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# Inner Product Spaces and Krein Spaces in the Quaternionic Setting

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# Inner Product Spaces and Krein Spaces in the Quaternionic Setting

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# INNER PRODUCT SPACES AND KREIN SPACES IN THE QUATERNIONIC SETTING

DANIEL ALPAY, FABRIZIO COLOMBO, AND IRENE SABADINI

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**ABSTRACT.** In this paper we provide a study of quaternionic inner product spaces. This includes ortho-complemented subspaces, fundamental decompositions as well as a number of results of topological nature. Our main purpose is to show that a closed uniformly positive subspace in a quaternionic Krein space is ortho-complemented, and this leads to our choice of the results presented in the paper.

## 1. INTRODUCTION

The purpose of this paper is to study quaternionic inner product spaces and, in particular, Krein spaces. Quaternionic Hilbert spaces are known for a long time, see for instance [12], and [1, 18, 17] for various applications to quantum mechanics. Some aspects of the theory of quaternionic Pontryagin spaces have been studied in [4]. The finite dimensional case is also of particular interest; see e.g., [20, 28, 3, 21, 22, 25]. Krein spaces are, roughly speaking, the direct sum of two in general *infinite dimensional* Hilbert spaces, and in particular the previous references do not treat this case. While preparing the work [2] on interpolation of Schur multipliers in the case of vector-valued slice-hyperholomorphic functions, we realized that no reference seemed to be available for a number of important results on quaternionic Krein spaces. The motivation of the present paper was to fill part of this gap. In the process, we found that we needed to

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prove several functional analysis results in the quaternionic setting.

In the complex case, the starting point is a complex vector space  $\mathcal{V}$ , endowed with a sesquilinear form  $[\cdot, \cdot]$ . The pair  $(\mathcal{V}, [\cdot, \cdot])$  is called an indefinite inner product space, and many important concepts are associated to such a pair, some algebraic and some topological. The combination of both is a main feature of the general theory developed in [10]. The form  $[\cdot, \cdot]$  defines an orthogonality: two vectors  $v, w \in \mathcal{V}$  are orthogonal if  $[v, w] = 0$ , and two linear subspaces  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $\mathcal{V}$  are orthogonal if every vector of  $\mathcal{V}_1$  is orthogonal to every vector of  $\mathcal{V}_2$ . Orthogonal sums will be denoted by the symbol  $[+]$ . Note that two orthogonal spaces may intersect. We will denote by the symbol  $[\oplus]$  a direct orthogonal sum. A complex vector space  $\mathcal{V}$  is a Krein space if it can be written (in general in a non-unique way) as  $\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-$ , where  $(\mathcal{V}_+, [\cdot, \cdot])$  and  $(\mathcal{V}_-, -[\cdot, \cdot])$  are Hilbert spaces. When the space  $\mathcal{V}_-$  (or, as in [19], the space  $\mathcal{V}_+$ ) is finite dimensional (note that this property does not depend on the decomposition),  $\mathcal{V}$  is called a Pontryagin space.

Krein spaces were introduced by Krein and later and independently by L. Schwartz in [24] where they were called "Hermitian spaces"; for historical remarks, we refer to [4, pp. 207-209]. Besides the book of Bogner [10], on which is based this work, we refer to [5, 15] for the theory of Krein spaces and of their operators, and to [19] for the case of Pontryagin spaces.

Among other topics, Krein spaces appear in a natural way in the theory of interpolation for operator-valued Schur functions (see e.g., [9, 8, 7, 6]). The motivation for the present work came in particular from the desire to extend interpolation theory for operator-valued Schur functions to the case of slice-hyperholomorphic functions, see the forthcoming work [2], as we now explain. Let  $\mathcal{Y}$  and  $\mathcal{U}$  be two Hilbert spaces. We denote by  $\mathcal{S}(\mathcal{U}, \mathcal{Y})$  the class of  $\mathbf{L}(\mathcal{U}, \mathcal{Y})$ -valued functions analytic and contractive in the open unit disk. To define the left-interpolation problem in this class we need a third Hilbert space, say  $\mathcal{X}$ , and two operators  $A \in \mathbf{L}(\mathcal{X})$  and  $C \in \mathbf{L}(\mathcal{X}, \mathcal{Y})$ . We assume that the series  $\sum_{n=0}^{\infty} A^{*n} C^* C A^n$  converges in the strong operator topology. The interpolation problem at hand is to find all (if any) functions  $S \in \mathcal{S}(\mathcal{U}, \mathcal{Y})$  such that

$$\sum_{k=0}^{\infty} A^{*k} C^* S_k = N^*,$$

where the  $S_k \in \mathbf{L}(\mathcal{U}, \mathcal{Y})$  are the coefficients of the power expansion of  $S$  at the origin and  $N \in \mathbf{L}(\mathcal{X}, \mathcal{U})$  is given. Krein spaces appear as follows in the solution of this problem. Let  $P$  be the solution of the Stein equation

$$P - A^* P A = (C^* \ N^*) J \begin{pmatrix} C \\ N \end{pmatrix}, \quad \text{where } J = \begin{pmatrix} I_{\mathcal{Y}} & 0 \\ 0 & -I_{\mathcal{U}} \end{pmatrix}$$

and assume  $P$  positive and boundedly invertible. We endow the space  $\mathcal{K} = \mathcal{X} \oplus \mathcal{Y} \oplus \mathcal{U}$  with the indefinite metric defined by

$$\tilde{J} = \begin{pmatrix} P & 0 \\ 0 & J \end{pmatrix}.$$

A key result in the arguments is that the space

$$\mathcal{K}_0 := \text{Ran} \begin{pmatrix} A \\ C \\ N \end{pmatrix}$$

is a closed uniformly positive subspace of  $\mathcal{K}$  and thus it is ortho-complemented. The proof of this last fact is in [10], and requires a long chain of preliminary results and we are not aware of any shortcut proof. As we previously remarked, the main purpose of this paper is to prove the counterpart of this fact in the quaternionic setting. To this end, we first need to prove some algebraic as well as topological results for quaternionic vector spaces which are of independent interest. The complex version of these results can be found in [10], [16], [23] (we will give more precise references where appropriate). In most cases, the proofs are not substantially different from the proofs of the corresponding results in the complex case. However, since we are not aware of any reference in which these results are explicitly proven in the quaternionic setting, we repeat them here.

The paper consists of nine sections besides the introduction, and its outline is as follows: Sections 2 and 3 are devoted to quaternionic topological vector spaces and to some basic functional analysis theorems in a quaternionic setting. These sections set the framework for the following sections, where one consider quaternionic vector spaces endowed with a possibly degenerate and non positive inner product. The algebraic aspects of such spaces are studied in Sections 4, 5 and 6. The notion of fundamental decomposition is studied in detail and plays a key role in the paper in the later sections. Topologies which make the inner product continuous (called *majorants*) are studied in Sections 6, 7 and 8. Finally, some aspects of Krein spaces are studied in Section 9.

## 2. QUATERNIONIC TOPOLOGICAL VECTOR SPACES

In this paper  $\mathbb{H}$  denotes the algebra of real quaternions. We send to [12, Chapter I] and to [4, p. 446] for the basic definitions of a vector space over  $\mathbb{H}$ . In this paper we will treat the case of right quaternionic vector spaces. The case of left quaternionic vector spaces may be treated in an analogous way. It is also useful to recall that if  $\mathcal{W}$  and  $\mathcal{V}$  are two right quaternionic vector spaces, an operator  $A : \mathcal{V} \rightarrow \mathcal{W}$  is right linear if

$$A(v_1q_1 + v_2q_2) = (Av_1)q_1 + (Av_2)q_2, \quad \forall v_1, v_2 \in \mathcal{V} \quad \text{and} \quad \forall q_1, q_2 \in \mathbb{H}.$$

For future reference we single out the following result, which is true for the case of vector spaces over any field or skew field. The claims are [11, Théorème 1 and Proposition 4, Ch. 2, §7] respectively.

**Theorem 2.1.** (a) *Every right quaternionic vector space has a basis.*  
 (b) *Every (right) linear subspace of a quaternionic vector space has a direct complement.*

We also recall the following: If  $\mathcal{V}$  is a right quaternionic vector space and  $\mathcal{V}_1 \subset \mathcal{V}$  is a (right) linear subspace of  $\mathcal{V}$ , the quotient space  $\mathcal{V}/\mathcal{V}_1$  endowed with

$$(v + \mathcal{V}_1)q = vq + \mathcal{V}_1$$

is also a right quaternionic vector space. Here  $v + \mathcal{V}_1$  denotes the equivalence class in the quotient space  $\mathcal{V}/\mathcal{V}_1$  of  $v \in \mathcal{V}_1$ .

Given a right quaternionic vector space  $\mathcal{V}$ , a semi-norm is defined (as in the complex case) as a map  $p : \mathcal{V} \rightarrow \mathbb{R}$  such that

$$p(v_1 + v_2) \leq p(v_1) + p(v_2), \quad \forall v_1, v_2 \in \mathcal{V}, \quad (2.1)$$

and

$$p(vc) = |c|p(v), \quad \forall v \in \mathcal{V} \text{ and } c \in \mathbb{H}. \quad (2.2)$$

**Remark 2.2.** Note that (2.2) implies that  $p(0) = 0$  and (2.1) implies

$$0 = p(v - v) \leq 2p(v),$$

so that a semi-norm has values in  $\mathbb{R}^+$ .

As it is well known, see [12, Ch. II, §1], given a vector space over a non discrete valued division ring it is possible to introduce the notion of semi-norm. We observe that one can give the notion of semi-norm in the framework of modules over a Clifford algebra, see [13]. However, in that case, (2.2) is required only when  $c \in \mathbb{R}$  while in general it has to be replaced by the weaker condition  $p(vc) \leq C|c|p(v)$ , where  $C$  is a suitable constant.

A family of semi-norms on  $\mathcal{V}$  gives rise to a topology which, at least in the cases of complex or real vector spaces, leads to a locally convex space.

Let  $p$  be a semi-norm and set

$$U_{v_0}(p, \alpha) \stackrel{\text{def.}}{=} \{v \in \mathcal{V} \mid p(v - v_0) < \alpha\}.$$

A family  $\{p_\gamma\}_{\gamma \in \Gamma}$  of semi-norms on  $\mathcal{V}$  defines a topology on  $\mathcal{V}$ , in which a subset  $U \subseteq \mathcal{V}$  is said to be open if and only if for every  $v_0 \in U$  there are  $\gamma_1, \dots, \gamma_n \in \Gamma$  and  $\varepsilon > 0$  such that  $v \in U_{v_0}(p_{\gamma_j}, \varepsilon)$ ,  $j = 1, \dots, n$ , implies  $v \in U$ .

**Remark 2.3.** All the spaces considered here will be right linear, and in general we will use the terminology *quaternionic vector space* rather than *right quaternionic vector space*. Similarly we will speak of linear operators rather than right linear operators.

A quaternionic vector space  $\mathcal{V}$  is also a vector space over  $\mathbb{R}$ . It is immediate to verify using (2.1) and (2.2) that when it is endowed with the topology induced by a family of semi-norms, it is a locally convex space.

**Definition 2.4.** A set  $U$  in a topological quaternionic vector space  $\mathcal{V}$  is called *balanced* if  $vc \in U$ , whenever  $v \in U$  and  $c \in \mathbb{H}$  with  $|c| \leq 1$ . A set  $U \in \mathcal{V}$  is said to be *absorbing* if for any  $v \in \mathcal{V}$  there exists  $c > 0$  such that  $vc^{-1} \in U$ .

