Gamma(\nu+\tfrac{1}{2})+\frac{\sqrt{\pi}}{2x}\left(\frac{x}{\pi}\right)^{\nu+1}\Gamma(\nu+\tfrac{1}{2})\sum_{n=1}^\infty(n^2+x^2)^{-\nu-1/2}. \tag{3.3}$$

Lemma 3.2 is due to G. N. Watson [30], who proved it by using the Poisson summation formula. H. Kober [15] generalized Lemma 3.2 in two different directions. In one of them, the index n on the left-hand side of (3.3) was replaced by $n+\alpha$, $0<\alpha<1$, and in the other $\cos(2\pi n\beta)$ was introduced into the summands on the left-hand side of (3.3). Berndt [4] generalized (3.3) by putting either even or odd periodic coefficients in the infinite series of (3.3).

Proof of Theorem 3.1. Using the definition (3.1), setting $u=x^2$, using Lemma 2.1, and setting $n=kd$, we find that

$$\begin{aligned}\alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\alpha) &= \alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\int_0^{\infty}x^{s-1}e^{-x^2-n^2\alpha^2/x^2}dx\\ &= \frac{1}{2}\alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\int_0^{\infty}u^{s/2-1}e^{-u-n^2\alpha^2/u}du\\ &= \sqrt{\alpha}\sum_{n=1}^{\infty}\sigma_{-s}(n)n^{s/2}K_{s/2}(2n\alpha) \qquad (3.4)\\ &= \sqrt{\alpha}\sum_{n=1}^{\infty}\sum_{d|n}d^{-s}n^{s/2}K_{s/2}(2n\alpha)\\ &= \sqrt{\alpha}\sum_{d=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{k}{d}\right)^{s/2}K_{s/2}(2dk\alpha).\end{aligned}$$

We now invoke Lemma 3.2 on the right-hand side above to deduce that, for Re $s>0$,

$$\begin{aligned}&\alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\alpha)\\ &= \sqrt{\alpha}\sum_{d=1}^{\infty}\frac{1}{d^{s/2}}\left(-\frac{1}{4}(d\alpha)^{-s/2}\Gamma\left(\frac{s}{2}\right)+\frac{1}{4}\sqrt{\pi}(d\alpha)^{-s/2-1}\Gamma\left(\frac{s+1}{2}\right)\right.\\ &\quad\left.+\frac{\pi^{3/2}}{2d\alpha}\left(\frac{d\alpha}{\pi^2}\right)^{s/2+1}\Gamma\left(\frac{s+1}{2}\right)\sum_{n=1}^{\infty}\frac{1}{(n^2+(d\alpha/\pi)^2)^{(s+1)/2}}\right)\\ &= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)\\ &\quad+\frac{1}{2}\alpha^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\alpha^2)^{(s+1)/2}} \qquad (3.5)\\ &= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)\\ &\quad+\frac{1}{2}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2/\pi^2+d^2)^{(s+1)/2}}\\ &= -\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)\end{aligned}$$

$$+\frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\beta^2+d^2\pi^2)^{(s+1)/2}}, \tag{3.6}$$

where we used the hypothesis $\alpha\beta=\pi^2$. By symmetry, from (3.5), for Re $s>0$,

$$\begin{aligned}&\beta^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\beta)\\&=-\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)\\&\quad+\frac{1}{2}\beta^{(s+1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\sum_{d=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(n^2\pi^2+d^2\beta^2)^{(s+1)/2}}.\end{aligned} \tag{3.7}$$

Reversing the roles of the summation variables d and n in (3.7), subtracting (3.7) from (3.6), and rearranging slightly, we deduce that

$$\begin{aligned}&\alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\alpha)-\beta^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\beta)\\&=-\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)\\&\quad+\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)-\frac{1}{4}\beta^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1).\end{aligned} \tag{3.8}$$

Therefore, using the functional equation (1.4) of $\zeta(s)$ and the fact that $\alpha\beta=\pi^2$, we find that

$$\begin{aligned}\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\Gamma\left(\frac{s+1}{2}\right)\zeta(s+1)&=\frac{1}{4}\alpha^{(-s-1)/2}\sqrt{\pi}\pi^{s+1/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\\&=\frac{1}{4}\alpha^{(-s-1)/2}(\alpha\beta)^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\\&=\frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s).\end{aligned} \tag{3.9}$$

Substituting (3.9) and its analogue with the roles of α and β reversed into (3.8), we find that

$$\begin{aligned}&\alpha^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\alpha)-\beta^{(1-s)/2}\sum_{n=1}^{\infty}\sigma_{-s}(n)\phi_s(n\beta)\\&=-\frac{1}{4}\alpha^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)+\frac{1}{4}\beta^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s)\\&\quad+\frac{1}{4}\beta^{(-s+1)/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)-\frac{1}{4}\alpha^{(s+1)/2}\Gamma\left(-\frac{s}{2}\right)\zeta(-s).\end{aligned} \tag{3.10}$$

The identity (3.10) is simply a rearrangement of (3.2), and so the proof of (3.2) is complete for Re $s>0$. By analytic continuation, (3.2) is valid for all complex numbers s. □

Using Lemma 2.1 and (3.4), we can restate Theorem 3.1 in the equivalent formulation given in Theorem 1.1 in the Introduction.

