Infinite Continued Fraction (CF) representations

of the exponential integral function,

Bessel functions and Lommel polynomials

Continued Fractions (CF) representations and orthogonal polynomials

In this section we recall the relation between specific recurrency relations of polynomials, corresponding orthogonality and appropriate infinite continued fractions representations, as the orthogonality of polynomials $\{p_n(x)\}$ originated in the theory of Continued Fractions [SzG].

In 1935 J. Favard [FaJ], proved the

Theorem 1: *If there is a sequence of real polynomial* $\{p_n(x)\}$ *possessing a recursion relation of the form*

$$p_{n+1}(x) = (x - a_n)p_n(x) - \lambda_n p_{n-1}(x) \quad , \quad p_0(x) = 1 \quad , \quad p_1(x) = x - a_0 \quad , \quad p_{-1}(x) = 0 \quad ,$$

then the polynomials are orthogonal.

For an infinite CF we use the notation ([SzG], 3.5)

(*) $$b_0 + \frac{a_1|}{|\mathrm{b}_1} + \frac{a_2|}{|\mathrm{b}_2} + \ldots\ldots + \frac{a_n|}{|\mathrm{b}_n} + \ldots\ldots \quad .$$

Hence, as usual, the convergent $\frac{R_n}{S_n}$ $n = 0,1,2.....$, is defined as the finite fraction obtained from (*) by stopping at the term b_n. We have for $n = 2,3,4....$ the recurrency formulas

$$R_0 = b_0 \quad R_1 = b_0 b_1 + a_1 \quad R_n = b_n R_{n-1} + a_n R_{n-2}$$

$$S_0 = 1 \quad S_1 = b_1 \quad S_n = b_n S_{n-1} + a_n S_{n-2} \,.$$

The Christoffel-Darboux (recurrency) formula ([SzG], 3.2) states

Theorem 2: *The following relation holds for any three consecutive orthogonal polynomials:*

$$(**)\quad p_n(x) = (A_n x + B_n)p_{n-1}(x) - C_n p_{n-2}(x) \quad \text{for} \quad n = 2,3,4....$$

Here A_n, B_n *and* C_n *are constans,* $A_n > 0$ *and* $C_n > 0$ *. If the highest corefficient of* $\{p_n(x)\}$ *is denoted by* k_n*, we have*

$$A_n = \frac{k_n}{k_{n-1}} \quad , \quad C_n = \frac{A_n}{A_{n-1}} = \frac{k_n k_{n-2}}{k_{n-1}^2} \cdot$$

The recurrency formula (**) suggests the consideration of the CF:

$$(***) \quad \frac{1|}{|\mathrm{A}_1 x + B_1} - \frac{C_2|}{|\mathrm{A}_2 x + B_2} - \frac{C_3|}{|\mathrm{A}_3 x + B_3} ..... - \frac{C_n|}{|\mathrm{A}_n x + B_n} + ...$$

Theorem 3 ([SzG], theorem 3.5.1): *Let the “”moments”* $c_n = \int_a^b x^n d\alpha(x)$ *and* $\alpha(x)$ *be a fixed non-decreasing function with infinitely many points of increase in the finite or infinite interval.*

The convergence $\frac{R_n}{S_n}$ $n = 0,1,2.....$ *, of (***) are determined by the formulas*

$$R_n = R_n(x) = c_0^{-1}(c_0 c_2 - c^2{}_1)^{1/2} \int_a^b \frac{p_n(x) - p_n(t)|}{x - t} d\alpha(t)$$

$$S_n = S_n(x) = c_0^{1/2} p_n(x) \cdot$$

The following decomposition into partial fractions holds for the convergence of (***):

$$\frac{R_n}{S_n} = c_0^{-2}(c_0 c_2 - c^2{}_1)^{1/2} \sum_{\nu=1}^{n} \frac{\lambda_{\nu,m}}{x - x_{\nu,m}} \quad ,$$

where $\lambda_{\nu,m}$ *and* $x_{\nu,m}$ *are defined by*

$$\lambda_{\nu,n} = \int_a^b \frac{p_m(x)}{p_n'(x_\nu)(x - x_\nu)} d\alpha(x) \cdot$$

Infinite CF representation I of the exponential integral function (Laguerre)

