Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 10, NUMBER 1 (2007) ISSN 1311-0454

THEOREMS ON THE CONVERGENCE OF SERIES IN GENERALIZED LOMMEL-WRIGHT FUNCTIONS

Jordanka Paneva-Konovska [1]

Abstract

The classical Cauchy-Hadamard, Abel and Tauber theorems provide useful information on the convergence of the power series in complex plane. In this paper we prove analogous theorems for series in the generalized Lommel-Wright functions with 4 indices. Results for interesting special cases of series involving Bessel, Bessel-Maitland, Lommel and Struve functions, are derived.We provide also a new asymptotic formula for the generalized Lommel-Wright functions in the case of large values of the index ν that are used in the proofs of the Cauchy-Hadamard, Abel and Tauber type theorems for the considered series.

Mathematics Subject Classification: 30B10, 30B30; 33C10, 33C20

Key Words and Phrases: Bessel, Bessel-Maitland, generalized Bessel-Maitland, Wright, generalized Lommel-Wright functions; Cauchy-Hadamard, Abel and Tauber theorems

1. Introduction

The Bessel, Lommel and Struve functions and their generalizations, as the Bessel-Maitland and generalized Lommel-Wright functions have originated from concrete problems in mechanics and astronomy. Recently they have proved themselves as some of the most frequently used special functions in mathematical analysis and its applications in physics, mechanics and engineering.

[1]This work is partially supported by National Science Research Fund - Bulgarian Ministry of Education and Science, under Grant MM 1305/2003

As it is known, investigation of the properties of the functions which are holomorphic in a certain domain of the complex plane, is often based on the possibility of their representation in series by particular countable systems of functions, holomorphic in the given region. The Taylor systems are the most frequently used in circular domains which leads to expansion in power series. Classical orthogonal polynomials and some special functions are used in the other regions. That is why it is useful to study the convergence of such series and to prove Cauchy-Hadamard, Abel and Tauber type theorems.

The classical Abel and Tauber type theorems give some important properties for the convergence of the power series $\sum_{n=0}^{\infty} a_n z^n$ on the boundary of their domain of convergence. Namely:

Theorem (Abel). *If the power series* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is convergent at a point* z_0 *of the boundary of the disk of convergence, then there exists the limit* $\lim_{z\to z_0} f(z) = f(z_0)$, *when* z *belongs to the angle domain* g_φ *with size* $2\varphi < \pi$ *and with a vertex at the point* $z = z_0$, *which is symmetric in the straight line defined by the points 0 and* z_0.

The example with the geometrical series $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots$ at the point $z_0 = 1$ ([15], p.92) shows that the inverse proposition is not true in general. That is, the existence of this limit does not imply the convergence of the power series $\sum_{n=0}^{\infty} a_n z_0^n$ without additional conditions on the growth of the coefficients. The corresponding result in this direction is given by the following theorem.

Theorem (Tauber). *If the coefficients of the power series satisfy the condition* $\lim_{n\to\infty} n a_n = 0$ *and if* $\lim_{z\to 1} f(z) = S$ *(*$z \to 1$ *radially), then the series* $\sum_{n=0}^{\infty} a_n$ *is convergent and* $\sum_{n=0}^{\infty} a_n = S$.

It turns out that the Abel theorem fails even for series of the kind $\sum_{k=1}^{\infty} a_{n_k} z^{n_k}$, where $(n_1, n_2, \ldots, n_k, \ldots)$ is a suitable permutation of the non-negative integers [15], p.92. Therefore, it is interesting to know if for series in a given sequence of holomorphic functions a statement like the Abel theorem is available. A positive answer to this question is given for series in Laguerre and Hermite polynomials by Rusev [13], §11.3; [14], Ch.4, §4; and by Boyadjiev [1], and resp. for series in Bessel and generalized Bessel-Maitland functions - by Paneva-Konovska [7], [10].

Let $J^{\mu,m}_{\nu,\lambda}(z)$ be the generalized Lommel-Wright function, introduced and studied by de Oteiza, Kalla & Conde [6] as a further (4-indices) generalization of the Bessel and Besel-Maitland (Wright) functions:

$$J^{\mu,m}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(\lambda+k+1))^m \, \Gamma(\nu+k\mu+\lambda+1)} \tag{1}$$

$$= (z/2)^{\nu+2\lambda} \, {}_1\Psi_{m+1} \left[(1,1); (\lambda+1,1), ..., (\lambda+1,1), (\nu+\lambda+1,\mu); -z^2/4\right]$$

$$z \in \mathbb{C} \setminus (-\infty, 0], \quad \mu > 0, \quad m \in \mathbb{N}, \quad \nu, \lambda \in \mathbb{C},$$

where ${}_p\Psi_q$ denotes the Wright generalization of the hypergeometric function ${}_pF_q$ (see [2]):

$$\begin{aligned} &{}_p\Psi_q \left[(a_1, A_1), ..., (a_p, A_p); (b_1, B_1), ..., (b_q, B_q); z\right] \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + A_1 k) ... \Gamma(a_p + A_p k) \, z^k}{\Gamma(b_1 + B_1 k) ... \Gamma(b_q + B_q k) \, k!}, \end{aligned}$$

$$A_j > 0 \ (j = 1, ..., p); \ B_j > 0 \ (j = 1, ..., q); \ 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0,$$

for suitably bounded values of $|z|$.