**Proposition 2.5.** *Let  $p$  be a semi-norm on a quaternionic vector space  $\mathcal{V}$ , let  $\alpha > 0$ . Then the set  $U_0(p, \alpha) = \{v \in \mathcal{V} \mid p(v) < \alpha\}$  is balanced and absorbing.*

*Proof.* By (2.2), if  $v \in U_0(p, \alpha)$  and  $|c| \leq 1$  then  $p(vc) = |c|p(v) < \alpha$  so  $U_0(p, \alpha)$  is balanced. Similarly one proves that  $U_0(p, \alpha)$  is absorbing.  $\square$

We recall the definition of the Minkowski functional  $p_U$  associated to a convex, balanced and absorbing set  $U$ :

$$p_U(v) = \inf A_v \quad \text{where} \quad A_v = \{a > 0 : va^{-1} \in U\}, \quad v \in V. \quad (2.3)$$

**Proposition 2.6.** *Let  $\mathcal{V}$  be a quaternionic vector space, and let  $U$  be a convex, balanced, absorbing set containing 0. Then the Minkowski functional  $p_U(v)$  is a semi-norm on  $\mathcal{V}$ .*

*Proof.* Let  $v_1, v_2 \in \mathcal{V}$  and  $c \in A_{v_1}, d \in A_{v_2}$ . Then  $v_1c^{-1} + v_2d^{-1} \in U$  or, equivalently,  $v_1 + v_2 \in cU + dU = (c+d)U$  since  $U$  is convex. Thus  $c+d \in A_{v_1+v_2}$  and  $p_U(v_1+v_2) \leq c+d$ , from which we conclude (2.1) since  $c$  and  $d$  are arbitrary.

To prove (2.2), we begin by considering  $\lambda > 0$  and  $v \in \mathcal{V}$ . Take any  $c \in A_v$ , then we have  $vc^{-1} \in U$  and  $v\lambda(\lambda c)^{-1} \in U$  and so  $\lambda c \in A_{v\lambda}$  and then  $p_U(v\lambda) \leq \lambda c$ . By the arbitrariness of  $c$  it follows that  $p_U(v\lambda) \leq \lambda p_U(v)$ . By replacing  $v$  by  $v\lambda$  and  $\lambda$  by  $\lambda^{-1}$  we obtain  $p_U(v) \leq \lambda^{-1} p_U(v\lambda)$  or, equivalently,  $\lambda p_U(v) \leq p_U(v\lambda)$ . Thus  $\lambda p_U(v) = p_U(v\lambda)$ . If we consider  $\lambda = 0$  then (2.2) is trivial since  $p_U(0) = 0$  by definition (2.3). Thus we assume now that  $\lambda \in \mathbb{H}$  and  $\lambda \neq 0$ . Let  $v \in \mathcal{V}$  and  $c \in A_v$ . Since  $U$  is balanced, then  $vc^{-1} \in U$  and also  $v\frac{\lambda}{|\lambda|}c^{-1} \in U$  and so  $p_U(v\lambda) \leq |\lambda|c$ . Since  $c$  is arbitrary, we have  $p_U(v\lambda) \leq |\lambda|p_U(v)$ . The reverse inequality is obtained by replacing  $v$  by  $v\lambda$  and  $\lambda$  by  $\lambda^{-1}$ . The statement follows.  $\square$

**Proposition 2.7.** *A topological quaternionic vector space is locally convex if and only if the topology is defined by a family of semi-norms.*

*Proof.* The "if" part of the statement has already been discussed. To show the "only if" part, consider a base  $B$  of neighborhood at 0 consisting of convex and balanced open sets. Since the multiplication by a scalar on right is continuous, each  $U \in B$  is absorbing. Then for  $U \in B$  we define  $p_U(v) = \inf A_v$  (see (2.3)) and so  $p_U$  is the Minkowski functional. The family  $\{p_U\}_{U \in B}$  is then a family of semi-norms such that

$$\{v \in \mathcal{V} : p_U(v) < 1\} \subseteq U \subseteq \{v \in \mathcal{V} : p_U(v) \leq 1\}$$

and the statement follows.  $\square$

We conclude this section by mentioning that the topology induced by the family of semi-norms  $\{p_\gamma\}_{\gamma \in \Gamma}$  is Hausdorff if and only if the condition  $p_\gamma(v) = 0$  for all  $\gamma \in \Gamma$  implies  $v = 0$ .

### 3. PRINCIPLES OF QUATERNIONIC FUNCTIONAL ANALYSIS

The material in this section is classical for complex Fréchet spaces and can be found e.g. in [16, Chapter II] or [23, Chapter 2].

Let  $\mathcal{V}$  be a quaternionic Fréchet space, that is a quaternionic locally convex topological vector space which is metrizable and complete, and let  $\rho$  be an associated metric. For the sake of simplicity, in the sequel we will write  $|u - w|$  instead of  $\rho(u, w)$ .

We now prove a result for continuous (not necessarily linear) maps which implies the principle of uniform boundedness.

**Theorem 3.1.** *For each  $a \in A$ , where  $A$  is a set, let  $S_a$  be a continuous map of a quaternionic Fréchet space  $\mathcal{V}$  into a quaternionic Fréchet space  $\mathcal{W}$ , which satisfies the following properties*

- (a)  $|S_a(u+w)| \leq |S_a(u)| + |S_a(w)|, \forall u, w \in \mathcal{V},$   
 (b)  $|S_a(w\alpha)| = |S_a(w)\alpha|, \forall w \in \mathcal{V}, \forall \alpha \geq 0.$

If, for each  $u \in \mathcal{V}$ , the set  $\{S_a v\}_{a \in A}$  is bounded, then  $\lim_{v \rightarrow 0} S_a v = 0$  uniformly in  $a \in A$ .

*Proof.* For  $\varepsilon > 0$ ,  $a \in A$  and a positive integer  $k$ , the set

$$\mathcal{V}_k \stackrel{\text{def.}}{=} \left\{ u \in \mathcal{V} : \left| \frac{1}{k} S_a(u) \right| + \left| \frac{1}{k} S_a(-u) \right| \leq \frac{\varepsilon}{2} \right\}$$

is closed since  $S_a$  are continuous. Moreover, by assumption, the sets  $\{S_a v\}_{a \in A}$  are bounded, so

$$\mathcal{V} = \bigcup_{k=1}^{\infty} \mathcal{V}_k.$$

By the Baire category theorem, there exists a  $\mathcal{V}_{k_0}$  that contains a ball  $B(v_0, \delta)$  with center at  $v_0$  and radius  $\delta > 0$ . Let  $|u| < \delta$ . Then  $v_0$  and  $v_0 + u$  both belong to  $B(v_0, \delta)$  and so they both are in  $\mathcal{V}_{k_0}$ . Thus we have

$$\left| \frac{1}{k_0} S_a(v_0 + u) \right| \leq \frac{\varepsilon}{2} \quad \text{and} \quad \left| \frac{1}{k_0} S_a(-v_0) \right| \leq \frac{\varepsilon}{2}.$$

Using assumption (a) we deduce

$$\left| \frac{1}{k_0} S_a(u) \right| \leq \left| \frac{1}{k_0} S_a(v_0 + u) \right| + \left| \frac{1}{k_0} S_a(-v_0) \right|,$$

and using assumption (b) we get

$$\left| \frac{1}{k_0} S_a(u) \right| = \left| S_a \left( \frac{1}{k_0} u \right) \right| \leq \varepsilon, \quad |u| < \delta, \quad a \in A.$$

Now observe that the mapping  $v \mapsto v/k_0$  is a homeomorphism of  $\mathcal{V}$  into itself since  $\mathcal{V}$  is a topological vector space and thus the multiplication by a quaternionic scalar (in particular, real) is continuous. Thus  $\lim_{v \rightarrow 0} S_a v = 0$  uniformly in  $a \in A$ .  $\square$

In the case of linear maps, Theorem 3.1 gives the following result which will be used in the proofs of Theorem 6.11 and Proposition 8.5 below.

**Theorem 3.2** (Principle of uniform boundedness). *For each  $a \in A$ , where  $A$  is a set, let  $T_a$  be continuous linear map of a quaternionic Fréchet space  $\mathcal{V}$  into a quaternionic Fréchet space  $\mathcal{W}$ . If, for each  $u \in \mathcal{V}$ , the set  $\{T_a v\}_{a \in A}$  is bounded, then  $\lim_{v \rightarrow 0} T_a v = 0$  uniformly in  $a \in A$ .*

For further reference, we repeat also the formulation of the principle of uniform boundedness for quaternionic Banach spaces.

**Theorem 3.3.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two quaternionic Banach spaces and let  $\{T_a\}_{a \in A}$  be bounded linear maps from  $\mathcal{V}$  to  $\mathcal{W}$ . Suppose that  $\sup_{a \in A} \|T_a v\| < \infty$  for any  $v \in \mathcal{V}$ . Then*

$$\sup_{a \in A} \|T_a\| < \infty.$$

The next result is the quaternionic counterpart of the open mapping theorem:



**Theorem 3.4** (Open mapping theorem). *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two quaternionic Fréchet spaces, and let  $T$  be a linear continuous quaternionic map from  $\mathcal{V}$  onto  $\mathcal{W}$ . Then the image of every open set is open.*

*Proof.* Let  $B_{\mathcal{V}}(r) \subset \mathcal{V}$  denote the open ball of radius  $r > 0$  and centered at the origin and let  $B_{\mathcal{V}}(r) - B_{\mathcal{V}}(r)$  be the set of elements of the form  $u - v$  where  $u, v \in B_{\mathcal{V}}(r)$ . Since the function  $u - v$  is continuous in  $u$  and  $v$ , there exists a ball  $B_{\mathcal{V}}(r')$ , for suitable  $r' > 0$ , such that  $B_{\mathcal{V}}(r') - B_{\mathcal{V}}(r') \subseteq B_{\mathcal{V}}(r)$ . For every  $v \in \mathcal{V}$  we have that  $v/n \rightarrow 0$  as  $n \rightarrow \infty$  so  $v \in nB_{\mathcal{V}}(r')$  for a suitable  $n \in \mathbb{N}$ . So

$$\mathcal{V} = \bigcup_{n=1}^{\infty} nB_{\mathcal{V}}(r') \quad \text{and} \quad \mathcal{W} = T\mathcal{V} = \bigcup_{n=1}^{\infty} nTB_{\mathcal{V}}(r').$$

By the Baire category theorem one of the closures  $\overline{nTB_{\mathcal{V}}(r')}$  contains a non empty open set. The map  $w \mapsto nw$  is a homeomorphism in  $\mathcal{W}$  and  $\overline{TB_{\mathcal{V}}(r')}$  contains a non empty open set denoted by  $\mathcal{B}$ , so

$$\overline{TB_{\mathcal{V}}(r)} \supseteq \overline{TB_{\mathcal{V}}(r') - TB_{\mathcal{V}}(r')} \supseteq \overline{TB_{\mathcal{V}}(r')} - \overline{TB_{\mathcal{V}}(r')} \supseteq \mathcal{B} - \mathcal{B}. \quad (3.1)$$

The map  $w \mapsto u - w$  is a homeomorphism and hence the set  $u - B_{\mathcal{V}}(r)$  is open. Since the set  $\mathcal{B} - \mathcal{B} = \bigcup_{u \in \mathcal{B}} (u - \mathcal{B})$  is open (as the union of open sets) and contains the origin, we conclude from (3.1) that  $\overline{TB_{\mathcal{V}}(r)}$  contains a neighborhood of the origin.