In this section we recall the key idea of M. Laguerre [LaE] to derive the CF representation of the exponential integral function $Ei(x)$:

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$$-Ei(x)=\int_x^\infty e^{-t}\frac{dt}{t}=e^{-x}\cfrac{1}{\frac{x+1}{1}-\cfrac{1}{\frac{x+3}{1}-\cfrac{1}{\frac{x+5}{4}-\cfrac{1/4}{\frac{x+7}{9}-\cfrac{1/9}{\frac{x+9}{16}--\cfrac{1/16}{\frac{x+..}{...}-......}}}}}}$$

His basic idea is to build appropriate polynomials $\varphi_m(x),\ f_m(x)$ and

$$F_n(x):=\sum_{k=1}^{n}(-1)^{k-1}\frac{(k-1)!}{x^k}$$

of degree m resp. n with $m\le n/2$ fulfilling the following relations

i) $F_n(x)=\dfrac{\varphi_m(x)}{f_m(x)}+O(\dfrac{1}{x^{2m+1}})$

i) $\dfrac{e^{-x}}{x}=\dfrac{d}{dx}\left[e^{-x}F_n(x)\right]-\dfrac{e^{-x}}{x}\dfrac{n!}{x^n}$.

This leads to

$$-Ei(x)=\int_x^\infty e^{-t}\frac{dt}{t}=e^{-x}\lim\frac{\varphi_m(x)}{f_m(x)} ,$$

whereby the polynomials $\varphi_m(x),\ f_m(x)$ fulfill the relations

$$\varphi_{m+1}(x)=(x+2m+1)\varphi_m(x)-m^2\varphi_{m-1}(x)$$

$$f_{m+1}(x)=(x+2m+1)f_m(x)-m^2 f_{m-1}(x) .$$

The functions

$$y_{1,m}(x)=f_m(x) \quad \text{and} \quad y_{2,m}(x)=\varphi_m(x)e^{-x}-f_m(x)\int_x^\infty e^{-t}\frac{dt}{t}$$

are shown to be the solutions of the differential equation

$$xy''+(x+1)y'-my=0 .$$

The functions $u_m(x) := y_{1,m}(x)$ fulfill the recurrency relations

$$u_{m+1}(x) = (x + 2m + 1)u_m(x) - m^2 u_{m-1}(x) \ .$$

The combination with the same relations for $\varphi_m(x),\ f_m(x)$ leads to

$$u_0 = -Ei(x) = \int_x^\infty e^{-t} \frac{dt}{t} \quad \text{and} \quad u_1 = e^{-x} - (x+1)\int_x^\infty e^{-t} \frac{dt}{t}$$

from which the CF representation follows.

Infinite CF representation II of the exponential integral function (Legendre)

In [NiN], §84 there is given an alternative proof of a CF representation for the exponential integral function. We recall its key properties, using the following notations:

$$Q_x(s) := \int_x^\infty e^{-t} t^s \frac{dt}{t} = \int_0^\infty e^{-t} t^s \frac{dt}{t} - \int_0^x e^{-t} t^s \frac{dt}{t} =: \Gamma(s) - P_x(s)$$

and

$$U^{\nu,\rho}(x) := \int_x^\infty e^{-tx} t^\nu (1+t)^\rho \, dt \quad .$$

It holds

$$\Gamma_x(s) := \frac{x^s}{s(1+\frac{s}{1})(1+\frac{s}{2})(1+\frac{s}{3})....(1+\frac{s}{x})} \underset{x\to\infty}{\to} \Gamma(s) = \frac{1}{s}\prod_1^\infty \frac{(k+1)^s}{k^{s-1}(k+s)}$$

and

$$P_x(s) = \sum_{k=1}^\infty \frac{(-1)^k}{k!} \frac{x^{s+k}}{s+k} \quad \text{resp.} \quad sP_x(s) = e^{-x} x^s + \sum_{k=1}^\infty \frac{(-1)^k}{k!} \frac{x^{s+k+1}}{s+k+1}$$