These functions and their special cases $J^{\mu}_{\nu}(z), J^{\mu}_{\nu,\lambda}(z)$, as depending on the arbitrary "fractional" parameter $\mu > 0$, present a fractional order extension of the Bessel function $J_\nu(z)$ and as such, are closely related to fractional order analogues of the Bessel operators and fractional order equations and systems modeling numerous real world phenomena arising in applied science. For some results related to fractional calculus' operators of functions (1), see for example A.I. Prieto, S.S. de Romero and H.M. Srivastava [12].

In this paper, first we obtain an asymptotic formula for the special functions (1) for large values of the index ν. Then, our main objective is to study series of the form:

$$\sum_{n=0}^{\infty} a_n J^{\mu,m}_{n-2\lambda,\,\lambda}(z), \quad z \in \mathbb{C}, \quad \mu > 0, \quad m \in \mathbb{N}, \quad \lambda \in \mathbb{C}, \tag{2}$$

with complex coefficients a_n (n=0, 1, 2, ...). For such series we prove theorems, corresponding to the classical Cauchy-Hadamard, Abel and Tauber theorems.

In particular, our results lead to corresponding convergence theorems for series in the Bessel, Bessel-Maitland, generalized Bessel-Maitland (Wright), Lommel and Struve functions.

2. An index-asymptotic formula

Theorem 1. *Let $\mu > 0$. Then for the generalized Lommel-Wright functions (1) the following asymptotic formula*

$$J_{\nu,\lambda}^{\mu,m}(z) = \frac{(z/2)^{\nu+2\lambda}}{(\Gamma(\lambda+1))^m\,\Gamma(\nu+\lambda+1)}\,(1+\theta_{\nu,\lambda}^{\mu,m}(z)), \quad z \in \mathbb{C} \setminus (-\infty, 0], \tag{3}$$

$$\text{with} \quad \theta_{\nu,\lambda}^{\mu,m}(z) \to 0 \ \text{ as } \ \mathcal{R}e\,\nu \to \infty,$$

holds. The functions $\theta_{\nu,\lambda}^{\mu,m}(z)$ are holomorphic functions of z in $\mathbb{C}$. The convergence is uniform on the compact subsets of the complex plane $\mathbb{C}$.

P r o o f. For the sake of brevity, denote

$$w_k(\nu,\lambda,\mu,m) = u_k(\nu,\lambda,\mu,m)\cdot\left(\frac{\Gamma(\lambda+2)}{\Gamma(\lambda+k+1)}\right)^m, \tag{4}$$

$$u_k(\nu,\lambda,\mu,m) = \frac{\Gamma(\lambda+\nu+\mu+1)}{\Gamma(\lambda+\nu+\mu k+1)}, \quad k \in \mathbb{N}\,;$$

$$w_0(\nu,\lambda,\mu,m) = \left(\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2)}\right)^m \cdot \frac{\Gamma(\nu+\lambda+1)}{\Gamma(\nu+\lambda+\mu+1)},$$

$$v_0(\nu,\lambda,\mu) = (\Gamma(\lambda+1))^m\,\Gamma(\lambda+\nu+1)).$$

Then (1) gets the form

$$J_{\nu,\lambda}^{\mu,m}(z) = \frac{(z/2)^{\nu+2\lambda}}{v_0(\nu,\lambda,\mu,m)} \tag{5}$$

$$\times\left(1 + w_0(\nu,\lambda,\mu,m)\sum_{k=1}^{\infty}(-1)^k\, w_k(\nu,\lambda,\mu,m)(z/2)^{2k}\right).$$

Using the analogues of the Stirling formula (see [2], p.62,(5))

$$\Gamma(z+1) \sim \sqrt{2\pi z}\, z^z \exp(-z), \quad \Gamma(z+\alpha) \sim z^\alpha\,\Gamma(z), \quad \mathcal{R}e\,z \to \infty,$$

we obtain for large values of ν and for all $k = 2,\, 3,\, \ldots$:

$$|u_k(\nu,\lambda,\mu,m)| = \left|\frac{\Gamma(\lambda+\nu+\mu+1)}{\Gamma(\lambda+\nu+\mu+1)+\mu(k-1)}\right|$$

$$\sim \left|\frac{1}{(\lambda+\nu+\mu+1)^{\mu(k-1)}}\right| \le \frac{1}{|(\lambda+\nu+\mu+1)|^{\mu}}.$$