Fix an arbitrary  $\varepsilon_0$  and let  $\varepsilon_{\ell} > 0$  be a sequence such that  $\sum_{\ell \in \mathbb{N}} \varepsilon_{\ell} < \varepsilon_0$ . Then there exists a sequence  $\theta_{\ell} > 0$  with  $\theta_{\ell} \rightarrow 0$  such that

$$\overline{TB_{\mathcal{V}}(\varepsilon_{\ell})} \supset B_{\mathcal{W}}(\theta_{\ell}), \quad \ell \in \mathbb{N} \cup \{0\}. \quad (3.2)$$

We now take an arbitrary  $w \in B_{\mathcal{W}}(\theta_0)$  and show that  $w = Tv$  for some  $v \in B_{\mathcal{V}}(2\varepsilon_0)$ . To this purpose we follow a recursive procedure. From (3.2) for  $\ell = 0$  there exists  $v_0 \in B_{\mathcal{V}}(\varepsilon_0)$  such that  $|w - Tv_0| < \theta_1$ . Since  $w - Tv_0 \in B_{\mathcal{W}}(\theta_1)$ , then  $w - Tv_0 \in TB_{\mathcal{V}}(\varepsilon_1)$  and again from (3.2) with  $\ell = 1$ , there exists  $v_1 \in B_{\mathcal{V}}(\varepsilon_1)$  such that  $|w - Tv_0 - Tv_1| < \theta_2$ . Iterating this procedure, we construct a sequence  $\{v_n\}_{n \in \mathbb{N}}$  such that  $v_n \in B_{\mathcal{V}}(\varepsilon_n)$  and

$$|w - T \sum_{\ell=0}^n v_{\ell}| < \theta_{n+1}, \quad n \in \mathbb{N} \cup \{0\}. \quad (3.3)$$

Let us denote  $p_m = \sum_{\ell=0}^m v_{\ell}$ . Then  $\{p_m\}$  is a Cauchy sequence since

$$|p_m - p_n| = |v_{n+1} + \dots + v_m| < \varepsilon_{n+1} + \dots + \varepsilon_m \quad \text{for } m > n.$$

Therefore the series  $\sum_{\ell=0}^{\infty} v_{\ell}$  converges to a point  $v \in \mathcal{V}$  with  $|v| \leq \sum_{\ell=0}^{\infty} \varepsilon_{\ell} = 2\varepsilon_0$ . Since  $T$  is continuous, we conclude from (3.3) that  $w = Tv$ . We thus showed that an arbitrary ball  $B_{\mathcal{W}}(2\varepsilon_0)$  is mapped onto the set  $TB_{\mathcal{V}}(2\varepsilon_0)$  which contains the ball  $B_{\mathcal{W}}(\theta_0)$ . So if  $\mathcal{N}$  is a neighborhood of the origin in  $\mathcal{V}$  then  $T\mathcal{N}$  contains a neighborhood of the origin of  $\mathcal{W}$ . Since  $T$  is linear then the above procedure works for every neighborhood of every point.  $\square$

**Theorem 3.5** (Banach continuous inverse theorem). *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two quaternionic Fréchet spaces and let  $T : \mathcal{V} \rightarrow \mathcal{W}$  be a one-to-one linear continuous quaternionic map. Then  $T$  has a linear continuous inverse.*

*Proof.* By Theorem 3.4  $T$  maps open sets onto open sets, so if we write  $T$  as  $(T^{-1})^{-1}$ , it is immediate that  $T^{-1}$  is continuous. Now take  $w_1, w_2 \in \mathcal{W}$  and  $v_1, v_2 \in \mathcal{V}$  such that  $Tv_1 = w_1, Tv_2 = w_2$  and  $p \in \mathbb{H}$ . Then

$$T(v_1 + v_2) = Tv_1 + Tv_2 = w_1 + w_2, \quad T(v_1p) = T(v_1)p = w_1p$$

and hence

$$T^{-1}(w_1 + w_2) = v_1 + v_2 \quad \text{and} \quad T^{-1}(w_1p) = v_1p,$$

so  $T^{-1}$  is linear quaternionic operator.  $\square$

**Definition 3.6.** *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two quaternionic Fréchet spaces. Suppose that  $T$  is a quaternionic operator whose domain  $\mathcal{D}(T)$  is a linear manifold contained in  $\mathcal{V}$  and whose range belongs to  $\mathcal{W}$ . The graph of  $T$  consists of all point  $(v, Tv)$ , with  $v \in \mathcal{D}(T)$ , in the product space  $\mathcal{V} \times \mathcal{W}$ .*

**Definition 3.7.** *We say that  $T$  is a closed operator if its graph is closed in  $\mathcal{V} \times \mathcal{W}$ .*

**Remark 3.8.** Equivalently we can say that  $T$  is closed if  $v_n \in \mathcal{D}(T)$ ,  $v_n \rightarrow v$ ,  $Tv_n \rightarrow y$  imply that  $v \in \mathcal{D}(T)$  and  $Tv = y$ .

The following theorem can be found also in [12, Corollaire 5, p. I.19].

**Theorem 3.9** (Closed graph theorem). *Let  $\mathcal{V}$  and  $\mathcal{W}$  be two quaternionic Fréchet spaces. Let  $T : \mathcal{V} \rightarrow \mathcal{W}$  be a linear closed quaternionic operator. Then  $T$  is continuous.*

*Proof.* Since  $\mathcal{V}$  and  $\mathcal{W}$  are two quaternionic Fréchet spaces we have that  $\mathcal{V} \times \mathcal{W}$  with the distance  $|(v, w)|_{\mathcal{V} \times \mathcal{W}} = |v|_{\mathcal{V}} + |w|_{\mathcal{W}}$  is a quaternionic Fréchet space. The graph of  $T$  denoted by  $\mathcal{G}(T) = \{(v, Tv), v \in \mathcal{D}(T)\}$  is a closed linear manifold in the product space  $\mathcal{V} \times \mathcal{W}$  so it is a quaternionic Fréchet space. The projection

$$P_{\mathcal{V}} : \mathcal{G}(T) \mapsto \mathcal{V}, \quad P_{\mathcal{V}}(v, Tv) = v$$

is one-to-one and onto, linear and continuous so by Theorem 3.5 its inverse  $P_{\mathcal{V}}^{-1}$  is continuous. Now consider the projection

$$P_{\mathcal{W}} : \mathcal{G}(T) \mapsto \mathcal{W}, \quad P_{\mathcal{W}}(v, Tv) = Tv,$$

observing that  $T = P_{\mathcal{W}}P_{\mathcal{V}}^{-1}$  we get the statement.  $\square$

#### 4. ORTHO-COMPLEMENTED SPACES

From this section on, we focus on quaternionic vector spaces endowed with an inner product, defined as follows:

**Definition 4.1.** *Let  $\mathcal{V}$  be a quaternionic vector space. The map*

$$[\cdot, \cdot] : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{H}$$

*is called an inner product if it is a (right) sesquilinear form:*

$$[v_1c_1, v_2c_2] = \overline{c_2}[v_1, v_2]c_1, \quad \forall v_1, v_2 \in \mathcal{V}, \text{ and } c_1, c_2 \in \mathbb{H},$$

*and Hermitian:*

$$[v, w] = \overline{[w, v]}, \quad \forall v, w \in \mathcal{V}.$$

We will call the pair  $(\mathcal{V}, [\cdot, \cdot])$  (or the space  $\mathcal{V}$  for short when the form is understood from the context) a (right) quaternionic indefinite inner product space. A form is called positive (or non-negative) if  $[v, v] \geq 0$  for all  $v \in \mathcal{V}$ .

**Remark 4.2.** Note that the Cauchy-Schwarz inequality holds for positive inner product spaces; see [10, Lemma 2.2, p. 5] for the classical case and [4, Lemma 5.6 and Remark 5.7, p. 447] and the references therein for the quaternionic case. Multiplying the inner product by  $-1$ , we see that the Cauchy-Schwarz inequality holds in inner product spaces for which the inner product is negative. We will call an inner product space definite, if it is either positive or negative.

The definitions on indefinite product spaces over  $\mathbb{C}$  reviewed in the introduction carry over when one considers the quaternions. In particular, two elements  $v$  and  $w$  in  $\mathcal{V}$  will be called orthogonal if  $[v, w] = 0$ , and two vector subspaces  $\mathcal{V}_1$  and  $\mathcal{V}_2$  of  $\mathcal{V}$  are orthogonal if every element of  $\mathcal{V}_1$  is orthogonal to every element of  $\mathcal{V}_2$ . Two orthogonal subspaces  $\mathcal{V}_1$  and  $\mathcal{V}_2$  may have a non trivial intersection. When their intersection reduces to the zero vector we denote by  $\mathcal{V}_1[\oplus]\mathcal{V}_2$  their direct orthogonal sum.

For  $\mathcal{L} \subset \mathcal{V}$  we set

$$\mathcal{L}^{[\perp]} = \{v \in \mathcal{V} : [v, w] = 0, \forall w \in \mathcal{L}\}.$$

The definition makes sense even when  $\mathcal{L}$  is not a linear space, but a mere subset of  $\mathcal{V}$ , and the set  $\mathcal{L}^{[\perp]}$  is always a linear space. It is called the *orthogonal companion* of  $\mathcal{L}$ . Note that

$$\mathcal{L} \subset (\mathcal{L}^{[\perp]})^{[\perp]} \stackrel{\text{def.}}{=} \mathcal{L}^{[\perp\perp]}. \quad (4.1)$$

A linear subspace  $\mathcal{L}$  is called non-degenerate if its *isotropic part*  $\mathcal{L}^0 \stackrel{\text{def.}}{=} \mathcal{L} \cap \mathcal{L}^{[\perp]}$  is trivial.

**Proposition 4.3.** *Let  $\mathcal{V}$  be a quaternionic inner product space, and let  $\mathcal{V}^0$  be its isotropic part. The formula*

$$[v + \mathcal{V}^0, w + \mathcal{V}^0]_q \stackrel{\text{def.}}{=} [v, w] \quad (4.2)$$

*defines a non-degenerate indefinite inner product on  $\mathcal{V}/\mathcal{V}^0$ .*

*Proof.* It suffices to note that formula (4.2) is well defined (that is, does not depend on the specific choice of  $v$  and  $w$ ).  $\square$

We now move to ortho-complemented spaces. Following [10, p. 18] we say that the space  $\mathcal{L}$  is *ortho-complemented* if  $\mathcal{V}$  is spanned by  $\mathcal{L}$  and  $\mathcal{L}^{[\perp]}$ . As explained in the introduction, the motivation for the present work was to prove in the quaternionic setting that a uniformly positive and closed subspace of a quaternionic Krein space is ortho-complemented (the definitions of these various notions appear in the sequel and the result itself is Theorem 9.13 below). In this section we will prove some results in the quaternionic setting whose counterparts for complex vector spaces can be found in [10, Chapter I]. We begin by stating the following direct consequence of (4.1).