Lemma: It holds

i) $P_x(s+1) = sP_x(s) - e^{-x}x^s \qquad Q_x(s+1) = sP_x(s) + e^{-x}x^s$

ii) $\lim_{k\to\infty} \frac{P_x(s+k)}{\Gamma(s+k)} = 0 \qquad \lim_{k\to\infty} \frac{Q_x(s+k)}{\Gamma(s+k)} = 1$

iii) $U^{s-1,-1}(x) = e^{-x}\Gamma(s) Q_x(1-s)$

iv) $\frac{U^{s,-1}(x)}{U^{s-1,-1}(x)} = \frac{\Gamma(s+1)Q_x(-s)}{\Gamma(s)Q_x(1-s)} = \frac{sQ_x(-s)}{Q_x(1-s)}$

v) $$\frac{U^{\nu+1,\rho}(x)}{U^{\nu,\rho}(x)} = \cfrac{\nu+1}{x - \cfrac{\rho}{1 + \cfrac{U^{\nu+2,\rho-1}(x)}{U^{\nu+1,\rho-1}(x)}}} \quad .$$

Proof: For the proofs of i)-v) we refer to [NiN], §9, §84. We just mention the relation

$$xU^{\nu+1,\rho}(x) = (\nu+1)U^{\nu,\rho}(x) + \rho U^{\nu+1,\rho-1}(x)$$

(partial integration) leading to

$$\frac{U^{\nu+1,\rho}(x)}{U^{\nu,\rho}(x)} = \cfrac{\nu+1}{x - \cfrac{\rho}{\cfrac{U^{\nu+1,\rho}(x)}{U^{\nu+1,\rho-1}(x)}}}$$

which is used in combination with the following identity

$$t^{\nu+2}(1+t)^{\rho-1} = t^{\nu+1}(1+t)^{\rho} - t^{\nu+1}(1+t)^{\rho-1} \ .$$

Corollary: [NiN], §84:

i) $$\frac{U^{s,-1}(x)}{U^{s-1,-1}(x)} = -1 + \frac{e^{-x}x^{-s}}{Q_x(1-s)}$$

ii) $$Q_x(s) = \cfrac{e^{-x}x^{-s}}{x + \cfrac{1-s}{1 + \cfrac{1}{x + \cfrac{2-s}{1 + \cfrac{2}{x + \cfrac{3-s}{1 + \ldots,}}}}}} \quad , \ \rho = -1 \quad \nu = s-1$$

Corollary: Putting $s = 0$ it follows

$$-Ei(x) = \int_x^\infty e^{-t}\frac{dt}{t} = e^{-x}\cfrac{1}{x + \cfrac{1}{1 + \cfrac{1}{x + \cfrac{2}{1 + \cfrac{2}{x + \cfrac{3}{1 + \ldots,}}}}}}$$

Some Properties of the Bessel Functions

The Bessel function of the first kind and order zero ([WaG], 6-5, 8-7, 13-6, 13-24) is defined by

$$J_0(2x) := \sum_0^\infty (-1)^k \frac{x^{2k}}{(k!)^2} = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \frac{\Gamma(1-s)}{\Gamma(s)} \frac{x^{2s}}{s^2} ds = \lim P_n(\frac{2x}{n}) \quad , \quad J_0(0) = 1$$

and

$$J_1(2x) := x\sum_0^\infty \frac{(-1)^k}{k+1} \frac{x^{2k}}{(k!)^2} \quad .$$

The Bessel equation of order zero can be interpreted as singular counterpart of the regular Schrödinger equation, which shows on the one hand side logarithmic singularity behavior in the form

$$\lim_{x \to 0^+} \frac{g(\mathrm{x})}{\sqrt{\mathrm{x}}\log\mathrm{x}} = 0 \quad .$$

C.L. Siegel (([WaG] 15-2) gave some early proofs about algebraic Bessel function values, e.g. he proved that $J_\nu(z)$ is not an algebraic number, if ν is a rational number and z is an algebraic number other than zero.

The analysis of the exponential integral function is linked to the concept of Laguerre polynomials. Hermite polynomials are the eigenvalue solutions of the *regular Schrödinger equation*. They build an orthogonal basis of $L_2(-\infty,\infty)$ as function of x^2. *Laguerre polynomials* build the corresponding polynomial system of $L_2(0,\infty)$ applying the variable transform $y = x^2$.

