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On Riemann's Zeta Function

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0. Introduction

In this paper, we will consider some techniques and ideas which are relevant to the study of the Riemann zeta function. We will generalize Riemann's proof of the functional equation by introducing series similar to theta series formed with modified Hermite polynomials. Introducing the generating function for these polynomials, we obtain a new expression for the Riemann zeta function involving an arbitrary parameter u. We will consider the significance of this relation if $\zeta(s)=0$. The vanishing of this new expression yields no information if $\mathrm{re}(s)=\frac{1}{2}$, but if $\mathrm{re}(s)\neq\frac{1}{2}$, it becomes interesting to study the behavior as $u\to\infty$. In this case, we obtain a new formulation of the Riemann hypothesis.

We have also a theorem about the zeros of certain polynomials which occur as integral transforms of the modified Hermite polynomials. That the zeros of these polynomials should lie on the critical line $\mathrm{re}(s)=\frac{1}{2}$ has apparently never been noticed previously. This fact is closely tied in with the Riemann hypothesis.

Let us begin with Riemann's second proof of the analytic continuation and functional equation (cf. [5]). Let $\theta(x)$ be the theta series:

$$\theta(x)=\sum_{-\infty}^{\infty} e^{-\pi n^2 x}.$$

Riemann proved that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)=\int_1^\infty \tfrac{1}{2}(\theta(x)-1)[x^{\frac{s}{2}}+x^{\frac{1-s}{2}}]\frac{dx}{x}-\frac{1}{s}-\frac{1}{1-s}. \tag{0.1}$$

Suppose $\zeta(s)=0$. Substituting x^2 for x and equating imaginary parts in the previous formula, we obtain

$$\int_1^\infty (\theta(x^2)-1)\,H(x)=K, \tag{0.2}$$

where, as usual $s=\sigma+it$,

$$K = \frac{t(2\sigma - 1)}{|s(1-s)|^2},$$

and

$$H(x) = [x^{\sigma-1} - x^{-\sigma}] \sin(t \log(x)).$$

This relation is trivial if $\sigma = \frac{1}{2}$ or if s is real, but not otherwise. We will generalize this relation. We will prove

Theorem 1. *For all real values of u we have*

$$\int_1^\infty \Psi(x, u) H(x)\, dx = K(1 + e^{-\pi u^2}) + \mathrm{im}\left\{\pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s) \Phi(s, u)\right\}, \tag{0.3}$$

where

$$\Psi(x, u) = \sum_{\varepsilon=0}^{3} \Psi_\varepsilon(x, u),$$

$$\Psi_\varepsilon(x, u) = e^{-\pi u^2/2} \sum_{n=1}^{\infty} e^{-\pi Q(nx, i^\varepsilon u)},$$

$$Q(x, u) = x^2 - 2xu + \frac{u^2}{2},$$

and Φ is an entire function of s and u.

If $u = 0$ and $\zeta(s) = 0$, this reduces to (0.2).

In particular, if $\zeta(s) = 0$,

$$\int_1^\infty \Psi(x, u) H(x)\, dx = K(1 + e^{-\pi u^2}) \tag{0.4}$$

for all real values of u. If $\sigma = \frac{1}{2}$ or if s is real, this is trivial; otherwise, one might attempt to prove the Riemann hypothesis by deriving a contradiction from this formula. We will study the behaviour of the left-hand side as $u \to \infty$.

Theorem 2. *Assume $\frac{1}{2} < \sigma < 1$. If $\zeta(s) \neq 0$, we have:*

$$\int_1^\infty \Psi_0(x, u) H(x)\, dx = K(1 + e^{-\pi u^2}) + \mathrm{im}\{\zeta(s) u^{s-1} + \zeta(1-s) u^{-s}\} + O(u^{\sigma-3}), \tag{0.5}$$

but if $\zeta(s) = 0$, we have:

$$\int_1^\infty \Psi_0(x, u) H(x)\, dx = K(1 + e^{-\pi u^2}) + \tfrac{1}{2} t(1 - 2\sigma) \pi^{-3} \left[\sum_{n=1}^{\infty} n^{-3} e^{-\pi n^2} \sin(2\pi n u)\right] u^{-3} + O(u^{-4}). \tag{0.6}$$

The contrast between these two situations is striking. The Riemann hypothesis asserts that the latter situation is impossible.