**Proposition 4.4.** *Let  $\mathcal{V}$  be a quaternionic vector space, and let  $\mathcal{L}$  denote a linear subspace of  $\mathcal{V}$  which is ortho-complemented. Then,  $\mathcal{L}^{[\perp]}$  is also ortho-complemented.*

A linear subspace  $\mathcal{M} \subset \mathcal{V}$  is called positive if  $[m, m] \geq 0$  for all  $m \in \mathcal{M}$ . It is called strictly positive if the inequality is strict for all  $m \neq 0$ . Similar definitions hold for negative and strictly negative subspaces. A linear subspace will be called definite if it is either positive or negative, and indefinite otherwise. It will be called neutral if  $[m, m] = 0$  for all  $m \in \mathcal{M}$ .

The following result is stated for future reference. Note that in the statement the spaces may have a non trivial intersection.

**Proposition 4.5.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be an indefinite quaternionic vector space.*

(a) *Let  $\mathcal{M}_1, \dots, \mathcal{M}_n$  be  $n$  pairwise orthogonal subspaces of  $\mathcal{V}$ . Assume that all the  $\mathcal{M}_i$  are positive (resp. neutral, negative, strictly positive, strictly negative). Then the space spanned by the  $\mathcal{M}_i$  has the same property.*

(b) *Let  $m_1, \dots, m_n \in \mathcal{V}$  be vectors which are positive (resp. neutral, negative, strictly positive, strictly negative). Then, for every choice of  $q_1, \dots, q_n \in \mathbb{H}$  the vector*

$$m = \sum_{j=1}^n m_j q_j$$

*is positive (resp. neutral, negative and when at least one of the  $q_j \neq 0$ , strictly positive, strictly negative).*

*Proof.*

(a) An element  $m$  is in the linear span of  $\mathcal{M}_1, \dots, \mathcal{M}_n$  if and only if it can be written (in general in a non-unique way) as

$$m = \sum_{j=1}^n m_j q_j, \tag{4.3}$$

where the  $m_j \in \mathcal{M}_j$  for  $j = 1, \dots, n$ . Then,

$$\begin{aligned} [m, m] &= \left[ \sum_{j=1}^n m_j q_j, \sum_{k=1}^n m_k q_k \right] \\ &= \sum_{j=1}^n [m_j, m_j] |q_j|^2 + \sum_{j \neq k} \bar{q}_k [m_j, m_k] q_j \\ &= \sum_{j=1}^n [m_j, m_j] |q_j|^2 \end{aligned}$$

since by hypothesis,  $[m_j, m_k] = 0$  for  $j \neq k$ . The result follows.

(b) This item follows from the fact that the linear span of a positive vector is a one dimensional subspace which is positive, and similarly for the other cases at hand.  $\square$

We now briefly discuss some properties of the isotropic part of an indefinite quaternionic inner product space.

**Proposition 4.6.** *Let  $\mathcal{V}$  denote a definite quaternionic inner product space. Then:*

(a) *Assume  $\mathcal{V}$  positive (resp. negative). Then, an element  $v$  belongs to the isotropic part  $\mathcal{V}^0$  of  $\mathcal{V}$  if and only if it is neutral:  $[v, v] = 0$ .*

- (b) Assume  $\mathcal{V}$  neutral. Then, the inner product vanishes identically in  $\mathcal{V}$ .  
 (c) A neutral subspace of  $\mathcal{N} \subset \mathcal{V}$  is ortho-complemented if and only if it is included in the isotropic part of  $\mathcal{V}$ .

*Proof.* The first two statements are direct consequences of the Cauchy-Schwarz inequality, which, as already remarked, holds in definite quaternionic inner product spaces. As for the third claim, item (b) implies that  $\mathcal{N} \subset \mathcal{N}^{[\perp]}$ . Thus,  $\mathcal{N}$  is ortho-complemented if and only if  $\mathcal{V} = \mathcal{N}^{[\perp]}$ , that is  $\mathcal{N}$  is orthogonal to  $\mathcal{V}$ , which is the claimed inclusion.  $\square$

**Proposition 4.7.** *Let  $\mathcal{V}$  denote a quaternionic inner product space, and let  $\mathcal{M}_1, \dots, \mathcal{M}_n$  be subspaces of  $\mathcal{V}$  which pairwise are orthogonal and have intersection reducing to  $\{0\}$ . Then,*

$$\left([\oplus]_{j=1}^n \mathcal{M}_j\right)^0 = [\oplus]_{j=0}^n \mathcal{M}_j^0, \quad (4.4)$$

where we recall that the symbol  $[\oplus]$  denotes direct and orthogonal sum.

Note that, since  $\mathcal{M}_j^0 \subset \mathcal{M}_j$ , the sum on the right side of (4.4) is indeed both direct and orthogonal.

*Proof of Proposition 4.7.* Let  $m$  and  $\ell$  be in the (direct and orthogonal) sum of the  $\mathcal{M}_j$ . They can be written (in a unique way) in the form (4.3):

$$m = \sum_{j=1}^n m_j \quad \text{and} \quad \ell = \sum_{j=1}^n \ell_j,$$

where  $m_j$  and  $\ell_j$  belong to  $\mathcal{M}_j$ ,  $j = 1, 2, \dots, n$ . Thus

$$[m, \ell] = \sum_{j=1}^n [m_j, \ell_j].$$

Thus,  $m$  is orthogonal to all elements in  $\mathcal{M}$  if and only if

$$[m_j, \ell_j] = 0, \quad \forall \ell_j \in \mathcal{M}_j, \quad j = 1, 2, \dots, n,$$

that is, if and only if  $\ell$  belongs to  $[\oplus]_{j=0}^n \mathcal{M}_j^0$ .  $\square$

In the statement of the following proposition, the existence of a direct complement is insured by Theorem 2.1.

**Proposition 4.8.** *Let  $\mathcal{V}$  denote a quaternionic inner product space, and let  $\mathcal{V}^0$  be its isotropic part. Let  $\mathcal{V}_1$  be a direct complement of  $\mathcal{V}^0$ . Then  $\mathcal{V}_1$  is non-degenerate and we have the direct sum decomposition*

$$\mathcal{V} = \mathcal{V}^0 [\oplus] \mathcal{V}_1. \quad (4.5)$$

*Proof.* Let  $v \in \mathcal{V}^0 \cap \mathcal{V}_1$  such that

$$[v, v_1] = 0, \quad \forall v_1 \in \mathcal{V}_1.$$

On the other hand, by definition of the isotropic part,

$$[v, v_0] = 0, \quad \forall v_0 \in \mathcal{V}^0.$$

Since  $\mathcal{V}_1$  is a direct complement of  $\mathcal{V}^0$  in  $\mathcal{V}$ , we have  $v \in \mathcal{V}^0$ , and so  $v = 0$  since  $\mathcal{V}^0 \cap \mathcal{V}_1 = \{0\}$ . The equality (4.5) follows.  $\square$

We now gather in form of a proposition [10, Lemmas 6.2, 6.3 and 6.4 p. 13]. As is remarked in [10, p. 13], the claims (b) and (c) in the proposition are not consequences one of the other. Note that on page 13 of that reference, the proof of Lemma 6.3 is in fact the proof of Lemma 6.4. Note also that we get three other claims when replacing positive by negative in the statements.

**Proposition 4.9.**

(a) Let  $\mathcal{V} = \mathcal{V}_1[\oplus]\mathcal{V}_2$  denote an orthogonal direct decomposition of the indefinite inner product quaternionic vector space  $\mathcal{V}$ , where  $\mathcal{V}_1$  is positive and  $\mathcal{V}_2$  is maximal strictly negative. Then,  $\mathcal{V}_1$  is maximal positive.

(b) The space orthogonal to a maximal positive subspace is negative.

(c) The space orthogonal to a maximal strictly positive subspace is negative.

*Proof.*

(a) Let  $\mathcal{W}_1 \supset \mathcal{V}_1$  be a positive subspace of  $\mathcal{V}$  containing  $\mathcal{V}_1$ , let  $v \in \mathcal{W}_1 \setminus \mathcal{V}_1$ , and write  $v = v_1 + v_2$ , where  $v_1 \in \mathcal{V}_1$  and  $v_2 \in \mathcal{V}_2$ . Then,  $v_2 = v - v_1 \in \mathcal{W}_1$  since  $\mathcal{W}_1$  is a subspace. On the other hand,  $v_2 \neq 0$  (otherwise  $v \in \mathcal{V}_1$ ) and so  $[v_2, v_2] < 0$ . This contradicts the fact that  $\mathcal{W}_1$  is positive.

(b) Let  $\mathcal{L}$  be a maximal positive subspace of  $\mathcal{V}$ , and let  $v \in \mathcal{L}^{\perp}$ . We distinguish three cases:

- (1) If  $v \notin \mathcal{L}$  and  $[v, v] = 0$ , there is nothing to prove.
- (2) If  $v \notin \mathcal{L}$  and  $[v, v] > 0$ , then the space spanned by  $v$  and  $\mathcal{L}$  is positive, contradicting the maximality of  $\mathcal{L}$ . So  $[v, v] \leq 0$ .
- (3) If  $v \in \mathcal{L}$ . Then,  $v \in \mathcal{L} \cap \mathcal{L}^{\perp}$ , and so  $[v, v] = 0$ , which is what we wanted to prove.

(c) Let now  $\mathcal{L}$  be a maximal positive definite subspace of  $\mathcal{V}$ , and let  $v \in \mathcal{L}^{\perp}$ , different from 0. If  $[v, v] \leq 0$  there is nothing to prove. If  $[v, v] > 0$ , the space spanned by  $v$  and  $\mathcal{L}$  is strictly positive, contradicting the maximality of  $\mathcal{L}$ .  $\square$

**Proposition 4.10.** Let  $\mathcal{V}$  denote a quaternionic inner product space, and let  $\mathcal{L} = [\oplus]_{j=1}^N \mathcal{L}_j$  be the direct orthogonal sum of  $\mathcal{L}_1, \dots, \mathcal{L}_N$ . Then,  $\mathcal{L}$  is ortho-complemented if and only if each of the  $\mathcal{L}_j$  is ortho-complemented.

*Proof.* Assume first that  $\mathcal{L}$  is ortho-complemented, and let  $\mathcal{M} \subset \mathcal{L}^{\perp}$  be such that

$$\mathcal{V} = \mathcal{L}[+]\mathcal{M},$$

where the sum is orthogonal, but need not be direct. Thus

$$\mathcal{V} = ([\oplus]_{j=1}^N \mathcal{L}_j) + \mathcal{M}.$$

For a given  $j \in \{1, \dots, N\}$ , the space

$$\mathcal{M}_j = ([\oplus]_{\substack{k=1 \\ k \neq j}}^N \mathcal{L}_k) + \mathcal{M}$$

is inside  $\mathcal{L}_j^{\perp}$  and such that  $\mathcal{V} = \mathcal{L}_j[+]\mathcal{M}_j$ . Thus,  $\mathcal{L}_j$  is ortho-complemented.