The recurrency formula for the Bessel function of order zero $J_0(2x)$ [WaG] 9-6, are given by

$$J_{\nu+1}(z) = \frac{2\nu}{z} J_\nu(z) - J_{\nu-1}(z) \quad .$$

It is used to express $J_{\nu+m}(z)$ linearly in terms of $J_\nu(z)$ and $J_{\nu-1}(z)$. The coefficients in this linear relation are polynomials in $\frac{1}{z}$ which are the Lommel polynomials defined by the recurrence relation

$$J_{\nu+m}(z) = J_\nu(x) R_{m,\nu}(z) - J_{\nu-1}(z) R_{m-1,\nu+1}(z) \quad .$$

There is a recurrency relation for the Lommel polynomials itself, i.e. ([WaG] 9-63)

$$R_{m-1.\nu+1}(x)+R_{m+1.\nu-1}(x)=\frac{2(\nu-1)}{\mathrm{x}}R_{m.\nu}(x) \ ,$$

as well as ([GWa] 9-65) for

$$\frac{R_{m+1.\nu}(2x)}{R_{m.\nu+1}(2x)}=\frac{\nu}{x}-\cfrac{1}{\frac{\nu+1}{x}-\cfrac{1}{\frac{\nu+2}{x}-\cfrac{1}{\frac{\nu+3}{x}-\cfrac{1}{\frac{\nu+4}{x}-.......\cfrac{1}{\cfrac{1}{\frac{\nu+m}{x}}}}}}}$$

and

$$\frac{J_{\nu-1}(x)}{J_{\nu}(x)}=\frac{2\nu}{x}-\cfrac{1}{\frac{2(\nu+1)}{x}-\cfrac{1}{\frac{2(\nu+2)}{x}-\cfrac{1}{J_{\nu+2}(x)/J_{\nu+3}(x)}}}$$

Putting

$$g_{n}(x):=g_{n,0}(x)\text{, and }\ h_{n}(\frac{1}{2x}):=h_{n,1}(\frac{1}{2x}):=x^{-n}g_{n}(x^{2})$$

the modified Lommel polynomials are defined by

$$g_{n+1}(x)=(n+1)g_{n}(x)-xg_{n-1}(x) \quad , \ g_{0}(x):=g_{1}(x):=1$$

resp.

$$g_{n}(x)=\sum_{0}^{\leq[n/2]}(-1)^{m}\frac{(\mathrm{n-m})!}{\mathrm{m}!(\mathrm{n-2m})!}\frac{\Gamma(\mathrm{n+1-m})}{\mathrm{m}!}x^{m}$$

with only real zeros [WaG], 9-71.

The relation of the modified Lommel polynomials $g_{m.\nu}(x)$ to the Lommel polynomials $R_{m.\nu}(x)$ are given by

$$g_{2n,0}(x)=x^{m}R_{2n,1}(2x) \ .$$

From [WaG], 13-33,15-41,15-5, we recall Euler's investigations in the zeros of $J_0(2x)$:

Using the abreviation $j_{-k} := -j_k$ we write the zeros of $J_0(x)$ in the form $\{j_k\}_{k\in Z-\{0\}}$. The zeros of $J_0(2\sqrt{x})$ are taking to be $\alpha_1, \alpha_2, \alpha_3, ...$i.e. $\{\alpha_k\}_{k\in N}$ and it holds for $\{\alpha_k = j_k^{\,2}/4\}_{k\in N}$.

In order to determinate the smallest zeros of $J_0(2\sqrt{x})$ Euler differentiated logarithmically the product formula to conclude

$$-\frac{d}{dx}\log J_0(2\sqrt{x}) = \sum_{n=1}^{\infty}\frac{1}{\alpha_n - x} = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{x^m}{\alpha_n^{\,m+1}}$$

provided that $|x| < \alpha_1$, and the last series is absolute convergent.