Since $K_s(z) = K_{-s}(z)$ [31, p. 79, Equation (8)], we see that (1.5) is invariant under the replacement of s by $-s$.

Ramanujan completes page 253 with two corollaries, which we now state and prove.

Corollary 3.3. *Let α and β be positive numbers such that $\alpha\beta = \pi^2$. Then*

$$\sum_{n=1}^{\infty} \sigma_{-1}(n)e^{-2n\alpha} - \sum_{n=1}^{\infty} \sigma_{-1}(n)e^{-2n\beta} = \frac{\beta-\alpha}{12} + \frac{1}{4}\log\frac{\alpha}{\beta}. \tag{3.11}$$

Proof. Let $s = 1$ in Theorem 1.1. From (2.2),

$$\sqrt{\alpha n}K_{1/2}(2n\alpha) = \frac{1}{2}\sqrt{\pi}e^{-2n\alpha}. \tag{3.12}$$

Using (3.12), the values $\Gamma(-\frac{1}{2}) = -2\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$ and $\zeta(-1) = -\frac{1}{12}$ [28, p. 19], and the Laurent expansion of $\zeta(s)$ about $s = 1$ [28, p. 16, Equation (2.1.16)] in (1.5), we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_{-1}(n)e^{-2n\alpha} - \sum_{n=1}^{\infty} \sigma_{-1}(n)e^{-2n\beta} - \frac{\beta-\alpha}{12}\\
&= \frac{1}{2\sqrt{\pi}}\lim_{s\to1}\Gamma\left(\frac{s}{2}\right)\zeta(s)\{\beta^{(1-s)/2} - \alpha^{(1-s)/2}\}\\
&= \frac{1}{2}\lim_{s\to1}\left(\frac{1}{s-1}+\gamma+\cdots\right)\left(\left\{1-\frac{s-1}{2}\log\beta+\cdots\right\} - \left\{1-\frac{s-1}{2}\log\alpha+\cdots\right\}\right)\\
&= \frac{1}{4}\log\frac{\alpha}{\beta}.
\end{aligned} \tag{3.13}$$

We easily see that (3.13) is equivalent to (3.11), and so the proof is complete. ∎

Corollary 3.3 is equivalent to the identity

$$\sum_{m=1}^{\infty} \frac{1}{m(e^{2m\alpha}-1)} - \sum_{m=1}^{\infty} \frac{1}{m(e^{2m\beta}-1)} = \frac{\beta-\alpha}{12} + \frac{1}{4}\log\frac{\alpha}{\beta}. \tag{3.14}$$

To see this, expand the summands in (3.14) in geometric series and collect together all terms with the same exponents in the resulting double series. The formula (3.14) (or (3.11)) is equivalent to the transformation formula for the logarithm of the Dedekind eta function. Ramanujan stated (3.14) twice in his second notebook [21], namely as Corollary (ii) in Section 8 of Chapter 14 [5, p. 256] and as Entry 27(iii) in Chapter 16 [6, p. 43]. He also stated (3.14) in an unpublished manuscript on infinite series reproduced along with Ramanujan's lost notebook [22], in particular, as formula (29) on page 320 of [22]. (See also [7, p. 65, Entry 3.5].)

We next demonstrate that Koshliakov's formula (1.1) is a corollary of Ramanujan's Theorem 3.1.

Corollary 3.4. *Let α and β denote positive numbers such that $\alpha\beta = \pi^2$. Then, if γ denotes Euler's constant,*

$$\sqrt{\alpha}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\beta) + \sum_{n=1}^{\infty} d(n)K_0(2n\alpha)\right) = \sqrt{\beta}\left(\frac{1}{4}\gamma - \frac{1}{4}\log(4\alpha) + \sum_{n=1}^{\infty} d(n)K_0(2n\beta)\right). \tag{3.15}$$