Conversely (and here we follow the proof of [10, Theorem 8.5, p. 17]), assume that  $\mathcal{L}$  is ortho-complemented. Let  $v \in \mathcal{V}$ . For  $j = 1, \dots, N$  we have

$$v = \ell_j + m_j, \quad \text{with } \ell_j \in \mathcal{L}_j \quad \text{and} \quad m_j \in \mathcal{L}_j^{\perp}. \quad (4.6)$$

Let  $\ell = \sum_{j=1}^N \ell_j \in \mathcal{L}$ , and let, for  $j = 1, \dots, N$

$$w_j = m_j - \left( \sum_{\substack{k=1 \\ k \neq j}}^N \ell_k \right).$$

Let  $j_1, j_2 \in \{1, \dots, N\}$ ,

$$\begin{aligned} w_{j_1} - w_{j_2} &= m_{j_1} - m_{j_2} - \left( \sum_{\substack{k=1 \\ k \neq j_1}}^N \ell_k \right) + \left( \sum_{\substack{k=1 \\ k \neq j_2}}^N \ell_k \right) \\ &= m_{j_1} - m_{j_2} - m_{j_2} + m_{j_1} \\ &= 0, \end{aligned}$$

in view of (4.6). Thus,  $w_j$  is independent of  $j$ . We set  $w_j = w$ . We have  $v = \ell + w$ . Furthermore, by its very definition, it is orthogonal to every  $\mathcal{L}_j$ , and hence orthogonal to  $\mathcal{L}$ , and this concludes the proof.  $\square$

If  $\mathcal{L}$ ,  $\mathcal{V}_1$  and  $\mathcal{V}_2$  are subspaces of the quaternionic vector space  $\mathcal{V}$ , and  $\mathcal{V}_1 \subset \mathcal{V}_2$ , then one can define a map  $I$  from  $\mathcal{L}/\mathcal{V}_1$  into  $\mathcal{V}/\mathcal{V}_2$  via

$$I(\ell + \mathcal{V}_1) = \ell + \mathcal{V}_2, \quad (4.7)$$

since  $\ell \in \mathcal{V}_1$  implies that  $\ell \in \mathcal{V}_2$ . In general the map  $I$  will not be one-to-one. If  $\ell + \mathcal{V}_2 = \ell' + \mathcal{V}_2$  where  $\ell$  and  $\ell'$  belong to  $\mathcal{L}$ , then  $\ell - \ell' \in \mathcal{V}_2 \cap \mathcal{L}$ . This need not imply that  $\ell - \ell' \in \mathcal{V}_1$  since we do not have in general

$$\mathcal{V}_2 \cap \mathcal{L} \subset \mathcal{V}_1. \quad (4.8)$$

We also note the following:

$$I(\mathcal{L}^{\perp}) = (I(\mathcal{L}))^{\perp} \quad (4.9)$$

where we denote by the same symbol orthogonality with respect to the original inner product and with respect the inner product (4.2).

We now prove the counterpart of [10, Theorem 9.4, p. 18].

**Theorem 4.11.** *Let  $\mathcal{V}$  denote a quaternionic inner product space. Then the subspace  $\mathcal{L}$  is ortho-complemented if and only if the following two conditions are in force:*

- (a) *The isotropic part of  $\mathcal{L}$  is included in the isotropic part of  $\mathcal{V}$ .*
- (b) *The image under the map  $I$  (defined by (4.7)) of the quotient space  $\mathcal{L}/\mathcal{L}^0$  is ortho-complemented in  $\mathcal{V}/\mathcal{V}^0$ .*

*Proof.* We first assume that  $\mathcal{L}$  is ortho-complemented, that is  $\mathcal{V} = \mathcal{L}[+] \mathcal{L}^{\perp}$ .

The inner product (4.2) preserves orthogonality, and thus

$$\mathcal{V}/\mathcal{V}_0 = (\mathcal{L}/\mathcal{V}_0)[+](\mathcal{L}^{\perp}/\mathcal{V}_0).$$

We now show that the map  $I$  is one-to-one and so

$$(\mathcal{L}/\mathcal{V}_0) = I(\mathcal{L}/\mathcal{L}_0),$$

and this will conclude the proof of the direct assertion. Every  $v$  in  $\mathcal{V}$  can be written as

$$v = \ell + m, \quad \ell \in \mathcal{L}, \quad m \in \mathcal{L}^{\perp}.$$

Let now  $\ell_0 \in \mathcal{L}^0$ . We have

$$[\ell_0, v] = [\ell_0, \ell] + [\ell_0, m] = 0$$

and thus  $\mathcal{L}^0 \subset \mathcal{V}^0$ . Equation (4.8) becomes

$$\mathcal{V}^0 \cap \mathcal{L} \subset \mathcal{L}^0, \tag{4.10}$$

which always holds, and by the discussion before the theorem the map  $I$  well defined and one-to-one and so (b) holds.

Conversely we assume now that (a) and (b) hold. We prove that  $\mathcal{L}$  is ortho-complemented. Taking  $\mathcal{V}_1 = \mathcal{L}^0$  and  $\mathcal{V}_2 = \mathcal{V}^0$ , (a) insures that the map  $I$  is well defined and equation (4.10) holds by definition of  $\mathcal{V}_0$ . Thus the map  $I$  is one-to-one. Using (b) we see that for every  $v \in \mathcal{V}$  there exist  $\ell \in \mathcal{L}$  and  $m \in \mathcal{L}^{\perp}$  such that

$$v + \mathcal{V}^0 = \ell + \mathcal{V}^0 + m + \mathcal{V}^0.$$

Thus we have  $v = \ell + m + v_0$ . This concludes the proof since  $\mathcal{V}^0 \subset \mathcal{L}^0 \subset \mathcal{L}^{\perp}$ .  $\square$

We conclude this section with results pertaining to a non-degenerate space (that is, when  $\mathcal{V}^0$  is trivial), and which are corollaries of the previous discussion.

**Proposition 4.12.** *Let  $\mathcal{V}$  be a quaternionic non-degenerate inner product space. Then:*

- (a) *Every ortho-complemented subspace is non-degenerate.*
- (b) *Let  $\mathcal{L} \subset \mathcal{V}$  be ortho-complemented. Then  $\mathcal{L} = \mathcal{L}^{\perp\perp}$ .*

*Proof.* (a) follows directly from Theorem 4.11 (a) since

$$\mathcal{L} \cap \mathcal{L}^{\perp} \subset \mathcal{V}^0 = \{0\}.$$

As for (b), we always have

$$\mathcal{L} \subset \mathcal{L}^{\perp\perp}. \tag{4.11}$$

We assume that  $\mathcal{L}$  is ortho-complemented. Let  $v \in \mathcal{L}^{\perp\perp}$ , with decomposition

$$v = v_1 + v_2, \quad v_1 \in \mathcal{L}, \quad \text{and} \quad v_2 \in \mathcal{L}^{\perp}.$$

Then, in view of (4.11),  $v_2 = v - v_1 \in \mathcal{L}^{\perp\perp}$ , and so  $v_2 \in \mathcal{L}^{\perp} \cap \mathcal{L}^{\perp\perp}$ . Since

$$\mathcal{L}^{\perp}[\perp]\mathcal{L}^{\perp\perp} = \mathcal{V}$$

(recall that  $\mathcal{L}^{\perp}$  is also ortho-complemented; see Proposition 4.4), this implies that  $v_2 = 0$  since  $\mathcal{V}$  is non-degenerate. Thus there is equality in (4.11).  $\square$



## 5. FUNDAMENTAL DECOMPOSITIONS

A quaternionic inner product space  $\mathcal{V}$  is decomposable if it can be written as a direct and orthogonal sum

$$\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-[\oplus]\mathcal{N} \quad (5.1)$$

where  $\mathcal{V}_+$  is a strictly positive subspace,  $\mathcal{V}_-$  is a strictly negative subspace, and  $\mathcal{N}$  is a neutral subspace. Representation (5.1) is called a fundamental decomposition. A quaternionic inner product space need not be decomposable, and the decomposition will not be unique (unless one of the spaces  $\mathcal{V}_\pm$  is trivial). A precise characterization of the decompositions is given in the following results.

**Proposition 5.1.** *Assume that (5.1) holds. Then  $\mathcal{N} = \mathcal{V} \cap \mathcal{V}^{[\perp]}$  (that is  $\mathcal{N}$  is equal to the isotropic part of  $\mathcal{V}$ ).*

*Proof.* We first show that  $\mathcal{N} \subset \mathcal{V}^0$ . Let  $m \in \mathcal{N}$ , and let  $v \in \mathcal{V}$  with decomposition

$$v = v_+ + v_- + n, \quad \text{where } v_\pm \in \mathcal{V}_\pm, n \in \mathcal{N}. \quad (5.2)$$

In view of (5.1) we have  $[m, v_+] = [m, v_-] = 0$ . Furthermore,  $[m, n] = 0$  since the inner product vanishes in a neutral subspace (this is a direct consequence of the Cauchy-Schwarz inequality). Thus  $[m, v] = 0$  and so  $m \in \mathcal{V}^0$ .

Conversely, let  $v_0 \in \mathcal{V}^0$ , with decomposition (5.2). Then,

$$0 = [v, v_+] = [v_+, v_+]$$

and so  $v_+ = 0$  since  $\mathcal{V}_+$  is positive definite. Similarly,  $v_- = 0$  and thus  $v_0 = n \in \mathcal{N}$ .  $\square$

By the definition of non-degenerate linear space, we have this immediate consequence of the previous result:

**Corollary 5.2.** *All the decompositions of a decomposable, non-degenerate inner product space  $\mathcal{V}$  are of the form*

$$\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-$$

where  $\mathcal{V}_+$  (resp.  $\mathcal{V}_-$ ) is a strictly positive (resp. negative) subspace.

The following is [10, Lemma 11.4, p. 24-25] in the present setting.

**Proposition 5.3.** *Let  $\mathcal{V}$  be a quaternionic non-degenerate inner product space, and let  $\mathcal{L}$  be a positive definite subspace of  $\mathcal{L}$ . There exists a fundamental decomposition of  $\mathcal{V}$  with  $\mathcal{V}_+ = \mathcal{L}$  if and only if  $\mathcal{L}$  is maximal positive definite and ortho-complemented.*

*Proof.* Assume first that  $\mathcal{V} = \mathcal{L}[\oplus]\mathcal{V}_-[\oplus]\mathcal{V}^0$ , where  $\mathcal{V}_-$  is negative definite and  $\mathcal{V}^0$  is the isotropic part of  $\mathcal{V}$ . Then  $\mathcal{L}$  is ortho-complemented. Let  $\mathcal{M} \supset \mathcal{L}$  be a positive definite subspace containing  $\mathcal{L}$  and let  $v \in \mathcal{M}$ , with decomposition

$$v = v_+ + v_- + n, \quad v_+ \in \mathcal{L}, \quad v_- \in \mathcal{L}_-, \quad n \in \mathcal{V}^0.$$

By linearity,  $v - v_+ = v_- + n \in \mathcal{M}$ . But

$$[v - v_+, v - v_+] = [v_-, v_-] + [n, n] < 0,$$

unless  $v_- = 0$ . But then  $[v - v_+, v - v_+] = 0$  implies  $v = v_+$  (and so  $n = 0$ ) since  $\mathcal{M}$  is positive definite. Thus  $v = v_+$  and  $\mathcal{L} = \mathcal{M}$ . Therefore,  $\mathcal{L}$  is maximal positive definite.

Conversely, if  $\mathcal{L}$  is ortho-complemented, then  $\mathcal{V} = \mathcal{L}[\oplus]\mathcal{L}^{[\perp]}$  and, since  $\mathcal{L}$  is positive definite, the latter sum is direct, that is,  $\mathcal{V} = \mathcal{L}[\oplus]\mathcal{L}^{[\perp]}$ .