The function $\Phi(s, u)$ in (0.3) has a Taylor expansion

$$\Phi(s, u) = 2e^{-\pi u^2/2} \sum_{k=0}^{\infty} \frac{(2\pi u)^{4k}}{(4k)!} p_{2k}(s) \tag{0.7}$$

in terms of certain polynomials $p_k(s)$, which may be defined recursively by:

$$p_0(s)=1, \qquad p_{n+1}(s)=\frac{1}{8\pi}\,[(2s-1)\,p_n(s)+s\,p_n(s+2)+(s-1)\,p_n(s-2)]. \tag{0.8}$$

The polynomials p_k satisfy the functional equation

$$p_n(1-s)=(-1)^n\,p_n(s). \tag{0.9}$$

We will prove the following very interesting property of the polynomials p_k, $k \geqq 1$.

Theorem 3. *All zeros of the polynomials* p_k *lie on the critical line* $\sigma=\frac{1}{2}$.

By (0.8), this follows from the following

Theorem 4. *If* $p(s)$ *is a polynomial all of whose zeros lie on the critical line* $\sigma=\frac{1}{2}$, *then all zeros of the polynomial*

$$(2s-1)\,p(s)+s\,p(s+2)+(s-1)\,p(s-2)$$

also lie on the critical line.

The polynomials p_k will be introduced differently in Sect. 1, as integral transforms of Hermite polynomials, and are directly connected with the Riemann zeta function by (1.9) below. It will be seen that Theorem 3 is closely tied in with the Riemann hypothesis.

1. A Differential Operator, and a Generalization of Riemann's Integral

Riemann's well-known "second proof" of the functional equation of the zeta function depends on the inversion formula for the theta function:

$$\theta(x)=\frac{1}{\sqrt{x}}\,\theta\left(\frac{1}{x}\right), \qquad \theta(x)=\sum_{-\infty}^{\infty} e^{-\pi n^2 x}. \tag{1.1}$$

This in turn depends on the Poisson summation formula, and on the "self-duality" property:

$$f_0(x)=e^{-\pi x^2}=\hat{f}_0(x), \tag{1.2}$$

where the Fourier transform of a function f is defined by the integral

$$\hat{f}(x)=\int_{-\infty}^{\infty} f(y)\,e^{2\pi i x y}\,dy.$$

Let us define recursively

$$f_n(x)=\frac{1}{2}\left(x-\frac{1}{2\pi}\frac{d}{dx}\right)f_{n-1}(x). \tag{1.3}$$

The differential operator $\frac{1}{2}\left(x-\frac{1}{2\pi}\frac{d}{dx}\right)$ is the "raising operator" in the quantum mechanical theory of the harmonic oscillator (cf. Messiah [2]). Denoting this operator temporarily as R, the relation

$$(Rf)^{\wedge} = iR\hat{f} \tag{1.4}$$

may be proved by integration by parts and differentiation under the integral sign. Therefore

$$\hat{f}_n = i^n f_n. \tag{1.5}$$

As in Messiah [2], Chap. XII, no. 7, it may be shown that

$$f_n(x) = (8\pi)^{-n/2} H_n(\sqrt{2\pi}\, x)\, e^{-\pi x^2},$$

where $H_n(x)$ is the n^{th} Hermite polynomial

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Let us denote:

$$2 \int_0^\infty f_n(x)\, x^s \frac{dx}{x} = M_n(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) p_{n/2}(s). \tag{1.6}$$

This Mellin transform is defined if $\mathrm{re}(s) > 1$. The function $p_{n/2}$ is defined whether or not n is even, but is a polynomial function of s if and only if n is even. Integrating by parts in the definition of M_n, we have

$$M_{n+1}(s) = \tfrac{1}{2} M_n(s+1) + \frac{s-1}{4\pi} M_n(s-1). \tag{1.7}$$

Applying this relation twice, and using the Gamma identity $\Gamma(s+1) = s\Gamma(s)$, we obtain the recursion (0.8), showing that p_n is a polynomial if n is an integer.

The functional equation (0.9) follows inductively from the recursion (0.8). This may also be proved directly as in Lemma 2.4.2 of Tate [6].

