Spectral Analysis
to prove the
Riemann Hypothesis

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The **Riemann Hypothesis** states that the non-trivial zeros of the Zeta function all have real part one-half. The **Hilbert-Polya conjecture** states that the imaginary parts of the zeros of the Zeta function corresponds to eigenvalues of an unbounded self adjoint operator. All attempts failed so far to represent the **Riemann duality equation** in the critical stripe as convergent (!) Mellin transforms of an underlying self adjoint integral operator equation.

The RH analysis is mainly built on the Fourier series representation of the Zeta function on the critical line ([EdH] 1.2). G. Polya defined a Zeta fake function, for which all its zeros lie on the critical line ([EdH] 12.5). If there would be a convolution representation of the Zeta function spectral theory can be applied to answer the RH positively ([CaD]).

Spectral analysis of convolution integral operators goes along with generalized Fourier analysis for linear, self-adjoint, positive definite operators, defined on appropriate Hilbert space domains. The kernel functions of our two convolution integral operators of the 1st and 2nd proof are built as Hilbert transforms of the Theta function (based on the Gauss-Weierstrass function) and the fractional part function. The corresponding domains are the Hilbert spaces H_0 and H_{-1}. The later one is proposed as alternative modeling framework for the quantum oscillator to overcome the issue of not convergent ground state energy series ([BeM], [FeR]).

The Mellin transforms of the corresponding operator duality equation (with respect to the inner product of the Hilbert space) give the Riemann duality equation in a complex-valued distributional sense ([PeB]). This can be interpreted as a representation of the Riemann duality equation built by a generalized (only weak, but self-adjoint) modified Müntz formula, defined in an appropriate Hilbert space framework to ensure convergent integrals ([TiE] 2.11). This proves the RH and the Hilbert-Polya conjecture in a weak sense. With Hilbert scale and spectral theory then the RH is also valid in a strong sense.

The (Hilbert) transformation from a Fourier integral representation to a convolution integral representation of the Zeta function goes along with the following introduction notes from ([EdH] 10.2 "ADJOINTS AND THEIR TRANSFORMS": *"In order to define the adjoint of an invariant operator (on the vector space* V *of complex-valued functions on the multiplicative group* R^+ *of positive real numbers), it is necessary to define an inner product on the vector space V. The natural definition would be*" ….based on a measure $d\mu$, … "*which is invariant on the group* R^+, *namely* $d\mu = d\log x$. *…However, the functional equation* $\xi(s) = \xi(1-s)$ *involves an inner product on* V *, which is natural with respect to the additive group* R^+, *….namely* dx."

The Hilbert transform properties ensure the self-adjoint property (appendix). The chosen (domain) Hilbert scale factors ("0"resp. "-1") ensure positive definite operators.

1st proof based on Hilbert space $H_0 := L_2(-\infty,\infty)$

The Jacobi $\vartheta-$function resp. Riemann's auxiliary function ψ are given by

$$\vartheta(x^2) := G(x) := \sum_{-\infty}^{\infty} f(nx) = G(1/x)/x \quad , \quad \psi(x^2) := \sum_{1}^{\infty} f(nx) \quad \text{with} \quad f(x) := e^{-\pi x^2} \; .$$

The Mellin transform M in the form

$$\Omega(s) := \Gamma(1+s/2)\pi^{-s/2} = \int_0^\infty x^s [xf'(x)] \frac{dx}{x}$$

gives the Riemann duality equation

$$\xi(s) := \zeta(s)(s-1)\Omega(s) = \xi(1-s) \quad , \quad s \in C \; .$$

The RH would be proven, if $\xi(s) = M(\vartheta)(s)$ would be valid. Then the duality equation $\xi(s) = \xi(1-s)$ would be a Mellin transform of an underlying dual integral equation relation (with respect to the (multiplicative) density $d\log x = dx/x$), whereby the Theta function $\vartheta(x)$ would be the density function. This means that by variable substitution $x \to 1/u$ (i.e. $d\log x = dx/x = -d\log u = -du/u$) the duality relation of the integral operator would reproduce the duality relation of the complex-valued Riemann duality equation. Unfortunately the integrals $M(\vartheta)(s)$ are not convergent for any s in opposite to $M(\psi)(s)$ ([EdH] 10). The root cause of this is the fact that the constant Fourier term of the Theta function series is not vanishing. As work around Riemann replaced the Theta function $\vartheta(x)$ by $\psi(x)$, which unfortunately is no Theta function anymore. To "generate" the duality equation based on $\psi(x)$, his "trick" was to replace the original Gauss-Weierstrass function by its derivative. This "cuts" out the constant Fourier term of the Theta function. To compensate this change and in order to keep the "duality equation balance" he then multiplied the resulting term again with x. The prize to be paid by this "trick" (appendix) is the "unnatural" factor $(s-1)$ of the Zeta function