Since $|(\lambda+\nu+\mu+1)|^{-\mu} \to 0$ when $\mathcal{R}e\,\nu \to \infty$ and $u_1 = 1$, therefore there exists a natural number N_0 such that for all $k \in \mathbb{N}$ and ν with $\mathcal{R}e\,\nu > N_0$, the inequalities $|u_k| \le 1$ hold, whence we conclude that $|w_k| \le |\Gamma(\lambda+2)/\Gamma(\lambda+k+1)|^m$. But $\sum_{k=1}^{\infty} (-1)^k\,(\Gamma(\lambda+2)/\Gamma(\lambda+k+1))^m\,(z/2)^{2k}$ is absolutely convergent on the complex plane $\mathbb{C}$ (and uniformly on the compact subsets of $\mathbb{C}$). Denoting

$$\theta_{\nu,\lambda}^{\mu,m}(z) = w_0(\nu,\lambda,\mu,m) \sum_{k=1}^{\infty} (-1)^k\, w_k(\nu,\lambda,\mu,m)(z/2)^{2k}$$

the rest immediately follows the same way as in [9]. ■

In particular, for $\nu = n - 2\lambda,\ n \in \mathbb{N}$, the index-asymptotic formula (3) takes the form:

$$J_{n-2\lambda,\lambda}^{\mu,m}(z) = \frac{(z/2)^n}{(\Gamma(\lambda+1))^m\,\Gamma(n-\lambda+1)}\,(1+\theta_{n-2\lambda,\lambda}^{\mu,m}(z));\ z,\lambda \in \mathbb{C},\ \mu > 0,$$
$$\theta_{n-2\lambda,\lambda}^{\mu,m}(z) \to 0 \ \text{ as } \ n \to \infty \quad (n \in \mathbb{N}). \tag{6}$$

3. A Cauchy-Hadamard type theorem

First, we prove an analogue of the Cauchy-Hadamard theorem.

Theorem 2 (Cauchy-Hadamard type). *The domain of convergence of the series (2), with complex coefficients a_n (n=0, 1, 2, ...), is the disk $|z| < R$ with a radius of convergence*

$$R = 1/\Lambda, \qquad \Lambda = 2^{-1} \limsup_{n\to\infty} \left(|a_n|\ |(\Gamma(\lambda+1))^m\,\Gamma(n-\lambda+1)|^{-1}\right)^{1/n}. \tag{7}$$

The cases $\Lambda = 0$ and $\Lambda = \infty$ are included in the common case, if $1/\Lambda$ is meant as ∞, respectively as 0.

P r o o f. Let us denote

$$u_n(z) = a_n J_{n-2\lambda,\lambda}^{\mu,\,m}(z), \quad b_n = 2^{-1}(|a_n||\,(\Gamma(\lambda+1))^m\,\Gamma(n-\lambda+1)|^{-1})^{1/n}.$$

Using the asymptotic formula (6), we get

$$u_n(z) = a_n \frac{(z/2)^n}{(\Gamma(\lambda+1))^m\Gamma(n-\lambda+1)}(1+\theta_{n-2\lambda,\lambda}^{\mu,\,m}(z)).$$

The proof goes separately in the three cases:

1. $\Lambda = 0$. Then $\lim\limits_{n\to\infty} b_n = \limsup\limits_{n\to\infty} b_n = 0$. Let us fix $z \neq 0$. Obviously, there exists a number N_1 such that for every $n > N_1$: $|1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z)| < 2$ and $2b_n < 1/|z|$ which is equivalent to $|u_n(z)| = b_n^n |z|^n |1 + \theta^{\mu,m}_{n-2\lambda,\lambda}(z)| < 2^{1-n}$. The absolute convergence of (2) follows immediately from this inequality.

2. $0 < \Lambda < \infty$. First, let z be inside the domain $|z| < R$ ($z \in \mathbb{C}$), i.e $|z|/R < 1$. Then $\limsup\limits_{n\to\infty} |z| b_n < 1$. Therefore, it exists a number $q < 1$ such that $\limsup\limits_{n\to\infty} |z| b_n \leq q$, whence $|z|^n b_n^n \leq q^n$. By using the asymptotic formula (6) for the common member $u_n(z)$ of the series (2), we obtain $|u_n(z)| = b_n^n |z|^n |1 + \theta^{\mu,m}_{n-2\lambda,\lambda}(z)| \leq q^n |1 + \theta^{\mu,m}_{n-2\lambda,\lambda}(z)|$. Since $\lim\limits_{n\to\infty} \theta^{\mu,m}_{n-2\lambda,\lambda}(z) = 0$ there exists N_2: for every $n > N_2$ $|1 + \theta^{\mu,m}_{n-2\lambda,\lambda}(z)| < 2$ and hence $|u_n(z)| \leq 2q^n$. Because the series $\sum\limits_{n=0}^{\infty} 2q^n$ is convergent, the series (2) is also convergent, even absolutely.

Now, let z lie outside this domain. Then $|z|/R > 1$ and $\limsup\limits_{n\to\infty} |z| b_n > 1$. Therefore there exists infinite number of values n_k of n: $|z|^{n_k} b_{n_k}^{n_k} > 1$. Since $\lim\limits_{n\to\infty} \theta^{\mu,m}_{n-2\lambda,\lambda}(z) = 0$, there exists N_3 so that for $n_k > N_3$; $|1 + \theta^{\mu,m}_{n_k-2\lambda,\lambda}(z)| \geq 1/2$, i.e. $|u_{n_k}(z)| \geq 1/2$ for infinite number of values of n. The necessary condition for convergence is not satisfied. Therefore the series (2) is divergent.