Let

$$\theta_j(x) = \sum_{-\infty}^{\infty} f_{2j}(n\sqrt{x}).$$

(We will not use the functions f_{2k+1}). By the Poisson summation formula, we have

$$\theta_j(x) = (-1)^j \frac{1}{\sqrt{x}}\, \theta_j\left(\frac{1}{x}\right). \tag{1.8}$$

Next we set

$$\psi_j(x) = \tfrac{1}{2}\{\theta_j(x) - f_{2j}(0)\}.$$

Using (1.6) and evaluating

$$\int_0^\infty \psi_j(x)\, x^{s/2} \frac{dx}{x}$$

as in Riemann [5], we obtain

$$p_j(s)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$$
$$=\int_1^\infty \psi_j(x)\left[x^{\frac{s}{2}}+(-1)^j x^{\frac{1-s}{2}}\right]\frac{dx}{x}-f_{2j}(0)\left[\frac{1}{s}+\frac{(-1)^j}{1-s}\right]. \tag{1.9}$$

This formula generalizes Riemann's second proof of the functional equation (note that the functional equation follows from (1.9) and (0.9)). If the Riemann hypothesis is true, then Theorem 3 above asserts that all zeros of this integral lie on the critical line $\sigma=\frac{1}{2}$. Thus, Theorem 3 may be regarded as an extension of the Riemann hypothesis.

2. Proof of Theorem 1

Our subsequent investigations depend on introducing the generating function for the functions f_n. We have

$$\sum_{k=0}^\infty \frac{(2\pi u)^k}{k!} f_k(x)=e^{-\pi Q(x,u)}, \tag{2.1}$$

where $Q(x,u)=x^2-2xu+\frac{u^2}{2}$ (cf. Messiah [2], XII.30). Recalling the definitions of $\Psi(x,u)$ and $\Psi_\varepsilon(x,u)$ in Theorem 1, we find

$$e^{\pi u^2/2}\,\Psi(x,u)=4\sum_{k=0}^\infty \frac{(2\pi u)^{4k}}{(4k)!}\psi_{2k}(x^2).$$

Interchanging integral and summation, and applying (1.9), we obtain

$$e^{\pi u^2/2}\int_1^\infty \Psi(x,u)[x^s+x^{1-s}]\frac{dx}{x}$$
$$=2\sum_{k=0}^\infty \frac{(2\pi u)^{4k}}{(4k)!}\frac{f_{4k}(0)}{s(1-s)}+2\sum_{k=0}^\infty \frac{(2\pi u)^{4k}}{(4k)!}p_{2k}(s)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

We may compute the first term on the right by simply noting that

$$\sum_{k=0}^\infty \frac{(2\pi u)^{4k}}{(4k)!}f_{4k}(0)$$
$$=\tfrac{1}{4}\{e^{-\pi Q(0,u)}+e^{-\pi Q(0,iu)}+e^{-\pi Q(0,-u)}+e^{-\pi Q(0,-iu)}\}=\cosh\left(\frac{\pi u^2}{2}\right).$$

Using (0.7), we thus obtain

$$\int_1^\infty \Psi(x,u)[x^{s-1}+x^{-s}]\,dx=\frac{1+e^{-\pi u^2}}{s(1-s)}+\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)\,\Phi(s,u). \tag{2.2}$$

Theorem 1 now follows by equating imaginary parts in this identity.

The advantage of equating imaginary parts in (2.2) is that the function $H(x)$ has a double zero at $x=1$. We will use this fact in the sequel.

3. The Minor Terms Estimated

The integral of Theorem 1 may be split into four parts, corresponding to $\varepsilon=0, 1, 2, 3$. It will be seen that the term corresponding to $\varepsilon=0$ is the most significant, and the most difficult to estimate. In this section, we will prove the following estimates for the minor terms.

Lemma. *We have*

$$\int_1^\infty \Psi_2(x, u)\, H(x)\, dx = O(e^{-\pi u^2}), \tag{3.1}$$

$$\int_1^\infty \Psi_1(x, u)\, H(x)\, dx + \int_1^\infty \Psi_3(x, u)\, H(x)\, dx$$
$$= \tfrac{1}{2} t(2\sigma-1)\, \pi^{-3} \left[\sum_{n=1}^\infty n^{-3} e^{-\pi n^2} \sin(2\pi n u)\right] u^{-3} + O(u^{-4}). \tag{3.2}$$

Proof. (3.1) follows from the fact that $H(x)$ is bounded along the line of integration. Next, we will show that:

$$\int_1^\infty \Psi_1(x, u)\, H(x)\, dx = \tfrac{1}{2} t(2\sigma-1)\, \pi^{-3} \left[\sum_{n=1}^\infty n^{-3} e^{-\pi n^2} \frac{1}{2i} e^{2\pi i n u}\right] u^{-3} + O(u^{-4}). \tag{3.3}$$

