1. PÓLYA'S THEOREM

In 1918, G. Pólya ([1], §4, I, p.364), obtained the following general theorem about zeros of the Fourier transform of a real function.

Theorem 1.1. *Let $a < b$ be finite real numbers and let $f(t)$ be a strictly positive continuous function on the open interval (a, b) and differentiable in (a, b) except possibly at finitely many points. Suppose that*

$$\alpha \le -\frac{f'(t)}{f(t)} \le \beta \tag{1.1}$$

at every point of (a, b) where $f(t)$ is differentiable. Then every zero of the integral

$$\widehat{f}(z) = \int_a^b f(t)e^{zt}\mathrm{d}t$$

lies in the open infinite strip

$$\alpha < \Re(z) < \beta,$$

with the exception of the function

$$f(t) = e^{ct+d} \tag{1.2}$$

with c, d, real constants.

Remark. If $f(t) = e^{ct+d}$ then

$$\widehat{f}(z) = 2e^{\frac{b+a}{2}(z+c)+d}\,\frac{\sinh\left(\frac{b-a}{2}(z+c)\right)}{z+c}.$$

For this function, any α and β with $\alpha \le -c \le \beta$ are admissible, while the zeros of $\widehat{f}(z)$ occur at $z = -c + 2\pi\imath k/(b-a)$ for $k = \pm 1, \pm 2, \dots$. Thus the theorem fails for this function.

After the change of variable $\log x \leftarrow t$ and approximating the integral over the half-line $(0, \infty)$ by integrals over finite intervals, one can obtain a theorem about zeros of Mellin transforms. This (after the above change of variable) is mentioned explicitly in [1], p.365.

Theorem 1.2. *Let $0 \le a < b \le +\infty$ and let $f(x)$ be a strictly positive continuous function on (a, b) and differentiable there except possibly at finitely many points. Suppose that*

$$\alpha \le -x\frac{f'(x)}{f(x)} \le \beta \tag{1.3}$$

at every point of (a, b) where $f(x)$ is differentiable. Suppose further that the integral

$$F(s) := \int_a^b f(x)x^s\frac{\mathrm{d}x}{x}$$

is convergent for[1] $\alpha^* < \Re(s) < \beta^*$. Then all zeros ρ of $F(s)$ in this strip satisfy $\alpha \le \Re(\rho) \le \beta$.

[1] By $\Re(z)$ and $\Im(z)$ we denote the real and imaginary part of the complex number z.

2. Applications

Attempts have been made to apply Pólya's theorems to the Riemann zeta function and to L-functions. So far, no positive information has been gleaned from such tries.

For the zeta function, the Müntz Formula has been used, namely

$$\widetilde{F}(s)\zeta(s) = \int_0^\infty \Big(\sum_{n=1}^\infty F(nx) - \frac{1}{x}\int_0^\infty F(t)\mathrm{d}t\Big)x^s\frac{\mathrm{d}x}{x}$$

with $\widetilde{F}(s)$ the Mellin transform, valid for $0 < \Re(s) < 1$ and $F(x)$ satisfying certain general conditions. A set of conditions under which the formula holds is spelled out in Titchmarsh's well-known monograph [2], 2.11. For example, the formula holds for $F(x)$ and $F'(x)$ continuous and bounded in every finite interval and $F(x) = O(x^{-1-\delta})$, $F'(x) = O(x^{-1-\delta})$ as $x \to +\infty$, for some $\delta > 0$. The last condition on $F(x)$ and $F'(x)$ can be relaxed to $|F|$ and $|F'|$ integrable on $[1,\infty)$.