3. $\Lambda = \infty$. Let $z \in \mathbb{C}\backslash\{0\}$. Then $b_{n_k} > 1/|z|$ for infinite number of values n_k of n. But, from here $|u_{n_k}(z)| = |z|^{n_k}\, b_{n_k}^{n_k}\, |1 + \theta^{\mu,m}_{n_k-2\lambda,\lambda}(z)| \geq 1/2$ and the necessary condition for the convergence of the series (2) is not satisfied and we conclude that the series (2) is divergent for every $z \neq 0$. ■

4. An Abel type theorem

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angle domain with size $2\varphi < \pi$ and with vertex at the point $z = z_0$, which is symmetric in the straight line defined by the points 0 and z_0. The following theorem is valid.

THEOREM 3 (ABEL TYPE). *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, the real number $0 < R < \infty$ be defined by (7) and K be the circle domain $|z| < R$. If $f(z)$ is the sum of the series (2) on the disk K and this series is convergent at the point z_0 of the boundary of K, then*

$$\lim_{z\to z_0} f(z) = \sum_{n=0}^{\infty} a_n J^{\mu,m}_{n-2\lambda,\lambda}(z_0), \quad \textit{when}\ \ |z| < R,\ z \in g_\varphi. \tag{8}$$

P r o o f. Let us consider the difference

$$\Delta(z) = \sum_{n=0}^{\infty} a_n J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - f(z) = \sum_{n=0}^{\infty} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z)) \quad (9)$$

and represent it in the form

$$\Delta(z) = \sum_{n=0}^{k} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z)) + \sum_{n=k+1}^{\infty} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J_{n-2\lambda,\lambda,\mu,m}(z)).$$

Let $p > 0$. Using the denotations

$$\beta_q = \sum_{n=k+1}^{q} a_n J^{\mu,m}_{n-2\lambda,\lambda}(z_0), \quad q > k, \quad \beta_k = 0,$$
$$\gamma_n(z) = 1 - J^{\mu,m}_{n-2\lambda,\lambda}(z)/J^{\mu,m}_{n-2\lambda,\lambda}(z_0),$$

and the Abel transformation (see [5], Vol.1, p.32, eq. (3.4:7)), we obtain consequently:

$$\sum_{n=k+1}^{k+p} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z)) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z)$$
$$= \beta_{k+p}\gamma_{k+p}(z) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z) - \gamma_n(z)),$$

i.e.

$$\sum_{n=k+1}^{k+p} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z))$$
$$= \left(1 - J^{\mu,m}_{k+p-2\lambda,\lambda}(z)/J^{\mu,m}_{k+p-2\lambda,\lambda}(z_0)\right) \sum_{n=k+1}^{k+p} a_n J^{\mu,m}_{n-2\lambda,\lambda}(z_0)$$
$$- \sum_{n=k+1}^{k+p-1} \left(\sum_{s=k+1}^{n} a_s J^{\mu,m}_{s-2\lambda,\lambda}(z_0) \right) \left(\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)} - \frac{J^{\mu,m}_{n+1-2\lambda,\lambda}(z)}{J^{\mu,m}_{n+1-2\lambda,\lambda}(z_0)} \right).$$

From asymptotic formula (6) it follows that there exists a natural number M such that $J^{\mu,m}_{n-2\lambda,\lambda}(z_0) \neq 0$ when $n > M$. Let $k > M$. Then, for every natural $n > k$:

$$J^{\mu,m}_{n-2\lambda,\lambda}(z)/J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n+1-2\lambda,\lambda}(z)/J^{\mu,m}_{n+1-2\lambda,\lambda}(z_0) = (z/z_0)^n \times \quad (10)$$

$$\frac{(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z))(1+\theta^{\mu,m}_{n+1-2\lambda,\lambda}(z_0))-(z/z_0)(1+\theta^{\mu,m}_{n+1-2\lambda,\lambda}(z))(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0))}{(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0))(1+\theta^{\mu,m}_{n+1-2\lambda,\lambda}(z_0))}.$$

For the right hand side of (10) we apply the Schwarz lemma. Then we get that there exists a constant C:

$$|J^{\mu,m}_{n-2\lambda,\lambda}(z)/J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n+1-2\lambda,\lambda}(z)/J^{\mu,m}_{n+1-2\lambda,\lambda}(z_0)| \leq C|z-z_0||z/z_0|^n.$$

Analogously, there exists a constant B:

$$|1 - J^{\mu,m}_{k+p-2\lambda,\lambda}(z)/J^{\mu,m}_{k+p-2\lambda,\lambda}(z_0)| \leq B|z-z_0| \leq 2B|z_0|.$$