We observe that $H(x)$ extends to a holomorphic function for $\mathrm{re}(x) > 0$ (choosing the principal branch of the logarithm), and is bounded in the region $\mathrm{re}(x) > 1$. The left-hand side of (3.3) equals

$$e^{-\pi u^2} \sum_{n=1}^\infty \int_1^\infty e^{-\pi(nx-iu)^2} H(x)\, dx.$$

For fixed n, let us apply Cauchy's theorem, moving the line of integration as follows. The integral will equal the integral over two segments, namely (I) from $x=1$ to $x=1+\frac{iu}{n}$; and (II) from $x=1+\frac{iu}{n}$ to $x=\infty+\frac{iu}{n}$. We shall consider the integrals over the two segments.

(I). Making the change of variable $x=1+\frac{i}{n}(u-v)$, v being the new variable, we find that

$$e^{-\pi u^2} \int_1^{1+\frac{iu}{n}} e^{-\pi(nx-iu)^2} H(x)\, dx$$
$$= e^{-\pi u^2} \frac{i}{n} e^{-\pi n^2} e^{2\pi i n u} \int_0^u e^{\pi v^2} e^{-2\pi i n(u-v)} H\left(1+\frac{i}{n}(u-v)\right) dv.$$

Using the Taylor expansion of $H(x)$ at $x=1$, it can be shown that

$$e^{-2\pi i n(u-v)} H\left(1+\frac{i}{n}(u-v)\right) = n^{-2} t(1-2\sigma)(u-v)^2 + O(n^{-1}(u-v)^3). \tag{3.4}$$

This estimate is uniform in n.

Thus our previous integral is

$$e^{-\pi u^2}\frac{i}{n}e^{-\pi n^2}e^{2\pi inu}\cdot\left[\frac{t}{n^2}(1-2\sigma)\int_0^u(u-v)^2e^{\pi v^2}dv+O\left(\frac{1}{n}\int_0^u(u-v)^3e^{\pi v^2}dv\right)\right],$$

uniformly in n.

The following asymptotic expansions may be proved by integration by parts:

$$\int_0^u(u-v)^2e^{v^2}dv=\frac{1}{4u^3}e^{u^2}+O(u^{-5}e^{u^2}),\qquad\int_0^u(u-v)^3e^{v^3}dv=\frac{3}{8u^4}e^{u^2}+O(u^{-6}e^{u^2}).$$

Consequently, our previous expression is

$$\tfrac{1}{2}t(2\sigma-1)\pi^{-3}n^{-3}e^{-\pi n^2}\frac{1}{2i}e^{2\pi inu}u^{-3}+O\left(\frac{1}{n^2}e^{-\pi n^2}u^{-4}\right),$$

uniformly in n.

(II). On the other hand, the integral over the second segment is small; this integral is

$$O(n^{-2}e^{-\pi n^2}e^{-\pi u^2}),$$

uniformly in n.

Combining these two estimates, we have

$$e^{-\pi u^2}\int_1^\infty e^{-\pi(nx-iu)^2}H(x)\,dx$$
$$=\tfrac{1}{2}t(2\sigma-1)\pi^{-3}n^{-3}e^{-\pi n^2}\frac{1}{2i}e^{2\pi inu}u^{-3}+O(n^{-2}e^{-\pi n^2}u^{-4}),$$

uniformly in n. Summing, we obtain (3.3).

Adding the similar estimate for the $\varepsilon=3$ term, we obtain (3.2), whence the Lemma.

4. A Method of Asymptotic Expansions

Let

$$\phi_0(x)=e^{-\pi x^2}$$

and define recursively

$$\phi_n(x)=\begin{cases}\int_{-\infty}^x\phi_{n-1}(y)\,dy & \text{if } n \text{ is even;}\\ \int_{-\infty}^x\{\phi_{n-1}(y)-c_{\frac{n-1}{2}}\Delta(y)\}\,dy & \text{if } n \text{ is odd,}\end{cases}$$

$$c_n=\int_{-\infty}^\infty\phi_{2n}(y)\,dy,$$

where $\Delta(x)$ is the Dirac distribution (unit mass at $x=0$). It can be shown inductively by integration by parts that

$$\int_{-\infty}^{\infty} t e^{-\pi t^2 x^2} f(x)\,dx = f(0) + c_1 t^{-2} f''(0) + \ldots + c_n t^{-2n} f^{(2n)}(0) + R_n, \tag{4.1}$$