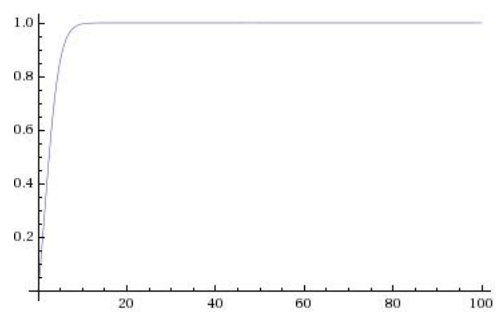
If we take $F(x) = -e^{-x}$ we obtain the well-known formula

$$-\Gamma(s)\zeta(s) = \int_0^\infty \Big(\frac{1}{x} - \frac{1}{e^x - 1}\Big)x^s\frac{\mathrm{d}x}{x}$$

valid for $0 < \Re(s) < 1$. If we apply Theorem 1.2 to this situation to the function

$$f(x) = -\sum_{n=1}^\infty e^{-nx} + \frac{1}{x}\int_0^\infty e^{-t}\mathrm{d}t$$

we see that $f(0) = \frac{1}{2}$, $f(\infty) = 0$ and $f(x)$ is decreasing. The function $-xf'(x)/f(x)$ is increasing from 0 to 1 in $(0,\infty)$, hence Theorem 1.2 applies only with $\alpha = 0$, $\beta = 1$, and no information is obtained about zeros of $\Gamma(s)\zeta(s)$ in the open strip $0 < \Re(s) < 1$ of validity of Müntz's formula.



The function $-xf'(x)/f(x)$

If Theorem 1.2 could be applied non trivially for an instance of Müntz's formula valid in the strip $0 < \Re(s) < 1$, then either $\alpha > 0$ or $\beta < 1$ must be satisfied. If $F(x)$ is analytic in a neighborhood of $x = 0$ and all its derivatives up to order $2J$ decay at

∞ as $F^{(j)}(x) = O(x^{-\alpha})$ with $\alpha > 1$, then by the Euler–MacLaurin Formula we have

$$-\sum_{n=1}^{\infty} F(nx) + \frac{1}{x}\int_0^{\infty} F(t)\mathrm{d}t = \frac{1}{2}F(0) - \sum_{j=1}^{J-1} \frac{B_{2j}}{(2j)!} F^{(2j-1)}(0)x^{2j-1} + \frac{x^{2J}}{(2J)!}\int_0^1 B_{2J}(t)\Big\{\sum_{n=0}^{\infty} F^{(2J)}(n(x+t))\Big\}\mathrm{d}t$$

where B_n denotes the n-th Bernoulli number and $B_n(t)$ is the n-th Bernoulli polynomial. Also, the integral in the right-hand side of this formula is $O(1/x)$ as $x \to 0$. Taking $J = 2$ we then see that

$$f(x) := -\sum_{n=1}^{\infty} F(nx) + \frac{1}{x}\int_0^{\infty} F(t)\mathrm{d}t = \frac{1}{2}F(0) - F'(0)\frac{x}{12} + O(x^2)$$

and in particular $f(0) = \frac{1}{2}F(0)$, $f'(0) = -F'(0)/12$, and finally $\alpha = 0$ if $F(0) \neq 0$. Moreover, the hypothesis about the behavior of $F(x)$ at infinity shows that

$$-\sum_{n=1}^{\infty} F(nx) + \frac{1}{x}\int_0^{\infty} F(t)\mathrm{d}t \sim x^{-1}\int_0^{\infty} F(t)\mathrm{d}t$$

which certainly implies $\beta \geq 1$. Therefore, application of Pólya's theorem to Müntz's formula yields no information about the location of non-trivial zeros of the zeta function, at least if $F(0) \neq 0$. If $F(0) = 0$ and $F'(0) \neq 0$ then $-xf'(x)/f(x) \to -1$ as $x \to 0$, so it is not positive and Pòlya's theorem does not apply.

REFERENCES

[1] G. PÓLYA, Über die Nullstellen gewisser ganzer Funktionen, *Math. Zeit.*, **2** (1918), 352–383. Also, Collected Papers, Vol II, 166–197.

[2] E.C. TITCHMARSH, *The Theory of the Riemann Zeta function*, Clarendon Press, Oxford 1951.