Since  $\mathcal{L}$  is maximal positive definite, it follows that  $\mathcal{L}^{\perp}$  is negative. Indeed, neither  $\mathcal{L}^{\perp} \setminus \mathcal{L}$  nor  $\mathcal{L}^{\perp} \cap \mathcal{L}$  contain positive vectors  $v$  since in the first case the space spanned by  $v$  and  $\mathcal{L}$  would be positive, contradicting the maximality of  $\mathcal{L}$  and in the second case we would have  $[v, v] = 0$  contradicting the positivity of  $v$ . An application of Lemma 4.8 allows then to write  $\mathcal{L}^{\perp}$  as a direct orthogonal sum of a negative definite space and of an isotropic space  $\mathcal{N}$ . Finally, the isotropic part  $\mathcal{N}$  of  $\mathcal{L}^{\perp}$  is the isotropic part of  $\mathcal{V}$ .  $\square$

To conclude this section we discuss some properties of linear operators between quaternionic inner product spaces. The linear operator  $A$  will be called invertible if it is one-to-one and its range is all of  $\mathcal{W}$ , or equivalently, if there exists a linear operator  $B : \mathcal{V} \rightarrow \mathcal{W}$  such that  $AB = I_{\mathcal{W}}$  and  $BA = I_{\mathcal{V}}$ .

Let  $\mathcal{V}$  be a quaternionic inner product space which is decomposable and non-degenerate, and let

$$\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-, \quad (5.3)$$

where  $\mathcal{V}_+$  is a strictly positive subspace and  $\mathcal{V}_-$  is a strictly negative subspace. The map

$$J(v) = v_+ - v_-$$

is called the associated fundamental symmetry. Since  $J(Jv) = v$ , it follows that  $J$  is invertible, and  $J = J^{-1}$ . It is readily seen that

$$[v, w] = [Jv, Jw], \quad v, w \in \mathcal{V}. \quad (5.4)$$

**Theorem 5.4.** *Let  $\mathcal{V}$  be a decomposable and non-degenerate quaternionic inner product space, and let (5.1) be a fundamental decomposition of  $\mathcal{V}$ , and let*

$$\langle v, w \rangle_J \stackrel{\text{def.}}{=} [Jv, w], \quad v, w \in \mathcal{V}.$$

Then,

$$\langle v, w \rangle_J = [v, Jw] = [v_+, w_+] - [v_-, w_-], \quad (5.5)$$

$$[v, w] = \langle v, Jw \rangle_J = \langle Jv, w \rangle_J, \quad (5.6)$$

and  $(\mathcal{V}, \langle \cdot, \cdot \rangle_J)$  is a pre-Hilbert space. Furthermore, with  $\|v\|_J = [v, Jv]$

$$|[v, w]|^2 \leq \|v\|_J^2 \|w\|_J^2, \quad v, w \in \mathcal{V}. \quad (5.7)$$

*Proof.* The first claim follows from the fact that both  $\mathcal{V}_+$  and  $\mathcal{V}_-$  are positive definite. In a quaternionic pre-Hilbert space, the Cauchy-Schwarz inequality holds and this implies (5.7) since

$$|[v, w]|^2 = |\langle v, Jw \rangle_J|^2 \leq \|v\|_J^2 \|Jw\|^2.$$

Equations (5.5) and (5.4) imply that  $\|w\|_J = \|Jw\|_J$ , and this ends the proof.  $\square$

**Remark 5.5.** Let  $\mathcal{V}$  be a quaternionic, non-degenerate, inner product vector space admitting a fundamental decomposition of the form  $\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_-$  and let  $J$  be the associated fundamental symmetry. Then  $\mathcal{V}_+$  is  $J$ -orthogonal to  $\mathcal{V}_-$ , i.e.  $\langle v_+, w_- \rangle_J = 0$  for every  $v_+ \in \mathcal{V}_+$  and  $w_- \in \mathcal{V}_-$ , as one can see from formula (5.5).

## 6. PARTIAL MAJORANTS

We now introduce and study some special topologies called partial majorants. A standard reference for the material in this section in the complex case is [10, Chapter III]. We begin by proving a simple fact (which, in general, is not guaranteed in a vector space over any field):

**Lemma 6.1.** *Let  $\mathcal{V}$  be a quaternionic inner product space and let  $w \in \mathcal{V}$ . The maps*

$$v \mapsto p_w(v) = |[v, w]|, \quad v \in \mathcal{V} \quad (6.1)$$

are semi-norms.

*Proof.* Property (2.1) is clear. Property (2.2) comes from the fact that the absolute value is multiplicative in  $\mathbb{H}$ :  $p_w(vc) = |[vc, w]| = |[v, w]c| = |[v, w]| \cdot |c| = |c|p_w(v)$ .  $\square$

**Definition 6.2.** *The weak topology on  $\mathcal{V}$  is the smallest topology such that all the semi-norms (6.1) are continuous.*

**Definition 6.3.** (a) *A topology on the quaternionic indefinite inner product space  $\mathcal{V}$  is called a partial majorant if it is locally convex and if all the maps*

$$v \mapsto [v, w] \quad (6.2)$$

are continuous.

(b) *A partial majorant is called admissible if every continuous linear function from  $\mathcal{V}$  to  $\mathbb{H}$  is of the form  $v \mapsto [v, w_0]$  for some  $w_0 \in \mathcal{V}$ .*

**Theorem 6.4.** *The weak topology of an inner product space is a partial majorant. A locally convex topology is a partial majorant if and only if it is stronger than the weak topology.*

*Proof.* To prove the first assertion, we have to show that in the weak topology the maps (6.2) are continuous. For any choice of  $\varepsilon > 0$ , and for any  $v_0, w \in \mathcal{V}$  the inequality  $|[v, w] - [v_0, w]| < \varepsilon$  is equivalent to  $p_w(v - v_0) < \varepsilon$  and the set  $\{v \in \mathcal{V} : p_w(v - v_0) < \varepsilon\}$  is a neighborhood  $U_{v_0}(p_w, \varepsilon)$  of  $v_0$ . Thus the weak topology is a partial majorant.

Let us now consider another locally convex topology stronger than the weak topology. Then we have already shown that the inequality  $|[v, w] - [v_0, w]| < \varepsilon$  holds for  $v \in U_{v_0}(p_w, \varepsilon)$  which is also an open set in the stronger topology and so any locally convex topology stronger than the weak topology is a partial majorant. Finally, we consider a partial majorant. Let  $v_0, w_1, \dots, w_n \in \mathcal{V}$ , let  $\varepsilon > 0$ . Then, by definition, there are neighborhoods  $U_\ell$  of  $w_\ell$ ,  $\ell = 1, \dots, n$  such that for any  $w \in U_\ell$  the inequality  $|[v, w_\ell] - [v_0, w_\ell]| < \varepsilon$ , i.e.  $p_{w_\ell}(v - v_0) < \varepsilon$  holds. Thus any  $w$  which belongs to the neighborhood of  $v_0$  given by  $\bigcap_{\ell=1}^n U_\ell$  belongs to  $U_{v_0}(p_{w_\ell}, \varepsilon)$  and the statement follows.  $\square$

As a consequence we have:

**Corollary 6.5.** *Every partial majorant of a non-degenerate inner product space  $\mathcal{V}$  is Hausdorff.*

*Proof.* Recall that any open set in the weak topology is also open in the partial majorant topology. The weak topology is Hausdorff if it separates points, i.e. if and only if for every  $w \in \mathcal{V}$  the condition  $p_w(v) = |[v, w]| = 0$  implies  $v = 0$ . But this is indeed the case since  $\mathcal{V}$  is non-degenerate.  $\square$

**Proposition 6.6.** *If a topology is a partial majorant of the quaternionic inner product space  $\mathcal{V}$  then the orthogonal companion of every subspace is closed.*

*Proof.* Let  $\mathcal{L}$  be a subspace of  $\mathcal{V}$  and let  $\mathcal{L}^{\perp}$  its orthogonal companion. We show that  $\mathcal{L}^{\perp}$  is an open set. Let  $v_0$  be in the complement  $(\mathcal{L}^{\perp})^c$  of  $\mathcal{L}^{\perp}$ ; then there is  $w \in \mathcal{L}$  such that  $[v_0, w] \neq 0$ . By continuity, there exists a neighborhood  $U$  of  $v_0$  such that  $[v, v_0] \neq 0$  for all  $v \in U$ , thus  $(\mathcal{L}^{\perp})^c$  is open.  $\square$

**Corollary 6.7.** *If a topology is a partial majorant of a non-degenerate inner product space  $\mathcal{V}$  then every ortho-complemented subspace of  $\mathcal{V}$  is closed.*

*Proof.* Consider the subspace  $\mathcal{L}^{\perp}$ , orthogonal to  $\mathcal{L}$ . Then  $\mathcal{L}^{\perp\perp}$  is closed by Proposition 6.6 and since  $\mathcal{L}^{\perp\perp} = \mathcal{L}$  by Proposition 4.12 the assertion follows.  $\square$

**Corollary 6.8.** *Let  $\tau$  be a partial majorant of the quaternionic inner product  $V$  and assume that  $V$  is non-degenerate. Then the components of any fundamental decompositions are closed with respect to  $\tau$ .*

*Proof.* This is a consequence of the previous corollary, since the two components are orthocomplemented.  $\square$

**Theorem 6.9.** *Let  $\mathcal{V}$  be a non-degenerate quaternionic inner product space and let  $\tau_1$  and  $\tau_2$  be two Fréchet partial majorants of  $\mathcal{V}$ . Then,  $\tau_1 = \tau_2$ .*

*Proof.* Let  $\tau$  be the topology  $\tau_1 \cup \tau_2$ . Then we can show following the proof of Theorem 3.3. p. 63 in [10] that  $\tau$  is a Fréchet topology stronger than  $\tau_1$  and  $\tau_2$ . We now consider the two topological vector spaces  $\mathcal{V}$  endowed with  $\tau$  and  $\mathcal{V}$  endowed with  $\tau_1$  and the identity map acting between them. By the closed graph theorem, see Theorem 3.9, we have that the identity map takes closed sets to closed sets and so  $\tau_1$  is stronger than  $\tau$ . A similar argument holds by considering  $\tau_2$  and thus  $\tau = \tau_1 = \tau_2$ .  $\square$

Assume now that a partial majorant  $\tau$  is defined by a norm  $\|\cdot\|$  on a non-degenerate inner product space  $\mathcal{V}$ . Let us define

$$\|v\|' \stackrel{\text{def.}}{=} \sup_{\|w\| \leq 1} |[v, w]|, \quad v \in \mathcal{V}. \quad (6.3)$$

Then  $\|\cdot\|'$  is a norm (called *polar* of the norm  $\|\cdot\|$ ), as it can be directly verified. As in the proof of Lemma 6.1 the fact that the modulus is multiplicative in  $\mathbb{H}$  is what matters. The topology  $\tau'$  induced by  $\|\cdot\|'$  is called the polar of the topology  $\tau$ . The definition (6.3) implies

$$|[v, w/\|w\|]| \leq \sup_{w \in \mathcal{V}} |[v, w/\|w\|]| \leq \sup_{\|w\| \leq 1} |[v, w]| = \|v\|', \quad (6.4)$$

from which we deduce the inequality  $|[v, w]| \leq \|v\|'\|w\|$ . Thus the polar of a partial majorant is a partial majorant since (6.2) holds and thus one can define  $\tau'' \stackrel{\text{def.}}{=} (\tau)'$  and so on, iteratively.