Proof. In order to let $s \to 0$ in Theorem 1.1, we need the well-known Laurent expansions [11, p. 944, formula 8.321, no. 1]

$$\Gamma(s) = \frac{1}{s} - \gamma + \cdots \tag{3.16}$$

and [28, pp. 19–20, Equations (2.4.3), (2.4.5)]

$$\zeta(s) = -\frac{1}{2} - \frac{1}{2}\log(2\pi)s + \cdots. \tag{3.17}$$

Hence, letting $s \to 0$ in (1.5) and using (3.16) and (3.17), we find that

$$\begin{aligned}
&\sqrt{\alpha}\sum_{n=1}^{\infty} d(n)K_0(2n\alpha) - \sqrt{\beta}\sum_{n=1}^{\infty} d(n)K_0(2n\beta)\\
&= \frac{1}{4}\lim_{s\to 0}\left(\left\{\left(\frac{1}{s/2} - \gamma + \cdots\right)\left(-\frac{1}{2} - \frac{1}{2}\log(2\pi)s + \cdots\right)\right.\right.\\
&\quad \times \left.\left(\sqrt{\beta}\left\{1 - \frac{1}{2}s\log\beta + \cdots\right\} - \sqrt{\alpha}\left\{1 - \frac{1}{2}s\log\alpha + \cdots\right\}\right)\right\}\\
&\quad + \left\{\left(\frac{1}{-s/2} - \gamma + \cdots\right)\left(-\frac{1}{2} + \frac{1}{2}\log(2\pi)s + \cdots\right)\right.\\
&\quad \times \left.\left.\left(\sqrt{\beta}\left\{1 + \frac{1}{2}s\log\beta + \cdots\right\} - \sqrt{\alpha}\left\{1 + \frac{1}{2}s\log\alpha + \cdots\right\}\right)\right\}\right)\\
&= \frac{1}{4}\gamma(\sqrt{\beta} - \sqrt{\alpha}) - \frac{1}{2}\log(2\pi)(\sqrt{\beta} - \sqrt{\alpha}) + \frac{1}{4}(\sqrt{\beta}\log\beta - \sqrt{\alpha}\log\alpha)\\
&= \frac{1}{4}\gamma(\sqrt{\beta} - \sqrt{\alpha}) - \frac{1}{4}\log(4\alpha\beta)(\sqrt{\beta} - \sqrt{\alpha}) + \frac{1}{4}(\sqrt{\beta}\log\beta - \sqrt{\alpha}\log\alpha),
\end{aligned} \tag{3.18}$$

where in the last step we used the equality $\alpha\beta = \pi^2$. A simplification and rearrangement of (3.18) yields (3.15) to complete the proof. □

4. Kindred Formulas on Page 254 of the Lost Notebook

Theorem 4.1. *For $a > 0$, define*

$$\psi(a) := \int_0^{\infty} \frac{1}{x}e^{-2\pi(x+a/x)}dx. \tag{4.1}$$

Recall that $d(n)$ denotes the number of positive divisors of the positive integer n. Then, if $a > 0$,

$$\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = \sum_{n=1}^\infty d(n)\psi(na)$$
$$= \frac{a}{\pi^2}\sum_{n=1}^\infty \frac{d(n)\log(a/n)}{a^2-n^2} - \frac{1}{2}\gamma - \left(\frac{1}{4}+\frac{1}{4\pi^2 a}\right)\log a - \frac{\log(2\pi)}{2\pi^2 a}. \quad (4.2)$$

Note, that by (2.1), we can write (4.2) in the form

$$\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = 2\sum_{n=1}^\infty d(n)K_0(4\pi\sqrt{an})$$
$$= \frac{a}{\pi^2}\sum_{n=1}^\infty \frac{d(n)\log(a/n)}{a^2-n^2} - \frac{1}{2}\gamma - \left(\frac{1}{4}+\frac{1}{4\pi^2 a}\right)\log a - \frac{\log(2\pi)}{2\pi^2 a}.$$

Proof. Expanding the integrand in geometric series and using (4.1), we find that

$$\begin{aligned}\int_0^\infty \frac{dx}{x(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= \sum_{m=1}^\infty\sum_{k=1}^\infty \int_0^\infty \frac{1}{x}e^{-2\pi(mx+ak/x)}dx\\ &= \sum_{m=1}^\infty\sum_{k=1}^\infty \int_0^\infty \frac{1}{u}e^{-2\pi(u+akm/u)}du\\ &= \sum_{n=1}^\infty d(n)\int_0^\infty \frac{1}{u}e^{-2\pi(u+an/u)}du\\ &= \sum_{n=1}^\infty d(n)\psi(na),\end{aligned}$$

which proves the first part of (4.2).