Let ε be an arbitrary positive number and choose $N(\varepsilon)$ so large that for $k > N(\varepsilon)$ the inequality

$$|\sum_{s=k+1}^{n} a_s J^{\mu,m}_{s-2\lambda,\lambda}(z_0)| < \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|))$$

holds for every natural $n > k$. Therefore, for $k > \max(M, N(\varepsilon))$:

$$|\sum_{s=k+1}^{\infty} a_s J^{\mu,m}_{s-2\lambda,\lambda}(z_0)| \leq \min(\varepsilon\cos\varphi/(12B|z_0|), \varepsilon\cos\varphi/(6C|z_0|),$$

and

$$\begin{aligned}|\sum_{n=k+1}^{\infty} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z))| \\ \leq (\varepsilon\cos\varphi/6)(1 + \sum_{n=k+1}^{\infty} |z_0|^{-1}|z-z_0||z/z_0|^n) \\ \leq (\varepsilon\cos\varphi/6)(1 + |z-z_0|/(|z_0|-|z|)).\end{aligned}$$

But near the vertex of the angle domain g_φ in the part d_φ closed between the angle's arms and the arc of the circle with center at the point 0 and touching the arms of the angle, we have $|z-z_0|/(|z_0|-|z|) < 2/\cos\varphi$, i.e. $|z-z_0|\cos\varphi < 2(|z_0|-|z|)$. That is why the inequality

$$|\sum_{n=k+1}^{\infty} a_n (J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z))| < (\varepsilon\cos\varphi)/6 + \varepsilon/3 \leq \varepsilon/2 \qquad (10)$$

holds for $z \in d_\varphi$ and $k > \max(M, N(\varepsilon))$. Fix some $k > \max(M, N(\varepsilon))$ and after that choose $\delta(\varepsilon)$ such that if $|z-z_0| < \delta(\varepsilon)$ then the inequality

$$|\sum_{n=0}^{k} a_n(J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z))| < \varepsilon/2 \tag{11}$$

holds inside d_φ. We get

$$|\Delta(z)| = |\sum_{n=0}^{\infty} a_n(J^{\mu,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,m}_{n-2\lambda,\lambda}(z))|$$

for the module of the difference (9). From (10) and (11) it follows that the equality (8) is satisfied. ∎

5. A Tauber type theorem

We consider the series $\sum\limits_{n=0}^{\infty} a_n,\ a_n \in \mathbb{C}$ and let

$$z_0 \in \mathbb{C}, \quad |z_0| = R, \quad 0 < R < \infty, \quad J^{\mu,m}_{n-2\lambda,\lambda}(z_0) \neq 0 \ \text{ for } n = 0, 1, 2, \ldots\ .$$

For shortness, denote

$$J^*_{n,\lambda,\mu,m}(z; z_0) = \frac{J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}.$$

Let the series $\sum\limits_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0)$ be convergent for $|z| < R$ and

$$F(z) = \sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0), \quad |z| < R.$$

Theorem 4 (Tauber type). *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with*

$$\lim_{n\to\infty} n a_n = 0, \tag{12}$$

and there exists

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R,\ z \to z_0 \text{ radially}),$$

then the series $\sum\limits_{n=0}^{\infty} a_n$ is convergent and

$$\sum_{n=0}^{\infty} a_n = S.$$

P r o o f. For a point z of the segment $[0, z_0]$ we have

$$\sum_{n=0}^{k} a_n - F(z) = \sum_{n=0}^{k} a_n - \sum_{n=0}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0)$$

$$= \sum_{n=0}^{k} a_n \frac{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)}{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)} - \sum_{n=0}^{\infty} a_n \frac{J^{\mu,\,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)}$$

$$= \sum_{n=0}^{k} a_n \frac{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,\,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)} - \sum_{n=k+1}^{\infty} a_n J^*_{n,\lambda,\mu,m}(z; z_0),$$

and therefore,

$$|\sum_{n=0}^{k} a_n - F(z)| \leq \sum_{n=0}^{k} |a_n| \left| \frac{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0) - J^{\mu,\,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)} \right| \tag{13}$$

$$+ \sum_{n=k+1}^{\infty} |a_n| \left| J^*_{n,\lambda,\mu,\,m}(z; z_0) \right| .$$

By using the asymptotic formula (6) for the generalized Lommel-Wright functions, we obtain:

$$a_n \frac{J^{\mu,\,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,\,m}_{n-2\lambda,\lambda}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \frac{1+\theta^{\mu,\,m}_{n-2\lambda,\lambda}(z)}{1+\theta^{\mu}_{n-2\lambda,\lambda}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1+\widetilde{\theta}_{n,\lambda,\mu,\,m}(z; z_0)\right).$$