**Proposition 6.10.** *Let  $\mathcal{V}$  be a non-degenerate inner product space.*

- (a) *If  $\tau_1$  and  $\tau_2$  are normed partial majorants of  $\mathcal{V}$ . If  $\tau_1$  is weaker than  $\tau_2$  then  $\tau_2'$  is weaker than  $\tau_1'$ .*
- (b) *If  $\tau$  be a normed partial majorant of  $\mathcal{V}$ , then its polar  $\tau'$  is a normed partial majorant on  $\mathcal{V}$ . Furthermore,  $\tau'' \leq \tau$ , and  $\tau''' = \tau'$ .*

*Proof.* Let  $\tau_1, \tau_2$  be induced by the norms  $\|\cdot\|_1$  and  $\|\cdot\|_2$ , respectively and let us assume that  $\tau_1 \leq \tau_2$ . Then for  $w \in \mathcal{V}$  there exists  $\lambda > 0$  such that  $\lambda\|w\|_2 \leq \|w\|_1$  and so, if we take  $\|w\|_1 \leq 1$  we have

$$\sup_{\|w\|_1 \leq 1} |[v, w]| \leq \sup_{\|w\|_2 \leq 1} |[v, \lambda w]| = \lambda \sup_{\|w\|_2 \leq 1} |[v, w]|,$$

so that  $\tau_2' \leq \tau_1'$ .

Moreover we have  $\sup_{\|y\|' \leq 1} |[x, y]| \leq \|x\|$  and so  $\tau'' \leq \tau$ . Let us now use this inequality by replacing  $\tau$  by  $\tau'$  and we get  $\tau''' \leq \tau'$ . By using point (a) applied to  $\tau_1 = \tau''$  and  $\tau_2 = \tau$  we obtain the reverse inequality and so  $\tau''' = \tau'$ .  $\square$

Among the partial majorants there are the admissible topology (see Definition 6.3). The next result shows that an admissible topology which is metrizable is uniquely defined. In order to prove the result, we recall that given a quaternionic vector space  $\mathcal{V}$ , its conjugate  $\mathcal{V}^*$  is defined to be the quaternionic vector space in which the additive group coincides with  $\mathcal{V}$  and whose multiplication by a scalar is given by  $(c, v) \mapsto v\bar{c}$ . An inner product  $(\cdot, \cdot)$  in  $\mathcal{V}^*$  can be assigned by  $(v, w) \stackrel{\text{def.}}{=} [w, v] = \overline{[v, w]}$ .

**Theorem 6.11.** *Let  $\tau_1, \tau$  be admissible topologies on a quaternionic inner product space  $\mathcal{V}$ . If  $\tau_1$  is given by a countable family of semi-norms, then  $\tau_1$  is stronger than  $\tau$ . Moreover, no more than one admissible topology of  $\mathcal{V}$  is metrizable.*

*Proof.* Assume that  $\tau_1$  and  $\tau$  are given by the families of semi-norms  $\{p_i\}, i \in \mathbb{N}$ , and  $\{q_\gamma\}, \gamma \in \Gamma$ , respectively. By absurd, suppose that  $\tau_1$  is not stronger than  $\tau$ . Then there exists an open set in  $\tau$  that does not contain any open set in  $\tau_1$  and, in particular, it does not contain

$$\{v \in \mathcal{V} \mid p_i(v) < \frac{1}{n}, \quad i = 1, \dots, n, \text{ for } n \in \mathbb{N}\}.$$

Thus, there exists a sequence  $\{v_n\} \subset \mathcal{V}$  such that  $p_i(v_n) < \frac{1}{n}$  but  $\max_{k=1, \dots, m} q_{\gamma_k}(v_n) = q_{\gamma_j}(v_n) \geq \varepsilon$  for some  $\varepsilon > 0$ . By choosing  $w_n = nv_n$  we have

$$\max_{i=1, \dots, n} p_i(w_n) < 1, \quad q_{\gamma_j}(w_n) \geq n\varepsilon, \quad n \in \mathbb{N}. \quad (6.5)$$

Let us consider the subspace of  $\mathcal{V}$  given by  $\mathcal{L} = \{v \in \mathcal{V} \mid q_{\gamma_j}(v) = 0\}$  and the quotient  $\hat{\mathcal{L}} \stackrel{\text{def.}}{=} \mathcal{V}/\mathcal{L}$ . We can endow  $\hat{\mathcal{L}}$  with the norm  $\|\hat{v}\| \stackrel{\text{def.}}{=} q_{\gamma_j}(v)$ , for  $\hat{v} = v + \mathcal{L} \in \hat{\mathcal{L}}$ . Let  $\hat{\varphi} : \hat{\mathcal{L}} \rightarrow \mathbb{H}$  be a linear function which is also continuous (bounded):

$$|\hat{\varphi}(\hat{v})| \leq \|\hat{\varphi}\| \|\hat{v}\|, \quad \hat{v} \in \hat{\mathcal{L}}.$$

Then the formula  $\varphi(v) \stackrel{\text{def.}}{=} \hat{\varphi}(\hat{v})$ ,  $v \in \mathcal{V}$ ,  $v \in \hat{v}$ , defines a linear and continuous function on  $\mathcal{V}$  since

$$|\varphi(v)| \leq \|\hat{\varphi}\| \|\hat{v}\| = \|\hat{\varphi}\| q_{\gamma_j}(v).$$

Thus  $\varphi$  is continuous in the topology  $\tau$  and since  $\tau$  is admissible,  $\varphi(v) = [v, w_0]$  for some suitable  $w_0 \in \mathcal{V}$ . We conclude that  $\varphi$  is also continuous in the topology  $\tau_1$ . So for some  $r \in \mathbb{N}$  and  $\delta > 0$  we have

$$|\varphi(v)| \leq \frac{1}{\delta} \max_{i=1, \dots, r} p_i(v), \quad v \in \mathcal{V}.$$

This last inequality together with (6.5) give  $|\varphi(w_n)| < 1/\delta$  for  $n > r$ . So the sequence  $\{\hat{\varphi}(\hat{w}_n)\}$  is bounded for any  $\hat{\varphi}$  fixed in the conjugate space  $\hat{\mathcal{L}}^*$  of the normed space

$\hat{\mathcal{L}}$ . However, we can look at  $\hat{\varphi}(\hat{w}_n)$  as the value of the functional  $\hat{w}_n$  acting on the elements of the Banach space  $\hat{\mathcal{L}}^*$ . Since we required that  $|\hat{\varphi}(\hat{v})| \leq \|\hat{\varphi}\| \|\hat{v}\|$ , for  $\hat{v} \in \hat{\mathcal{L}}$  the functional  $\hat{w}_n$  is continuous. By the quaternionic version of the Hahn-Banach theorem, see e.g. [14, Theorem 4.10.1], we deduce that  $\|\hat{w}_n\| = q_{\gamma_j}(\hat{w}_n)$ . From (6.5), more precisely from  $q_{\gamma_j}(w_n) \geq n\varepsilon$ , we obtain a contradiction with the principle of uniform boundedness, see Theorem 3.2.  $\square$

## 7. MAJORANT TOPOLOGIES AND INNER PRODUCT SPACES

The material in this section can be found, in the complex case, in [10, Chapter IV].

**Definition 7.1.** *A locally convex topology on  $(\mathcal{V}, [\cdot, \cdot])$  is called a majorant if the inner product is jointly continuous in this topology. It is called a complete majorant if it is metrizable and complete. It is called a normed majorant if it is defined by a single (semi-)norm, and a Banach majorant if it is moreover complete with respect to this norm. It is called a Hilbert majorant if it is a complete normed majorant, and the underlying norm is defined by an inner product.*

Of course, the norm defining a Banach majorant (and hence the inner product defining a Hilbert majorant) is not unique. But it follows from Theorem 3.4 that any two such norms are equivalent.

### Proposition 7.2.

- (a) *Given a majorant, there exists a weaker majorant defined by a single semi-norm.*
- (b) *A normed partial majorant  $\tau$  on the non-degenerate inner product space  $\mathcal{V}$  is a majorant if and only if it is stronger than its polar:  $\tau' \leq \tau$ .*

*Proof.* (a) From the definition of a majorant, there exist semi-norms  $p_1, \dots, p_N$  and  $\epsilon > 0$  such that

$$|[u, v]| \leq 1, \quad \forall u, v \in U,$$

where

$$U = \{v \in \mathcal{V}; p_j(v) \leq \epsilon, j = 1, \dots, N\}.$$

It follows that the inner product is jointly continuous with respect to the semi-norm  $\max_{j=1, \dots, N} p_j$ .

- (b) Recall that the polar  $\tau'$  is defined by (6.3). We have  $\tau' \leq \tau$  if and only if the identity map from  $(\mathcal{V}, \tau)$  into  $(\mathcal{V}, \tau')$  is continuous, that is if and only if there exists  $k > 0$  such that

$$\|v\|' \leq k\|v\|, \quad \forall v \in \mathcal{V}. \quad (7.1)$$

This in turn holds if and only if

$$|[v, u]| \leq k\|v\|, \quad \forall v, u \in \mathcal{V} \quad \text{with} \quad \|u\| \leq 1. \quad (7.2)$$

The result follows since any such  $w \neq 0$  is such that  $\|w\| \leq 1$  if and only if it be written as  $\frac{w}{\|w\|}$ , for some  $w \neq 0 \in \mathcal{V}$ .  $\square$

**Proposition 7.3.** *Let  $\mathcal{V}$  be a non-degenerate inner product space, admitting a normed majorant. Then there exists a weaker normed majorant which is self-polar.*

*Proof.* We briefly recall the proof of [10, p. 85]. The key is that the polar norm (defined in (6.3)) is still a norm in the quaternionic case. By maybe renormalizing we assume that

$$|[u, v]| \leq \|u\| \|v\|, \quad u, v \in \mathcal{V}, \quad (7.3)$$

where  $\|\cdot\|$  denotes a norm defining majorant. Define a sequence of norms  $(\|\cdot\|_n)_{n \in \mathbb{N}}$  by  $\|\cdot\|_1 = \|\cdot\|$  and

$$\|u\|_{n+1} = \left( \frac{1}{2} (\|u\|_n^2 + (\|u\|'_n)^2) \right)^{\frac{1}{2}}, \quad n = 1, 2, \dots, \quad (7.4)$$

where we recall that  $\|\cdot\|'$  denotes the polar norm of  $\|\cdot\|$ ; see (6.3). An induction shows that each  $\|\cdot\|_n$  satisfies (7.3) and that the sequence  $(\|\cdot\|_n)_{n \in \mathbb{N}}$  is decreasing, and thus defining a semi-norm  $\|\cdot\|_\infty = \lim_{n \rightarrow \infty} \|\cdot\|_n$ . One readily shows that  $\|\cdot\|_\infty \geq \frac{1}{\sqrt{2}} \|\cdot\|_1$ , and hence  $\|\cdot\|_\infty$  is a norm, and a majorant since it also satisfies (7.3) by passing to the limit the corresponding inequality for  $\|\cdot\|_n$ .