The second identity in (4.2) was actually first proved in print in 1966 by Soni [26]. Her proof is short, depends on Koshliakov's formula (1.1), and uses the integral evaluations (2.3) with $\nu = 0$ and (2.4). We use her idea to prove the second major claim of Ramanujan on page 254. □

In contrast to the claims on the top and bottom thirds of page 254, the one claim in the middle of page 254 seems to be missing one element, and so we shall proceed as we think Ramanujan might have done. Define (as Ramanujan did not)

$$\rho(a) := \int_0^\infty \frac{1}{\sqrt{x}}e^{-2\pi(x+a/x)}dx, \qquad a > 0. \quad (4.3)$$

Proceeding as we did above, using (4.3), and employing Lemma 2.1, we find that

$$\int_0^\infty \frac{dx}{\sqrt{x}(e^{2\pi x}-1)(e^{2\pi a/x}-1)} = \sum_{m=1}^\infty\sum_{k=1}^\infty \frac{1}{\sqrt{m}}\int_0^\infty \frac{1}{\sqrt{u}}e^{-2\pi(u+akm/u)}du$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \sigma_{-1/2}(n) \int_0^{\infty} \frac{1}{\sqrt{u}} e^{-2\pi(u+an/u)} du \\
&= \sum_{n=1}^{\infty} \sigma_{-1/2}(n) \rho(an) \\
&= 2\sum_{n=1}^{\infty} \sigma_{-1/2}(n) (an)^{1/4} K_{1/2}(4\pi\sqrt{an}) \\
&= \frac{1}{\sqrt{2}} \sum_{n=1}^{\infty} \sigma_{-1/2}(n) e^{-4\pi\sqrt{an}}, \qquad (4.4)
\end{aligned}
$$

where we have used (2.2). Ramanujan's next claim gives an identity for the last series above, with a replaced by $a/4$.

Entry 4.2. *For $a > 0$,*

$$
\sum_{n=1}^{\infty} \sigma_{-1/2}(n) e^{-2\pi\sqrt{an}} = Ka \sum_{n=1}^{\infty} \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} + \textit{two trivial terms.} \qquad (4.5)
$$

Evidently, K on the right-hand side of (4.5) represents an unspecified constant. Ramanujan does not divulge the identities of the "two trivial terms." Our calculation in (4.4) showing a discrepancy with the series on the left-hand side of (4.5) actually provides a clue that this series in (4.5) *should be replaced by the series on the right-hand side of* (4.4). We next state a corrected version of Entry 4.2 providing the identities of the constant and the "trivial" terms.

Theorem 4.3. *If $a > 0$, then*

$$
\begin{aligned}
&\sum_{n=1}^{\infty} \sigma_{-1/2}(n) e^{-4\pi\sqrt{an}} - \frac{a}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})} \\
&\qquad = \frac{1}{2}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{\pi\sqrt{a}} - 1\right) + \frac{1}{2}\zeta\left(-\frac{1}{2}\right)\left(4\pi\sqrt{a} - \frac{1}{\pi a}\right). \qquad (4.6)
\end{aligned}
$$

Proof. In (1.5), set $s = \frac{1}{2}$ and $\alpha = x$, so that $\beta = \pi^2/x$. Hence,

$$
\begin{aligned}
&\sqrt{x} \sum_{n=1}^{\infty} \sigma_{-1/2}(n) n^{1/4} K_{1/4}(2nx) - \frac{\pi}{\sqrt{x}} \sum_{n=1}^{\infty} \sigma_{-1/2}(n) n^{1/4} K_{1/4}(2n\pi^2/x) \\
&\qquad = \frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\frac{\sqrt{\pi}}{x^{1/4}} - x^{1/4}\right) + \frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\frac{\pi^{3/2}}{x^{3/4}} - x^{3/4}\right). \qquad (4.7)
\end{aligned}
$$

Multiply both sides of (4.7) by

$$
\frac{1}{x^{5/2}} K_{1/4}(2a\pi^2/x)
$$

and integrate over $(0,\infty)$. Inverting the order of summation and integration by absolute convergence, we find that

$$\begin{aligned}&\sum_{n=1}^{\infty}\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^2}K_{1/4}(2nx)K_{1/4}(2a\pi^2/x)dx\\&\quad-\pi\sum_{n=1}^{\infty}\sigma_{-1/2}(n)n^{1/4}\int_0^{\infty}\frac{1}{x^3}K_{1/4}(2n\pi^2/x)K_{1/4}(2a\pi^2/x)dx\\&=\frac{1}{4}\Gamma\left(\frac{1}{4}\right)\zeta\left(\frac{1}{2}\right)\left(\sqrt{\pi}I_3-I_1\right)+\frac{1}{4}\Gamma\left(-\frac{1}{4}\right)\zeta\left(-\frac{1}{2}\right)\left(\pi^{3/2}I_5-I_{-1}\right),\end{aligned}\tag{4.8}$$

where

$$I_j=\int_0^{\infty}u^{j/4}K_{1/4}(2a\pi^2u)du,\tag{4.9}$$

and where, to obtain the four integrals on the right-hand side of (4.8), we made the change of variable $x=1/u$ in each one.