Let ε be an arbitrary positive number. We choose a number N_1 so large that the inequalities $|1 + \widetilde{\theta}_{k,\lambda,\mu}(z; z_0)| < 2, |ka_k| < \frac{\varepsilon}{6}$ hold as $k \geq N_1$. If $k > N_1$ and z is on the segment $[0, z_0]$, then for the second summand in (13) the following estimate is valid:

$$\sum_{n=k+1}^{\infty} |a_n| \left| J^*_{n,\lambda,\mu,m}(z; z_0) \right| = \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^n |1 + \widetilde{\theta}_{n,\lambda,\mu,m}(z; z_0)| \tag{14}$$

$$\leq 2 \left| \frac{z}{z_0} \right|^{k+1} \sum_{n=k+1}^{\infty} |a_n| \left| \frac{z}{z_0} \right|^{n-k-1} \leq 2 \sum_{n=0}^{\infty} |a_{n+k+1}| \left| \frac{z}{z_0} \right|^n$$

$$= 2 \sum_{n=0}^{\infty} \frac{|(n+k+1)a_{n+k+1}|}{n+k+1} \left| \frac{z}{z_0} \right|^n < 2 \sum_{n=0}^{\infty} \frac{\varepsilon/6}{n+k+1} \left| \frac{z}{z_0} \right|^n$$

$$< \frac{2}{k} \frac{\varepsilon}{6} \frac{1}{1 - |z/z_0|} = \frac{\varepsilon}{3} \frac{1}{k} \frac{|z_0|}{|z_0| - |z|} .$$

Now let us consider the first summand in (13). We have:

$$\sum_{n=0}^{k}|a_n|\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}\mu,m(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|$$

$$=\sum_{n=0}^{q}|a_n|\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|+\sum_{n=q+1}^{k}|a_n|\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|.$$

According to Schwarz's lemma, there exists a constant C such that

$$\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|<C|z-z_0|.$$

Moreover, there exists a number N_2 such that the following inequality

$$\sum_{n=0}^{q}|a_n|\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|\le C\,|z-z_0|\,k\,\frac{\sum\limits_{n=0}^{q}|a_n|}{k} \tag{15}$$

$$<C\,|z-z_0|\,k\,\frac{\varepsilon}{3RC}=|z-z_0|\,k\,\frac{\varepsilon}{3R}.$$

holds as $k>N_2$. It remains to estimate the sum

$$\sum_{n=q+1}^{k}|a_n|\left|\frac{J^{\mu}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|.$$

To this end, using asymptotic formula (6), we find consequently:

$$\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}=\frac{(z_0)^n(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0))-z^n(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z))}{z_0^n(1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0))}$$

$$=1-\left(\frac{z}{z_0}\right)^n\frac{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}=1-\left(\frac{z}{z_0}\right)^n\left[1+\frac{\theta^{\mu,m}_{n-2\lambda,\lambda}(z)-\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right]$$

$$=1-\left(\frac{z}{z_0}\right)^n-\left(\frac{z}{z_0}\right)^n\frac{\theta^{\mu,m}_{n-2\lambda,\lambda}(z)-\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\,.$$

Therefore,

$$\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right| \tag{16}$$

$$\leq \left|1-\left(\frac{z}{z_0}\right)^n\right| + \left|\frac{z}{z_0}\right|^n \left|\frac{\theta^{\mu,m}_{n-2\lambda,\lambda}(z)-\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right|.$$

We obtain the following inequalities

$$\left|1-\left(\frac{z}{z_0}\right)^n\right| = \left|1-\frac{z}{z_0}\right| \left|1+\frac{z}{z_0}+\left(\frac{z}{z_0}\right)^2+\cdots+\left(\frac{z}{z_0}\right)^{n-1}\right| \leq n\left|1-\frac{z}{z_0}\right|$$

for the first summand of (16). According to Schwarz's lemma, there exists a constant ρ such that

$$\left|\frac{\theta^{\mu,m}_{n-2\lambda,\lambda}(z)-\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right| \leq 1 \qquad \text{as} \qquad |z-z_0|<\rho.$$

Then, for such $|z|$, we obtain for the second summand of (16):

$$\left|\frac{z}{z_0}\right|^n \left|\frac{\theta^{\mu,m}_{n-2\lambda,\lambda}(z)-\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}{1+\theta^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right| \leq \left|\frac{z}{z_0}\right|^n |z-z_0|.$$

From (12) it follows that

$$\lim_{n\to\infty} a_n = 0, \quad \lim_{k\to\infty}\frac{\sum\limits_{n=1}^{k} n|a_n|}{k}=0, \quad \lim_{k\to\infty}\frac{\sum\limits_{n=1}^{k}|a_n|}{k}=0.$$

Then a number N_3 exists such that

$$\frac{\sum\limits_{n=q+1}^{k} n|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{and} \quad \frac{\sum\limits_{n=q+1}^{k}|a_n|}{k} < \frac{\varepsilon}{3(1+R)} \quad \text{as} \quad k>N_3.$$

Therefore,

$$\sum_{n=q+1}^{k}|a_n|\left|\frac{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)-J^{\mu,m}_{n-2\lambda,\lambda}(z)}{J^{\mu,m}_{n-2\lambda,\lambda}(z_0)}\right| \leq \sum_{n=q+1}^{k} n|a_n|\left|1-\frac{z}{z_0}\right| \tag{17}$$

$$+\sum_{n=q+1}^{k}|a_n|\left|\frac{z}{z_0}\right|^n|z-z_0| \leq k\,\frac{|z-z_0|}{R}\,\frac{\sum\limits_{n=q+1}^{k} n|a_n|}{k}+k\,|z-z_0|\,\frac{\sum\limits_{n=q+1}^{k}|a_n|}{k}$$

$$< k\,|z-z_0|\,\frac{1+R}{R}\,\frac{\varepsilon}{3(1+R)} = k\,|z-z_0|\,\frac{\varepsilon}{3R}.$$