where $f(x)$ satisfies certain reasonable conditions and

$$R_n = t^{-2n-1} \int_{-\infty}^{\infty} \phi_{2n+2}(tx) f^{(2n+2)}(x)\,dx.$$

The constants c_n may be evaluated by considering $f(x)=x^{2n}$. We find

$$c_n = \frac{1}{n!(4\pi)^n}.$$

5. Proof of Theorem 2

(0.6) of Theorem 2 follows from Theorem 1 and the Lemma of Section 3. (0.5) will follow from Theorem 1 and the Lemma of Section 3 if we prove that, for $\frac{1}{2}<\sigma<1$,

$$\zeta(s)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\Phi(s,u) = \zeta(s)\,u^{s-1} + \zeta(1-s)\,u^{-s} + O(u^{\sigma-3}). \tag{5.1}$$

Let

$$\phi(s,u) = \sum_{k=0}^{\infty} \frac{(2\pi u)^k}{k!}\, p_{k/2}(s), \tag{5.2}$$

so that

$$\Phi(s,u) = \tfrac{1}{4}[\phi(s,u)+\phi(s,iu)+\phi(s,-u)+\phi(s,-iu)]\,2e^{-\pi u^2/2}.$$

From the functional equation (0.9) we have

$$\phi(s,iu)+\phi(s,-iu) = \phi(1-s,u)+\phi(1-s,-u).$$

Therefore, by the functional equation of the Riemann zeta function,

$$\zeta(s)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\,\Phi(s,u) = \frac{1}{4}\left[\zeta(s)\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\{\phi(s,u)+\phi(s,-u)\}\right.$$
$$\left. + \zeta(1-s)\,\pi^{-\frac{1-s}{2}}\,\Gamma\left(\frac{1-s}{2}\right)\{\phi(1-s,u)+\phi(1-s,-u)\}\right] 2e^{-\pi u^2/2}.$$

Using (1.6) and (2.1), we find

$$2e^{-\pi u^2/2}\,\pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\phi(s,u) = 4\int_0^{\infty} e^{-\pi(x-u)^2} x^{s-1}\,dx,$$

so it follows that

$$2e^{-\pi u^2/2}\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\{\phi(s,u)+\phi(s,-u)\}=4\int_{-\infty}^{\infty}e^{-\pi(x-u)^2}|x|^{s-1}\,dx.$$

(5.1) now follows from (4.1).

6. Proof of Theorem 3

We now prove Theorems 3 and 4. It was previously noted that, because of the recursion (0.8), Theorem 3 follows from Theorem 4. It is convenient to make use of the substitution $s=\frac{1}{2}+ix$. Thus s lies on the critical line if and only if x is real, and Theorem 4 is then equivalent to

Theorem 4′. *Let $p(x)$ be a polynomial all of whose roots are real. Then all roots of*

$$q(x)=\left(x+\frac{i}{2}\right)p(x+2i)+2xp(x)+\left(x-\frac{i}{2}\right)p(x-2i)$$

are real.

We will deduce this from the following Proposition:

Proposition. *Let $f(x)$ be a polynomial all of whose roots have positive imaginary part, and let $\bar{f}(x)$ be the polynomial whose coefficients are the complex conjugates of those of $f(x)$. Then all the roots of $f+\bar{f}$ are real.*

This may be proved along the lines of Titchmarsh [7], Sect. 10.23.

Corollary 1. *Let $p(x)$ be a polynomial all of whose roots are real. Then all roots of*

$$p(x+i)+p(x-i)$$

are also real.

Corollary 2. *Let $p(x)$ be a polynomial all of whose roots are real. Let t be nonnegative. Then all roots of*

$$\tfrac{1}{2}((x+it)\,p(x+i)+(x-it)\,p(x-i))$$

are also real.

Now, let us see how Theorem 4′ follows from Corollaries 1 and 2. Indeed, we can write

$$q(x)=\left(x+\frac{i}{2}\right)r(x+i)+\left(x-\frac{i}{2}\right)r(x-i),$$

where

$$r(x)=p(x+i)+p(x-i).$$

$r(x)$ has only real roots by Corollary 1, and so $q(x)$ has only real roots by Corollary 2. Theorem 4′ now follows, whence also Theorems 3 and 4.

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Note Added in Proof.

Professor J. Vaaler observed that if n is odd,

$$\frac{\Gamma\left(\frac{s+1}{2}\right)\pi^{-\frac{s+1}{2}}}{\Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}}\, p_{n/2}(s)$$

is a polynomial and has zeros on the critical line.