We examine each of the six integrals in (4.8) in turn. First, using (2.3) and (2.2), we find that

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^2}K_{1/4}(2nx)K_{1/4}(2a\pi^2/x)dx&=\frac{1}{2a\pi}K_{1/2}(4\pi\sqrt{an})\\&=\frac{1}{4\sqrt{2}a^{5/4}n^{1/4}\pi}e^{-4\pi\sqrt{an}}.\end{aligned}\tag{4.10}$$

Secondly, making the change of variable $u=\pi^2/x$ and using (2.5), we deduce that

$$\begin{aligned}\int_0^{\infty}\frac{1}{x^3}K_{1/4}(2n\pi^2/x)K_{1/4}(2a\pi^2/x)dx&=\frac{1}{\pi^4}\int_0^{\infty}uK_{1/4}(2nu)K_{1/4}(2au)du\\&=\frac{1}{\pi^4}\frac{\pi(4na)^{-1/4}(\sqrt{2n}-\sqrt{2a})}{2\sin(\pi/4)(4n^2-4a^2)}\\&=\frac{\sqrt{2}(an)^{-1/4}}{8\pi^3(n+a)(\sqrt{n}+\sqrt{a})}.\end{aligned}\tag{4.11}$$

In our calculations of I_j, $j=3,1,5,-1$, we employ (2.6). Thus,

$$I_3=2^{-1/4}(2a\pi^2)^{-7/4}\Gamma(1)\Gamma\left(\frac{3}{4}\right)=\frac{1}{4a^{7/4}\pi^{7/2}}\Gamma\left(\frac{3}{4}\right),\tag{4.12}$$

$$I_1=2^{-3/4}(2a\pi^2)^{-5/4}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)=\frac{1}{4a^{5/4}\pi^{2}}\Gamma\left(\frac{3}{4}\right),\tag{4.13}$$

$$I_5=2^{1/4}(2a\pi^2)^{-9/4}\Gamma\left(\frac{5}{4}\right)\Gamma(1)=\frac{1}{4a^{9/4}\pi^{9/2}}\Gamma\left(\frac{5}{4}\right),\tag{4.14}$$

$$I_{-1}=2^{-5/4}(2a\pi^2)^{-3/4}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)=\frac{1}{4a^{3/4}\pi}\Gamma\left(\frac{1}{4}\right).\tag{4.15}$$

Using (4.10)–(4.15) in (4.8) and making frequent use of the reflection formula

$$\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)},$$

we deduce that

$$\frac{1}{4\sqrt{2}a^{5/4}\pi}\sum_{n=1}^{\infty}\sigma_{-1/2}(n)e^{-4\pi\sqrt{an}}-\frac{1}{4\sqrt{2}a^{1/4}\pi^2}\sum_{n=1}^{\infty}\frac{\sigma_{-1/2}(n)}{(n+a)(\sqrt{n}+\sqrt{a})}$$
$$=\frac{\sqrt{2}}{16}\zeta\left(\frac{1}{2}\right)\left(\frac{1}{a^{7/4}\pi^2}-\frac{1}{a^{5/4}\pi}\right)+\frac{\sqrt{2}}{16}\zeta\left(-\frac{1}{2}\right)\left(-\frac{1}{a^{9/4}\pi^2}+\frac{4}{a^{3/4}}\right). \quad (4.16)$$

If we multiply both sides of (4.16) by $4\sqrt{2}a^{5/4}\pi$ and rearrange slightly, we obtain (4.6) to complete the proof. ∎

We record the last two results on page 254 as Ramanujan wrote them. The constant K and the "two trivial terms" are not necessarily the same as they are in Entry 4.2.

Entry 4.4. *If*

$$\chi(a) := \int_0^{\infty} e^{-2\pi(x+a/x)}dx, \qquad a>0, \quad (4.17)$$

then

$$\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)}=\sum_{n=1}^{\infty}\sigma_{-1}(n)\chi(an) \quad (4.18)$$
$$=Ka^2\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)}+\textit{two trivial terms}. \quad (4.19)$$

Proof. We prove (4.18). Expanding the integrand in geometric series, setting $mx=u$, and invoking (2.1), we find that

$$\begin{aligned}\int_0^{\infty}\frac{dx}{(e^{2\pi x}-1)(e^{2\pi a/x}-1)} &= \sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{m}\int_0^{\infty}e^{-2\pi(u+akm/u)}du\\ &= \sum_{n=1}^{\infty}\sigma_{-1}(n)\int_0^{\infty}e^{-2\pi(u+an/u)}du\\ &= \sum_{n=1}^{\infty}\sigma_{-1}(n)\chi(an)\\ &= 2\sqrt{a}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}).\end{aligned}$$

∎

Lastly, we provide and prove a more precise version of (4.19) giving the identities of the missing terms.