$$\xi(s) := \zeta(s)(s-1)\Gamma(1+s/2)\pi^{-s/2} \; .$$

Alternatively, we proposed to apply the Hilbert transformed Gauss-Weierstrass function to define $\vartheta_H(x)$ in order to achieve the same ("duality equation creation") effect, but without "destroying" the structure of the Theta function itself (enabled by the Hilbert transform property: $(Hf,f)=0$). As the Hilbert transform is an isomorphism onto the L_2-Hilbert space its application to solve the RH requires an upfront analysis in a weak L_2-sense ([PeB] I §15). Then with standard density arguments the RH follow in a strong sense. The Hilbert transforms changes the constant Fourier term to zero, i.e. $H(\vartheta)(x)$ has a vanishing constant Fourier term. At the same time the purely Gauss-Weierstrass structure of the Theta function series elements does not go lost. **The crucial equations are given by (appendix)**

$$[H(f)](x) = 4\pi \int_0^\infty f(\xi)\sin(2\pi\xi x)d\xi$$

$$M[f_H](s) = \pi^{(1-s)/2}\tan(\pi\frac{s}{2})\Gamma(\frac{s}{2}) = \pi^{(1-s)/2}\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \pi^{(1-s)/2}\cot(\pi\frac{1-s}{2})\Gamma(\frac{s}{2}) = 2\sqrt{\pi}\tan(\pi\frac{s}{2})M[f](s) \, .$$

As a consequence there is a Mellin transformation equation in a weak L_2-sense in the form $T(s) = T(1-s)$, where all its zeros lie on the critical line. This proves the RH as there cannot be two different complex-valued functions with same set of zeros. We also note that a duality equation relation in the form $T(s) = T(1-s)$ is equivalent to a Theta function property of the underlying density function ([HaH]).

2nd proof based on
Hilbert space $H_0^{\#} := L_2^{\#}(0,1)$

Let $\Gamma := S^1(R^2)$, i.e. Γ is the boundary of the unit sphere. Let $u(s)$ being a $2\pi-$periodic function and $\oint$ denotes the integral from 0 to 2π in the Cauchy-sense. Then for $u \in H := L_2^*(\Gamma)$ with $\Gamma := S^1(R^2)$ and for real β the Fourier coefficients

$$u_\nu := \frac{1}{2\pi}\oint u(x)e^{-i\nu x}dx$$

enable the definitions of the norms

$$\|u\|_\beta^2 := \sum_{-\infty}^{\infty}|\nu|^{2\beta}|u_\nu|^2 \quad .$$

Beside the Hilbert transform $(Hu)(x)$ (see also appendix) the integral operator

$$(Au)(x) := -\oint \log 2\sin\frac{x-y}{2}u(y)dy \quad , \quad D(A) = L_2^*(\Gamma) \ ,$$

is applied, which is related to the Hilbert transform ([StE]) by

$$(Au')(x) = -\oint \log 2\sin\frac{x-y}{2}du(y) = \oint \cot\frac{x-y}{2}u(y)dy = (Hu)(x) \ .$$

The operator A can be extended to an isomorphism $A: H_{-1} \to H_0$ ([BrK]). For its kernel function it holds ([GrI] 4.2247/8):

$$\frac{1}{2\pi}\int_0^\infty \log(2\sin\frac{x}{2})dx = 0 \quad , \quad \frac{1}{2\pi}\int_0^\infty \log^2(2\sin\frac{x}{2})dx = \frac{\pi^2}{12}$$

i.e.

$k(x) := \log(2\sin\frac{x}{2}) \in H_0$ and $\hat{k}_0 = 0$ i.e. $k(x)$ is a wavelet (appendix) .