Finally, let us note that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| \leq \sum_{n=0}^{q} |a_n| \left|\frac{J_{n-2\lambda,\lambda}^{\mu,m}(z_0) - J_{n-2\lambda,\lambda}^{\mu,m}(z)}{J_{n-2\lambda,\lambda}^{\mu,m}(z_0)}\right|$$

$$+ \sum_{n=q+1}^{k} |a_n| \left|\frac{J_{n-2\lambda,\lambda}^{\mu,m}(z_0) - J_{n-2\lambda,\lambda}^{\mu,m}(z)}{J_{n-2\lambda,\lambda}^{\mu,m}(z_0)}\right| + \sum_{n=k+1}^{\infty} |a_n| \left|J_{n,\lambda,\mu,m}^{*}(z;z_0)\right| .$$

Let $N = \max(N_1, N_2, N_3)$, $k > N$ and $|z - z_0| < \rho$. Then by using (14), (15), (17), we can conclude that

$$\left|\sum_{n=0}^{k} a_n - F(z)\right| < |z - z_0|\, k\, \frac{\varepsilon}{3R} + k\,|z - z_0|\, \frac{\varepsilon}{3R} + \frac{\varepsilon}{3}\,\frac{1}{k}\,\frac{|z_0|}{|z_0| - |z|}$$

$$= \frac{\varepsilon}{3}\left[\frac{2k}{R}\,|z - z_0| + \frac{1}{k}\,\frac{|z_0|}{|z_0| - |z|}\right] .$$

If we substitute z by $z_0(1 - \frac{1}{k})$, then

$$\left|\sum_{n=0}^{k} a_n - F\left(z_0(1 - \frac{1}{k})\right)\right| < \frac{\varepsilon}{3}\, 3 = \varepsilon.$$

This proves that $\lim\limits_{k\to\infty} \sum\limits_{n=0}^{k} a_n$ exists and equals $\lim\limits_{k\to\infty} F\left(z_0(1 - \frac{1}{k})\right)$, i.e.

$$\sum_{n=0}^{\infty} a_n = \lim_{k\to\infty} F\left(z_0(1 - \frac{1}{k})\right) = S.$$

Thus the theorem is proved. ■

6. Special cases

Obviously for $\underline{m = 1}$, special function (1) turns into the generalization of the Bessel function $J_\nu(z)$, introduced by Pathak [11] (for details see [3], p.353 and [4], eq.(8.2)):

$$J_{\nu,\lambda}^{\mu,1}(z) = J_{\nu,\lambda}^{\mu}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{\Gamma(\lambda + k + 1)\Gamma(\nu + k\mu + \lambda + 1)} . \quad (18)$$

Then Theorem 1 gives the asymptotic formula, given in [9]. Theorems 2, 3, 4 provide analogues of the classical Cauchy-Hadamard, Abel and Tauber theorems for the functions (18).

Fixing $m = 1$, for particular choices of the other parameters λ and μ we obtain results for more special cases of series of the form (2).

i. Let $\underline{\lambda = 0}$, then special functions (1), (18) give the generalization of the Bessel-Clifford function $C_\nu(z) = z^{-\nu/2} J_\nu(2\sqrt{z})$, introduced by E. M. (Maitland) Wright [16], and called Wright function or Bessel-Maitland function (see Marichev [4], p.109; Kiryakova [3], p.336):

$$J_\nu^\mu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(\nu + \mu k + 1)}\,, \quad \mu > -1. \tag{19}$$

Namely,

$$J_{\nu,0}^{\mu,1}(z) = (z/2)^\nu J_\nu^\mu(z^2/4), \tag{20}$$

therefore our results, for $m = 1$ and $\lambda = 0$, yield the corresponding theorems from Paneva-Konovska [10]. Additionally, if $\underline{\mu = 1}$, then $J_{\nu,0}^{1,1}(z) = J_\nu(z)$ and we get the theorems for convergence of series in Bessel functions, see Paneva-Konovska [7], [8].

ii. Let now $\underline{\mu = 1}$, then (see Marichev [4], p.109; Kiryakova [3], p.336)

$$J_{\nu,\lambda}^{1,1}(z) = \frac{2^{2-2\lambda-\nu}}{\Gamma(\lambda)\Gamma(\lambda+\nu)}\; s_{2\lambda+\nu-1,\,\nu}(z), \tag{21}$$

where $s_{\alpha,\nu}(z)$, $\alpha, \nu \in \mathbb{C}$ denotes the Lommel function, [2, Vol.2, p.50,(69)]:

$$s_{\alpha,\nu}(z) = \frac{z^{\alpha+1}}{(\alpha-\nu+1)(\alpha+\nu+1)}\; {}_1F_2\left(1; \frac{\alpha-\nu+3}{2}; \frac{\alpha+\nu+3}{2}; -\frac{z^2}{4}\right). \tag{22}$$