Theorem 4.5. *Let* $\chi(n)$ *be denoted by* (4.17) *for* $a > 0$*, and let* γ *denote Euler's constant. Then*

$$\sum_{n=1}^{\infty} \sigma_{-1}(n)\chi(an) = 2\sqrt{a}\sum_{n=1}^{\infty} \sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an})$$
$$= -\frac{a^2}{2\pi}\sum_{n=1}^{\infty}\frac{\sigma_{-1}(n)}{n(n+a)} + \frac{a}{2\pi}\left((\log a+\gamma)\zeta(2)+\zeta'(2)\right) + \frac{1}{4\pi}(\log 2a\pi+\gamma) + \frac{1}{48a\pi}. \tag{4.20}$$

Proof. In (3.11), set $\alpha = x$, so that $\beta = \pi^2/x$. Recalling (2.2), we find that

$$\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{nx}K_{1/2}(2nx) = \left(\sum_{n=1}^{\infty}\sigma_{-1}(n)e^{-2n\pi^2/x} - \frac{x}{12}\right) + \frac{1}{2}\log\frac{x}{\pi} + \frac{\pi^2}{12x}$$
$$=: I_1 + I_2 + I_3. \tag{4.21}$$

Next, multiply both sides of (4.21) by

$$\frac{1}{x^{5/2}}K_{1/2}(2a\pi^2/x)$$

and integrate over $(0,\infty)$.

Consider first the series arising on the left-hand side of (4.21). Inverting the order of summation and integration on the left-hand side by absolute convergence, we arrive at

$$\frac{2}{\sqrt{\pi}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}\int_0^{\infty}\frac{1}{x^2}K_{1/2}(2nx)K_{1/2}(2a\pi^2/x)dx = \frac{1}{a\pi^{3/2}}\sum_{n=1}^{\infty}\sigma_{-1}(n)\sqrt{n}K_1(4\pi\sqrt{an}), \tag{4.22}$$

where we have employed (2.3).

Second, the contribution from I_3 in (4.21) is given by

$$\frac{\pi^2}{12}\int_0^{\infty}x^{-7/2}K_{1/2}(2a\pi^2/x)dx = \frac{\pi^2}{12}\int_0^{\infty}u^{3/2}K_{1/2}(2a\pi^2u)du = \frac{1}{96a^{5/2}\pi^{5/2}}, \tag{4.23}$$

where we used (2.6) in the last step with $\mu = 3/2$, $\nu = 1/2$, and a replaced by $2a\pi^2$.

Third, using (2.2), we find that the contribution from I_2 in (4.21) is equal to allowdisplaybreaks

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty x^{-5/2}\log(x/\pi)K_{1/2}(2a\pi^2/x)dx \\
&= \frac{1}{4\sqrt{a\pi}}\int_0^\infty x^{-2}\log(x/\pi)e^{-2a\pi^2/x}dx \\
&= \frac{1}{8a^{3/2}\pi^{5/2}}\int_0^\infty \log(2a\pi/u)e^{-u}du \\
&= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\int_0^\infty e^{-u}\log(2a\pi)du - \int_0^\infty e^{-u}\log u\;du\right\} \\
&= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi) - \int_0^\infty e^{-u}\log u\;du\right\} \\
&= \frac{1}{8a^{3/2}\pi^{5/2}}\left\{\log(2a\pi)+\gamma\right\}, \qquad (4.24)
\end{aligned}$$

since $\gamma = -\int_0^\infty e^{-u}\log u\;du$ [11, p. 602, formula 4.331, no. 1].

Finally, the contribution from I_1 in (4.21) is given by

$$J := \int_0^\infty \left(\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x} - \frac{1}{12}x\right)x^{-5/2}K_{1/2}(2a\pi^2/x)dx. \qquad (4.25)$$

Note that $\zeta(2) = \pi^2/6$. Thus, we can write

$$\begin{aligned}
\sum_{n=1}^\infty \sigma_{-1}(n)e^{-2n\pi^2/x} - \frac{1}{12}x &= \sum_{n=1}^\infty\sum_{d|n}\frac{1}{d}e^{-2n\pi^2/x} - \frac{1}{12}x \\
&= \sum_{d=1}^\infty\sum_{m=1}^\infty\frac{1}{d}e^{-2md\pi^2/x} - \frac{1}{12}x \\
&= \sum_{d=1}^\infty\frac{1}{d}\frac{1}{e^{2d\pi^2/x}-1} - \left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}. \qquad (4.26)
\end{aligned}$$