Let $[x]$ denote the largest integer not exceeding the real number x. The fractional part (saw tooth) function of x is defined by

$$\rho(x) = \{x\} = x - [x] = \frac{1}{2} - \sum_1^\infty \frac{\sin 2\pi\nu x}{2\pi|\nu|}$$

It holds ([ZyA] I, theorem 2-7):

$$2\pi H[\rho](\frac{x}{2\pi}) := 2\pi\rho_H(\frac{x}{2\pi}) = -\sum_1^\infty \frac{\cos\nu x}{\nu} = \log 2\sin\frac{x}{2} \in L_2^{\#}(0,2\pi) \quad , \quad 0 < x < 2\pi \ .$$

The operator A is applied to put the following criticism of a proposition of Ramanujan ([BeB] 8, Entry 17 (iv)) into the context of Hilbert space theory:

"*Ramanujan informs us to note that*

$$\sum_1^\infty \sin(2\pi\nu x) = \frac{1}{2}\cot(\pi x) \text{ ,}$$

which also is devoid of meaning, ... may be formally established by differentiating the equality

$$\sum_1^\infty \frac{\cos 2\pi\nu x}{\nu} = -\log 2\sin(\pi x) \quad \text{".}$$

We also note from [BeB] Chapter 6:

"*Ramanujan's theory of divergent series emanates from the Euler-Maclaurin summation formula,* building on sophisticated constant "c" (1.1) of the series $\sum_1^\infty f(k)$.

He claims that the constant of a series is like the center of gravity of a "body" (p. 63).

Putting

$$(A\sigma)(x) := \sum_1^\infty \frac{\sin(2\pi\nu x)}{\nu} \in H_0^\#$$

it follows

$$\sigma = \sum_1^\infty \sin(2\pi\nu\circ) \in H_{-1}^\# \text{ .}$$

Then "differentiating" of a not defined "function" is no longer only formally as it´s defined in a weak sense with respect to the inner product of the Hilbert space $H_{-1}^\#$ i.e.

$$(u,v)_{-1} = (Au,Av)_0 \text{ , } (u,v)_{-1/2} = (Au,v)_0 \text{ , } (u',v')_{-1} = (Au',Av')_0 = (Hu,Hv)_0 = (u,v)_0 \text{ .}$$

We note that the Hilbert space $H_{-1}^\#$ (resp. the smaller Hilbert space $H_{-1/2}^\#$ for a weak variational form representation) is proposed as answer to the still open question from B. Riemann ([RiB]) about a characterization of all functions represented by its trigonometric series ([LaD] 2.2).

The fractional part function is linked to the Zeta function by [TiE] (2.1.5)

$$\frac{\zeta(s)}{s} = -\int_0^\infty x^{-s}([x]-x)\frac{dx}{x} = \int_0^\infty x^{-s}\rho(x)\frac{dx}{x} \quad \text{in the critical stripe } 0 < \mathrm{Re}(s) < 1 \text{ .}$$

From ([TiE] 2.1) we recall the duality relation in the critical stripe

$$\zeta(s) = \chi(s)\zeta(1-s) \quad \text{resp.} \quad \xi(s) = \xi(1-s)$$

with

$$\zeta(1-s) = 2\cdot(2\pi)^{-s}\cos(\frac{\pi}{2}s)\Gamma(s)\zeta(s)$$

and

$$\chi(s) := (\frac{2}{\pi})^{1-s}\Gamma(1-s)\sin(\frac{\pi}{2}s) = \pi^{s-1/2}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = \frac{1}{\chi(1-s)} \text{ .}$$

The crucial relationship of the Hilbert transformed fractional part function and the Zeta function is given by (appendix):

$$-(2\pi)\int_0^\infty y^s\rho_H(\frac{y}{2\pi})\frac{dy}{y} = -\int_0^\infty x^s\log(2\sin\frac{x}{2})\frac{dx}{x} = \zeta(s)\Gamma(s)\cos(\frac{\pi s}{2}) = \frac{1}{2}(2\pi)^s\zeta(1-s) \text{ .}$$

Appendix

The two relevant formulas from [Grl] 3.621, 3.761 are:

$$\int_0^{\pi/2} \sin^{\mu-1} x \cos^{\nu-1} x dx = \frac{1}{2} B(\frac{\mu}{2},\frac{\nu}{2}) \quad , \quad \mathrm{Re}(\mu), \mathrm{Re}(\nu) > 0$$

$$\int_0^{\infty} x^{\mu-1}\cos(ax)dx = \frac{\Gamma(\mu)}{a^{\mu}}\cos(\frac{\mu\pi}{2}) \quad , \quad 0 < \mathrm{Re}(\mu) < 1, a > 0 \cdot$$

Those result for $\mathrm{Re}(1-s) > 0$ into

$$\int_0^{\pi(2} \frac{1}{\cos^s \tau} d\tau = \frac{\sqrt{\pi}}{2}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{2-s}{2})}$$

and for $0 < \mathrm{Re}(s) < 1$ and $c := \sqrt{\frac{2}{\pi}}$ into

$$\int_0^{\infty} x^{s-1}\cos(x)dx = \Gamma(s)\cos(\frac{\pi s}{2}) = \sqrt{\pi}2^{s-1}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1-s}{2})} = (\frac{\pi}{2})^{1-s}\chi(s) = \frac{1}{c}\frac{c^s}{c^{1-s}}\chi(s) \cdot$$