Puttting

$$\sigma_{m+1} := \sum_{1}^{\infty}\frac{1}{\alpha_n^{\,m+1}}$$

and change the order of summations results into

(*) $$-\frac{d}{dx}J_0(2\sqrt{x}) = \frac{J_1(2\sqrt{x})}{\sqrt{x}} = J_0(2\sqrt{x})\sum_{m=0}^{\infty}\sigma_{m+1}x^m$$

Based on the formula (*) above Euler obtained a system of equations, which allow to calculate the σ_k and from that to deduce the smallest values of α_k, i.e. Euler calculated

$$\sigma_1 = 1, \sigma_2 = 1/2, \sigma_3 = 1/3, \sigma_4 = 11/48, \sigma_5 = 19/120, \sigma_6 = 473/4320,$$

to deduce e.g. $\alpha_1 = 1.445795..\ \ \alpha_2 = 7.6658..\ \ \alpha_3 = 18.72...$

Lemma Being $\{j_k\}_{k\in Z-\{0\}}$ the zeros of $J_0(y)$ and $\{\alpha_k = j_k^{\,2}/4\}_{k\in N}$ the zeros of $J_0(2\sqrt{x})$ it holds

i) $$J_0(2\sqrt{x}) = J_0(2\sqrt{0})\prod_{n=1}^{\infty}\left(1-\frac{x}{\alpha_n}\right)$$

ii) $$-\frac{d}{dx}\left[\log J_0(2\sqrt{x})\right] = \frac{J_1(2\sqrt{x})}{\sqrt{x}J_0(2\sqrt{x})} = 1+\sum_{m=1}^{\infty}\sigma_{m+1}x^m = 1+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{j_n}\left[\frac{x}{j_n}\right]^k$$

iii) $$-\frac{d}{dx}\left[\log J_0(2\sqrt{x})\right] = \sum_{n=1}^{\infty}\frac{1}{\alpha_n - x} \qquad \text{for} \quad |x| < \alpha_1 = 1.445795..$$

iv) $\sigma_1 = \sum_{1}^{\infty}\frac{1}{j_n} = 1$ is the only integer value for the σ_m .

v) $J_0'(x) = -J_1'(x)$ and $\frac{1}{2}\int_0^{\infty}J_0(x)J_1(x)\frac{dx}{x} = 1$ resp. $\int_0^{\infty}\left[\frac{J_0(2\sqrt{x})}{2\sqrt{x}}\right]^2\frac{J_1(2\sqrt{x})dx}{xJ_0(2\sqrt{x})} = 1$.

The orthogonality relations of the Lommel polynomials ([ChT], ch. VI, §6, [DiD], [ScH]) we summaries in

Lemma (*Lommel polynomials)*: Being $\{j_k\}_{k\in Z-\{0\}}$ the zeros of $J_0(y)$. For the *Lommel polynomials* it holds the following orthogonality relation

$$\sum_{k=1}^{\infty}\frac{1}{j_k^{\,2}}h_m(\frac{\pm 1}{j_k})h_n(\frac{\pm 1}{j_k})=\sum_{\substack{k=-\infty\\k\neq 0}}^{\infty}\frac{1}{j_k^{\,2}}h_m(\frac{1}{j_k})h_n(\frac{1}{j_k})=\frac{\delta_{n,m}}{(n+1)} \qquad \text{with} \quad h_n(\frac{1}{2x})=x^{-n}g_n(x^2)$$

whereby $\{j_k\}_{k\in Z-\{0\}}$ are the zeros of $J_0(x)$ and the recurrency relation

$$h_{m+1}(x)=2xmh_m(x)-h_{m-1}(x) \ , \ h_{-1}(x)=0 \ , \ h_0(x)=1 \ .$$

We summaries A. Hurwitz's limit of Bessel functions ([HuA], [WaG] 9-65, 9-7) in

Lemma (***Hurwitz limit for Bessel function***):

It holds:

$$J_0(2x)=\lim_{n\to\infty}\frac{g_n(x^2)}{n!}=\lim_{n\to\infty}\frac{x^n}{n!}h_n(\frac{1}{2x}) \qquad \text{resp.} \qquad J_0(\frac{2}{x})=\lim_{n\to\infty}\frac{x^{-n}}{n!}h_n(\frac{x}{2})$$

$$\lim\frac{z^{\nu+\mathrm{m}}R_{m.\nu+1}(2z)}{\Gamma(\nu+\mathrm{m}+1)}=J_\nu(2z)$$

with

(*)
$$g_{2n,0}(x):=\sum_0^n(-1)^m\frac{(2\mathrm{n}-\mathrm{m})!}{\mathrm{m}![2(\mathrm{n}-\mathrm{m})]!}\frac{\Gamma(2\mathrm{n}+1-\mathrm{m})}{\mathrm{m}!}x^m \ .$$

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