Using (4.26) and (2.2) in (4.25), we see that

$$\begin{aligned}
J &= \int_0^\infty \left(\sum_{n=1}^\infty \frac{1}{n}\frac{1}{e^{2n\pi^2/x}-1} - \left(\sum_{n=1}^\infty\frac{1}{n^2}\right)\frac{x}{2\pi^2}\right)\frac{1}{2\sqrt{a\pi}}e^{-2a\pi^2/x}\frac{dx}{x^2} \\
&= \frac{1}{2\sqrt{a\pi}}\int_0^\infty\sum_{n=1}^\infty\frac{1}{n}\left(\frac{1}{e^{2n\pi^2/x}-1} - \frac{1}{2n\pi^2/x}\right)e^{-2a\pi^2/x}\frac{dx}{x^2}. \qquad (4.27)
\end{aligned}$$

Since, for $z>0$,

$$\frac{1}{e^z-1} - \frac{1}{z} < 0,$$

we can change the order of summation and integration by the monotone convergence theorem. Hence,

$$J = \frac{1}{2\sqrt{a\pi}} \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{\infty} \left(\frac{1}{e^{2n\pi^2/x} - 1} - \frac{1}{2n\pi^2/x} \right) e^{-2a\pi^2/x} \frac{dx}{x^2}$$
$$= \frac{1}{4\sqrt{a}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du. \tag{4.28}$$

Consider now two different expressions for the logarithmic derivative of the gamma function, namely [11, p. 952, formula 8.362, no. 1; formula 8.361, no. 8]

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(n+z)}$$
$$= \log z - \frac{1}{z} - \int_0^{\infty} \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) e^{-tz} dt,$$

where Re $z > 0$. Hence,

$$\int_0^{\infty} \left(\frac{1}{e^u - 1} - \frac{1}{u} \right) e^{-au/n} du = \log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)}. \tag{4.29}$$

Putting (4.29) in (4.28), we find that

$$J = \frac{1}{4a^{1/2}\pi^{5/2}} \sum_{n=1}^{\infty} \frac{1}{n^2} \left(\log(a/n) + \gamma - \sum_{m=1}^{\infty} \frac{a}{m(mn+a)} \right)$$
$$= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) - \sum_{n=1}^{\infty} \frac{\log n}{n^2} - \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a}{n^2 m(mn+a)} \right)$$
$$= \frac{1}{4a^{1/2}\pi^{5/2}} \left((\log a + \gamma)\zeta(2) + \zeta'(2) - a \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} \right)$$
$$= -\frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \tag{4.30}$$

We now combine all our calculations that arose from (4.21), namely, (4.22), (4.23), (4.24), (4.25), and (4.30), to deduce that

$$\frac{1}{a\pi^{3/2}} \sum_{n=1}^{\infty} \sigma_{-1}(n)\sqrt{n} K_1(4\pi\sqrt{an}) = \frac{1}{96a^{5/2}\pi^{5/2}} + \frac{1}{8a^{3/2}\pi^{5/2}} \{\log(2a\pi) + \gamma\}$$
$$- \frac{a^{1/2}}{4\pi^{5/2}} \sum_{n=1}^{\infty} \frac{\sigma_{-1}(n)}{n(n+a)} + \frac{1}{4a^{1/2}\pi^{5/2}} ((\log a + \gamma)\zeta(2) + \zeta'(2)). \tag{4.31}$$

Finally multiply both sides of (4.31) by $2\pi^{3/2}a^{3/2}$ to deduce (4.20) and complete the proof. □

The authors are pleased to acknowledge the helpful comments of Scott Ahlgren, Carlos Moreno, and Yoshio Tanigawa.

References

[1] G. E. Andrews and B. C. Berndt, *Ramanujan's Lost Notebook*, Part II, Springer, New York, to appear.

[2] P. T. Bateman and E. Grosswald, *On Epstein's zeta function*, Acta Arith. **9** (1964), 365–373.

[3] B. C. Berndt, *Identities involving the coefficients of a class of Dirichlet series.* V., Trans. Amer. Math. Soc. **160** (1971), 139–156.

[4] B. C. Berndt, *Periodic Bernoulli numbers, summmation formulas and applications*, in *Theory and Application of Special Functions*, R. A. Askey, ed., Academic Press, New York, 1975, pp. 143–189.

[5] B. C. Berndt, *Ramanujan's Notebooks, Part* II, Springer–Verlag, New York, 1989.

[6] B. C. Berndt, *Ramanujan's Notebooks, Part* III, Springer–Verlag, New York, 1991.