Lemma: The Mellin transform of $f_H(x)$ is given by

$$M[f_H](s) = 2\pi^{-s/2}\Gamma(\frac{1+s}{2})\int_0^{\pi(2}\frac{1}{\cos^s\tau}d\tau = \pi^{(1-s)/2}\tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) \qquad \text{for } \mathrm{Re}(1-s) > 0$$

and it holds

$$\frac{\Gamma(\frac{1+s}{2})\Gamma(\frac{1-s}{2})}{\Gamma(1-\frac{s}{2})} = \frac{\pi\sin(\frac{\pi}{2}s)}{\pi\cos(\frac{\pi}{2}s)}\Gamma(\frac{s}{2}) = \tan(\frac{\pi}{2}s)\Gamma(\frac{s}{2}) = \cot(\frac{\pi}{2}(1-s))\Gamma(\frac{s}{2}) \cdot$$

Proof: It follows by variable substitutions $\pi x^2(1-t) = y$, $t = z^2$ and $z = \sin\tau$

$$\int_0^{\infty} x^s f_H(x)\frac{dx}{x} = 2\sqrt{\pi}\int_0^1\int_0^{\infty} x^{s+1}e^{-\pi x^2(1-t)}\frac{dx}{x}\sqrt{t}\frac{dt}{t}$$

$$= 2\sqrt{\pi}\int_0^1\frac{1}{2}\left[\frac{1}{\pi(1-t)}\right]^{(s+1)/2}\sqrt{t}\frac{dt}{t}\int^{\infty} y^{(s+1)/2}e^{-y}\frac{dy}{y} = \pi^{-s/2}\Gamma(\frac{1+s}{2})(\int_0^1\left[\frac{1}{(1-t)}\right]^{(s+1)/''}\sqrt{t}\frac{dt}{t} \cdot$$

Therefore

$$= \pi^{-s/2}\Gamma(\frac{1+s}{2})\int_0^1\left[\frac{1}{(1-x^2)}\right]^{(s+1)/2}2x\frac{dx}{x} = \pi^{-s/2}\Gamma(\frac{1+s}{2})\int_0^{\pi/2}\left[\frac{1}{(1-x^2)}\right]^{(s+1)/2}2x\frac{dx}{x}$$

$$= \pi^{-s/2}\Gamma(\frac{1+s}{2})2\int_0^{\pi/2}\left[\frac{1}{\cos^2\tau}\right]^{(s+1)/2}\cos\tau d\tau \cdot$$

Lemma: It holds

$$-(2\pi)\int_0^{\infty} y^s\rho_H(\frac{y}{2\pi})\frac{dy}{y} = -\int_0^{\infty} x^s\log(2\sin\frac{x}{2})\frac{dx}{x} = \zeta(s)\Gamma(s)\cos(\frac{\pi s}{2}) = \frac{1}{2}(2\pi)^s\zeta(1-s)$$

Proof: With the formula from ([Grl] 3.761) above it holds for $0 < \mathrm{Re}(S) < 1$

$$2\cdot(2\pi)^{-s}\int_0^{\infty} x^s\cos(nx)\frac{dx}{x} = 2\cdot(2\pi)^{-s}\int_0^{\infty}\frac{1}{n^1}y^s\cos(y)\frac{dy}{y} = 2\cdot(2\pi)^{-s}\int_0^{\infty} y^s\cos y\frac{dy}{y} = 2\cdot(2\pi)^{-s}\Gamma(s)\cos(\frac{\pi}{2}s) = 2\cdot(2\pi)^{-s}(\frac{\pi}{2})^{1-s}\chi(s) \cdot$$

$$-2\pi\int_0^{\infty} x^s\rho_H(\frac{x}{2\pi})\frac{1}{x} = \sum_1^{\infty}\frac{1}{n}\int_0^{\infty} x^s\cos(nx)\frac{dx}{x} = \sum_1^{\infty}\frac{1}{n^s}\int_0^{\infty}\frac{1}{n^1}y^s\cos(y)\frac{dy}{y} = \zeta(s)\int_0^{\infty} y^s\cos y\frac{dy}{y} = \zeta(s)\Gamma(s)\cos(\frac{\pi}{2}s) \cdot$$