For $\nu = n + 1 - 2\lambda$, relation (21) becomes

$$J_{n+1-2\lambda,\lambda}^{1,1}(z) = \frac{2^{1-n}}{\Gamma(\lambda)\Gamma(n+1-\lambda)}\, s_{n,n+1-2\lambda}(z),$$

and Theorems 2, 3, 4 provide, as special cases, results on the convergence of series in Lommel functions ($\widetilde{a}_n := c_n a_n$):

$$\sum_{n=0}^{\infty} \widetilde{a}_n J_{n+1-2\lambda,\lambda}^{1,1}(z) = \sum_{n=0}^{\infty} a_n s_{n,n+1-2\lambda}(z). \tag{23}$$

Additionally, let us take $\lambda = 1/2$, then we obtain the Struve functions ([2, Vol.2, p.51,(84)]):

$$\mathbf{H}_n(z) = \frac{2^{1-\nu}}{\sqrt{\pi}\Gamma(\nu+1/2)}\; s_{\nu,\nu}(z) \tag{24}$$

and our results turn into Cauchy-Hadamard, Abel and Tauber type theorems for series in Struve functions ($\widetilde{a}_n := c_n a_n^* = d_n a_n$):

$$\sum_{n=0}^{\infty} \widetilde{a}_n J_{n,1/2}^{1,1}(z) = \sum_{n=0}^{\infty} a_n^* s_{n,n}(z) = \sum_{n=0}^{\infty} a_n \mathbf{H}_n(z). \tag{25}$$

References

[1] L. Boyadjiev, Abel's theorems for Laguerre and Hermite series. *Compt. Rend. Acad. Bulg. Sci.* **39**, No 4 (1986), 13-15.

[2] A. Erdélyi et al. (Ed-s), *Higher Transcendental Functions*, Vols. 1 – 3. McGraw-Hill, New York-Toronto-London (1953).

[3] V. Kiryakova, *Generalized Fractional Calculus and Applications.* Longman & J. Wiley, Harlow and N. York (1994).

[4] O. I. Marichev, *Method of Evaluation of Integrals of Special Functions* (In Russian). Nauka i Technika, Minsk (1978); Engl. translation: O.I. Marichev, *Handbook of Integral Transforms of Higher Transcendental Functions: Theory and Algorithmic Tables.* Ellis Horwood Ltd., N. York etc.; Halsted Press (1983).

[5] A. Markushevich, *Theory of Analytic Functions.* Vol. 1, 2. Nauka, Moscow (1967/68) (In Russian).

[6] M.B.M. de Oteiza, S. Kalla and S. Conde, Un estudio sobre la función Lommel-Maitland. *Revista Técnica de la Facultad de Ingenieria de la Universidad del Zulia* **9**, No 2 (1986), 33-40.

[7] J. Paneva-Konovska, Cauchy-Hadamard and Abel type theorems for Bessel functions series. In: *Proc. 19-th Summer School "Applications of Mathematics in Engineering", Varna, 24.08-2.09, 1993*, Sofia (1994), 165-170.

[8] J. Paneva-Konovska, A Tauber type theorem for series in Bessel functions. *Fractional Calculus & Applied Analysis* **2**, No 5 (1999), 683-688.

[9] J. Paneva-Konovska, Index-asymptotic formulae for Wright's generalized Bessel functions. *Math. Science Research J.*, **11**, No 5 (May 2007), 424-431.

[10] J. Paneva-Konovska, Theorems on the convergence of series in generalized Bessel-Maitland functions. *Kuwait J. Science and Eng.*, Submitted.

[11] R.S. Pathak, Certain convergence theorems and asymptotic properties of a generalization of Lommel and Maitland transformations. *Proc. Nat. Acad. Sci. India* **A-36**, No 1 (1966), 81-86.

[12] A.I. Prieto, S.S. de Romero, H.M. Srivastava. Some fractional-calculus results involving the generalized Lommel-Wright and related functions. *Applied Mathematics Letters* **20** (2007), 17-22.

[13] P. Rusev, *Analytic Functions and Classical Orthogonal Polynomials.* Publ. House Bulg. Acad. Sci., Sofia (1984).

[14] P. Rusev, *Classical Orthogonal Polynomials and Their Associated Functions in Complex Domain.* Publ. House Bulg. Acad. Sci., Sofia (2005).

[15] L. Tchakalov, *Introduction in Theory of Analytic Functions.* Nauka i Izkustvo, Sofia (1972) (In Bulgarian).

[16] E.M. Wright, On the coefficients of power series having exponential singularities. *J. London Math. Soc.* **8** (1933), 71-79.

Faculty of Applied Mathematics and Informatics
Technical University of Sofia
Sofia 1156, BULGARIA

e-mail: yorry77@mail.bg

Received: April 20, 2007