[7] B. C. Berndt, *An unpublished manuscript of Ramanujan on infinite series identities*, J. Ramanujan Math. Soc. **19** (2004), 57–74.

[8] B. C. Berndt and A. J. Yee, *Ramanujan's contributions to Eisenstein series, especially in his lost notebook*, in *Number Theoretic Methods – Future Trends*, C. Jia and S. Kanemitsu, eds., Kluwer, Dordrecht, 2002, pp. 31–53; abridged version, *A survey on Eisenstein series in Ramanujan's lost notebook*, in *New Aspects of Analytic Number Theory*, Y. Tanigawa, ed., Research Institute for Mathematical Sciences, Kyoto University, Kyoto, 2002, pp. 130–141.

[9] A. L. Dixon and W. L. Ferrar, *On the summation formulae of Voronoï and Poisson*, Quart. J. Math. (Oxford) **8** (1937), 66–74.

[10] W. L. Ferrar, *Some solutions of the equation* $F(t) = F(t^{-1})$, J. London Math. Soc. **11** (1936), 99–103.

[11] I. S. Gradshteyn and I. M. Ryzhik, eds., *Table of Integrals, Series, and Products*, 5th ed., Academic Press, San Diego, 1994.

[12] A. P. Guinand, *Some rapidly convergent series for the Riemann* ξ*-function*, Quart. J. Math. (Oxford) **6** (1955), 156–160.

[13] S. Kanemitsu, Y. Tanigawa, H. Tsukada, and M. Yoshimoto, *On Bessel series expressions for some lattice sums*: II, J. Physics A: Mathematics and General, **37** (2004), 719–734.

[14] S. Kanemitsu, Y. Tanigawa, and M. Yoshimoto, *On rapidly convergent series for the Riemann zeta-values via the modular relation*, Abh. Math. Sem. Univ. Hamburg **72** (2002), 187–206.

[15] H. Kober, *Transformationsformeln gewisser Besselscher Reihen Beziehungen zu Zeta-functionen*, Math. Z. **39** (1934), 609–624.

[16] N. S. Koshliakov, *On Voronoï's sum-formula*, Mess. Math. **58** (1929), 30–32.

[17] T. Kubota, *Elementary Theory of Eisenstein Series*, Kodansha, Tokyo, 1973.

[18] H. Maass, *Über eine neue Art von nichtanalytischen automorphen Funktionen and die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949), 141–183.

[19] C. J. Moreno, *Advanced Analytic Number Theory: L-Functions*, Math. Surveys and Monographs, Vol. 115, American Mathematical Society, Providence, RI, 2005.

[20] F. Oberhettinger and K. L. Soni, *On some relations which are equivalent to functional equations involving the Riemann zeta function*, Math. Z. **127** (1972), 17–34.

[21] S. Ramanujan, *Notebooks* (2 volumes), Tata Institute of Fundamental Research, Bombay, 1957.

[22] S. Ramanujan, *The Lost Notebook and Other Unpublished Papers*, Narosa, New Delhi, 1988.

[23] A. Selberg, *Collected Papers*, Vol. I, Springer-Verlag, Berlin, 1989.

[24] A. Selberg and S. Chowla, *On Epstein's zeta-function* (I), Proc. Nat. Acad. Sci. (USA) **35** (1949), 371–374.

[25] A. Selberg and S. Chowla, *On Epstein's zeta-function*, J. Reine Angew. Math. **227** (1967), 86–110.

[26] K. Soni, *Some relations associated with an extension of Koshliakov's formula*, Proc. Amer. Math. Soc. **17** (1966), 543–551.

[27] A. Terras, *Harmonic Analysis on Symmetric Spaces and Applications* I, Springer-Verlag, New York, 1985.

[28] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Clarendon Press, Oxford, 1951.

[29] M. G. Voronoï, *Sur une fonction transcendante et ses applications à la sommation de quelques séries*, Ann. École Norm. Sup. (3) **21** (1904), 207–267, 459–533.
[30] G. N. Watson, *Some self-reciprocal functions*, Quart. J. Math. (Oxford) **2** (1931), 298–309.
[31] G. N. Watson, *Theory of Bessel Functions*, 2nd ed., University Press, Cambridge, 1966.
[32] S. Zwegers, *Mock Theta Functions*, Doctoral Dissertation, Universiteit Utrecht, 2002.

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, IL 61801, USA
E-mail address: berndt@math.uiuc.edu

Department of Mathematics, Yonsei University, 134 Shinchon-dong, Seodaemun-gu, Seoul, 120-749, Korea
E-mail address: leeyb@yonsei.ac.kr

Department of Mathematics, Yonsei University, 134 Shinchon-dong, Seodaemun-gu, Seoul, 120-749, Korea
E-mail address: jsohn@yonsei.ac.kr