Hilbert, Fourier (wave), wavelet and Mellin transformations

Let and $H_0 := L_2(-\infty,\infty)$ the Hilbert space of (complex-valued) L_2 –functions and let $f,g,\psi \in H_0$, $\overline{\psi}$ the conjugate of ψ and $(y,z) \in R^+ xR$. Then the Fourier F, the Hilbert H and the wavelet W_ψ transformation operators are defined by

Definition ((wave) Fourier (inverse) transformation): The Fourier transform is defined by

$$F(f)(v) := \hat{f}(v) := f_v := \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ivx}dx \quad \text{and it holds} \quad f(x) = \int_{-\infty}^{\infty} \hat{f}(x)e^{-ivx}dx$$

Definition (Hilbert (inverse) transformation) ([PeB] example 9.11): The Hilbert transform is defined by

$$(Hf)(x) := \lim_{\varepsilon \to 0} \frac{1}{\pi} \int_{|x-t|>\varepsilon} \frac{f(t)}{x-t}dt \quad \text{and it holds} \quad f(x) = -(H^2 f)(x) \ .$$

Lemma: Let $\hat{f}(\sigma), \hat{g}(\sigma)$ denote the Fourier coefficients of L_2 –function f,g. Then it holds

i) $H(f), H(g)$ are L_2 –functions and $H^2(f) = -f$, $(Hf,g) = -(f,Hg)$, $\|Hf\| = \|f\|$, $(Hf,f) = 0$

ii) For $g := H(f)$ it holds $\hat{g}(0) = 0$ and $[xH - Hx](f)(x) = constant = \hat{f}(o)$

iii) If f is odd, then $[xH - Hx](f)(x) = 0$

iv) H commutes with translations and homothesis

v) H anticommutes with reflections.

A function $\psi \in L_2(-\infty,\infty)$ with compact domain is called a normalized wavelet if it has vanishing mean value, i.e.

$$\int_{-\infty}^{\infty} \psi(x)d = 0 \quad \text{and} \quad \|\psi\|_0 = 1 \ .$$

A wavelet is called “valid”, if it holds

$$0 < c_\psi := \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \ .$$

Definition (wavelet (inverse) transformations): For a normalized, valid wavelet ψ and for $f \in L_2(R)$ it holds

$$W_\psi(f)(y,z) := \frac{1}{\sqrt{y}} \int_{-\infty}^{\infty} f(x) \cdot \overline{\psi}\left(\frac{x-y}{z}\right)dx \quad \text{and it holds} \quad f(x) = \int_{-\infty}^{\infty} W_\psi(f) \cdot \frac{1}{\sqrt{y}} \psi\left(\frac{x-y}{z}\right)\frac{dy}{y^2}dz \ , \ (y,z) \in R^+ xR \ .$$

In case of a real valued ψ *then* $\hat{\psi}$ is symmetric.
For $(n = 1)$

$$\psi_y(x) := \frac{1}{\sqrt{y}} \psi\left(\frac{x}{y}\right)$$

(which is a Theta function-like structure) it holds $\|\psi_y\|_0 = \|\psi\|_0$ and the Calderón (reproducing) formula

$$f = \int_0^{\infty} \psi_y * \psi_y * f \frac{dy}{y} \ .$$

Definition (Mellin (inverse) transformation): If $F(s)$ is analytic in the strip $a < \mathrm{Re}(s) < b$, and if it tends to zero uniformly as $\mathrm{Im}(s) \to \pm\infty$ for any real value c between a and b, with its integral along such a line converging absolutely, then if

$$f(x) = \left[M^{-1}F\right](x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}F(s)ds .$$

We have that

$$F(s) = \left[Mf\right](s) = \int_0^\infty x^s f(x) d\log x .$$

Conversely, suppose $f(x)$ is piecewise continuous on the positive real numbers, taking a value halfway between the limit values at any jump discontinuities, and suppose the integral

$$F(s) = \left[Mf\right](s) = \int_0^\infty x^s f(x) d\log x$$

is absolutely convergent when $a < \mathrm{Re}(s) < b$. Then $f(x)$ is recoverable via the inverse Mellin transform from its Mellin transform $F(s)$.

Lemma: It holds

i) $M\left[h'\right](s) = (1-s)M\left[h\right](s-1)$

ii) $M\left[xh'\right](s) = -sM\left[h\right](s)$

iii) $M\left[(xh)'\right](s) = (1-s)M\left[h\right](s)$

iv) $M\left[h''\right](s) = (s-1)(s-2)M\left[h\right](s-2)$

v) $(h * M\left[g\right])(s) = \int_0^\infty h(\frac{x}{u})g(u)\frac{du}{u} = \int_0^\infty u^{-s}h(\frac{x}{u})g(u)u^s \frac{du}{u}$.

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