We now show that the topology defined by  $\|\cdot\|_\infty$  is self-polar. We first note that the sequence of polars  $(\|\cdot\|'_n)_{n \in \mathbb{N}}$  is increasing, and bounded by the polar  $\|\cdot\|'_\infty$ . Set  $\|\cdot\|_e = \lim_{n \rightarrow \infty} \|\cdot\|'_n$ . Applying inequality (6.4) to  $\|\cdot\|_n$  and taking limits leads to

$$|[u, v]| \leq \|u\|_e \|v\|_\infty, \quad u, v \in \mathcal{V}.$$

Thus  $\|\cdot\|'_\infty \leq \|\cdot\|_e$ , and we get that  $\|\cdot\|_\infty = \|\cdot\|_e$ . Letting  $n \rightarrow \infty$  in (7.4) we get  $\|\cdot\|_\infty = \|\cdot\|'_\infty$ .  $\square$

**Proposition 7.4.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic non-degenerate inner product space. Then a partial majorant is a minimal majorant if and only if it is normed and self-polar.*

*Proof.* Assume first that the given partial majorant  $\tau$  is a minimal majorant. By item (a) of Proposition 7.2 there is a weaker majorant  $\tau_a$  defined by a single semi-norm. Moreover by Corollary 6.5 any partial majorant (and in particular any majorant) is Hausdorff, and so the  $\tau_a$  is Hausdorff and the above semi-norm is in fact a norm. By Proposition 7.3 there exists a self-polar majorant  $\tau_\infty$  which is weaker than  $\tau_1$ . The minimality of  $\tau$  implies that  $\tau_\infty = \tau$ .

Conversely, assume that the given partial majorant  $\tau$  is normed and self-polar. Then  $\tau$  is a majorant in view of item (b) of Proposition 7.2. Assume that  $\tau_a \leq \tau$  is another majorant. Then, by part (b) in Lemma 7.2,  $\tau_a \geq \tau'_a$ , and by item (a) of Proposition 6.10 we have  $\tau'_a \geq \tau'$ . This ends the proof since  $\tau$  is self-polar.  $\square$

**Theorem 7.5.** *Let  $\mathcal{V}$  be a quaternionic non-degenerate inner product space, and let  $\tau$  be an admissible topology which is moreover a majorant. Then  $\tau$  is minimal, it defines a Banach topology and is the unique admissible majorant on  $\mathcal{V}$ . Finally,  $\tau$  is stronger than any other admissible topology on  $\mathcal{V}$ .*

We now introduce the Gram operator. It will play an important role in the sequel. Recall that Hilbert majorants have been defined in Definition 7.1.

**Proposition 7.6.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic inner-product space, admitting a Hilbert majorant, with associated inner product  $\langle \cdot, \cdot \rangle$ , and corresponding norm  $\|\cdot\|$ . There exists a linear continuous operator  $G$ , self-adjoint with respect to the inner product  $\langle \cdot, \cdot \rangle$ , and such that*

$$[v, w] = \langle v, Gw \rangle, \quad v, w \in \mathcal{V}.$$

*Proof.* The existence of  $G$  follows from Riesz' representation for continuous functionals, which still holds in quaternionic Hilbert spaces (see [13, p. 36], [18, Theorem II.1, p. 440]); the fact that  $G$  is Hermitian follows from the fact that the form  $[\cdot, \cdot]$  is Hermitian. In the complex case, an everywhere defined Hermitian operator in a Hilbert space is automatically bounded; rather than proving the counterpart of this fact in the quaternionic setting we note, as in [10, p. 88] that there exists a constant  $k$  such that

$$|[u, w]| \leq k\|u\| \cdot \|v\|, \quad \forall u, v \in \mathcal{V}. \quad (7.5)$$

The boundedness of  $G$  follows from (7.5) and  $[v, Gv] = \|Gv\|^2$ .  $\square$

The semi-norm

$$v \mapsto \|Gv\| \quad (7.6)$$

defines a topology called the Mackey topology. As we remarked after Definition 7.1 the inner product defining a given Hilbert majorant is not unique, and so to every inner product will correspond a different Gram operator.

**Proposition 7.7.** *The Mackey topology is admissible and is independent of the choice of the inner product defining the Hilbert majorant.*

*Proof.* The uniqueness will follow from Theorem 6.11 once we know that the topology, say  $\tau_G$ , associated to the semi-norm (7.6) is admissible. From the inequality

$$|[u, v]| = \langle Gu, v \rangle \leq \|Gu\| \cdot \|v\|$$

we see that  $\tau_G$  is a partial majorant. To show that it is admissible, consider a linear functional  $f$  continuous with respect to  $\tau_G$ . There exists  $k > 0$  such that

$$|f(u)| \leq k\|Gu\|, \quad \forall u \in \mathcal{V}.$$

Thus the linear relation

$$(kGu, f(u)), \quad u \in \mathcal{V}$$

is the graph of a contraction, say  $T$ ,

$$T(Gu) = \frac{1}{k}f(u), \quad \forall u \in \mathcal{V},$$

in the pre-Hilbert space  $(\text{Ran } G) \times \mathbb{H}$ , the latter being endowed with the inner product

$$\langle (Gu, p), (Gv, q) \rangle_{\mathcal{V} \times \mathbb{H}} = \langle Gu, Gv \rangle + \bar{q}p = [Gu, v] + \bar{q}p.$$

The operator  $T$  admits a contractive extension to all of  $\mathcal{V} \times \mathbb{H}$ , and by Riesz representation theorem, there exists  $f_0 \in \mathcal{V}$  such that

$$T(u) = \langle u, f_0 \rangle, \quad \forall u \in \mathcal{V}.$$

Thus

$$f(u) = kT(Gu) = k\langle Gu, f_0 \rangle = [u, kf_0],$$

which ends the proof.  $\square$

Consider a subspace  $\mathcal{L}$  of a quaternionic inner-product space  $(\mathcal{V}, [\cdot, \cdot])$ , the latter admitting an Hilbert majorant with associated inner product  $\langle \cdot, \cdot \rangle$  and associated norm  $\|\cdot\|$ . We denote by  $P_{\mathcal{L}}$  the orthogonal projection onto  $\mathcal{L}$  in the Hilbert space  $(\mathcal{V}, \langle \cdot, \cdot \rangle)$ , and set

$$G_{\mathcal{L}} = P_{\mathcal{L}}G|_{\mathcal{L}}. \quad (7.7)$$



**Proposition 7.8.** *Consider  $\mathcal{V}$  be a quaternionic inner-product space, admitting an Hilbert majorant, let  $\mathcal{L}$  be a closed subspace of  $\mathcal{V}$  and let  $G_{\mathcal{L}}$  be defined by (7.7). Then:*

(a) *An element  $v \in \mathcal{V}$  admits a projection onto  $\mathcal{L}$  if and only if*

$$P_{\mathcal{L}}v \in \text{ran } G_{\mathcal{L}}. \quad (7.8)$$

(b)  *$\mathcal{L}$  is ortho-complemented in  $(\mathcal{V}, [\cdot, \cdot])$  if and only if*

$$\text{ran } P_{\mathcal{L}}G = \text{ran } G_{\mathcal{L}}.$$

*Proof.* (a) The vector  $v \in V$  has a (not necessarily unique) projection, say  $w$  on  $\mathcal{L}$  if and only if

$$[v - w, u] = 0, \quad \forall u \in \mathcal{L},$$

that is, if and only if

$$\langle G(v - w), u \rangle = 0, \quad \forall u \in \mathcal{L}.$$

This last condition is equivalent to  $P_{\mathcal{L}}Gv = G_{\mathcal{L}}w$ , which is equivalent to (7.8).

(b) The second claim is equivalent to the fact that every element admits a projection on  $\mathcal{L}$ , and therefore follows from (a).  $\square$

## 8. THE SPECTRAL THEOREM AND DECOMPOSABILITY

The spectral theorem for Hermitian operators is stated in [17], [26], [27] in which, however, a proof is not provided. Moreover, in these works, the spectrum used is not the  $S$ -spectrum, see [14, p. 141], thus for the sake of completeness we state and prove the result. To this end, we need some preliminaries.

We first note that any linear quaternionic Hilbert space  $\mathcal{V}$  can be also considered as a complex Hilbert space, its so-called symplectic image denoted by  $\mathcal{V}_s$ , which coincides with  $\mathcal{V}$  as Abelian additive group and whose multiplication by a scalar is the multiplication given in  $\mathcal{V}$  restricted to  $\mathbb{C}$ . Here we identify  $\mathbb{C}$  with the set of quaternions of the form  $x_0 + ix_1$ . Any linear operator  $T$  on  $\mathcal{V}$  is obviously also  $\mathbb{C}$ -linear and so it is a linear operator on  $\mathcal{V}_s$ . We denote by  $T_s$  the operator  $T$  when it acts on  $\mathcal{V}_s$ . The converse is not true, i.e. if  $S$  is a  $\mathbb{C}$ -linear operator acting on  $\mathcal{V}_s$  then  $S$  is not, in general, a linear operator on  $\mathcal{V}$ , unless additional hypothesis are given. It is immediate to verify that if  $T$  is Hermitian then  $T_s$  is Hermitian (see also [26]).

We now state the spectral theorem:

**Theorem 8.1.** *Let  $A$  be a Hermitian operator on the quaternionic Hilbert space  $\mathcal{V}$ . Then there exists a spectral measure  $E$  defined on the Borel sets in  $\mathbb{R}$  such that*

$$A = \int_{-\infty}^{+\infty} \lambda dE(\lambda). \quad (8.1)$$

*Proof.* We observe that if  $A$  is a Hermitian linear operator, then its  $S$ -spectrum is real. Then we consider the symplectic image  $\mathcal{V}_s$  of  $\mathcal{V}$  and the operator  $A_s$  which is Hermitian and whose (real) spectrum coincide with the spectrum of  $A$ . Then we can use the classical spectral theorem to write

$$A_s = \int_{-\infty}^{+\infty} \lambda dE_s(\lambda)$$

where  $dE_s(\lambda)$  is a spectral measure with values in the lattice of projections in  $\mathcal{V}_s$ . Since the support of  $E$  is contained in  $\mathbb{R}$ , we use Corollary 6.1 in [26] to guarantee that  $E$

is a spectral measure with values in the lattice of projections in  $\mathcal{V}$ . This concludes the proof since  $A_s$  is in fact  $A$ .  $\square$

**Theorem 8.2.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic inner-product space, admitting a Hilbert majorant. Then  $\mathcal{V}$  is decomposable, and there exists a fundamental decomposition such that all three components and any sum of two of them are complete with respect to the Hilbert majorant.*

*Proof.* As in the proof of the corresponding result in the complex case (see [10, p. 89]) we apply the spectral theorem to the Gram operator  $G$  associated to the form  $[\cdot, \cdot]$ , and write  $G$  as (8.1):

$$G = \int_{-\infty}^{+\infty} \lambda dE(\lambda),$$

where the spectral measure is continuous and its support is finite since  $G$  is bounded. We then set

$$\mathcal{V}_- = E(0^-)\mathcal{V}, \quad \mathcal{V}_0 = (E(0) - E(0^-))\mathcal{V}, \quad \text{and} \quad \mathcal{V}_+ = (I - E(0))\mathcal{V}.$$

We have

$$\mathcal{V} = \mathcal{V}_-[\oplus]\mathcal{V}_0[\oplus]\mathcal{V}_+.$$

Each of the components and each sum of pairs of components of this decomposition is an orthogonal companion, and therefore closed for the Hilbert majorant in view of Proposition 6.6.  $\square$

In the next result, the space is non-degenerate, but the majorant is a Banach majorant rather than a Hilbert majorant.

**Proposition 8.3.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic non-degenerate inner-product space, admitting a Banach majorant  $\tau$  and a decomposition majorant  $\tau_1$ . Then,  $\tau_1 \leq \tau$ .*

*Proof.* Let  $\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-$  be a fundamental decomposition of  $\mathcal{V}$ . By Corollary 6.8 the space  $\mathcal{V}_+$  is closed in the topology  $\tau$ . Let  $P_+$  denote the map

$$P_+v = v_+ \tag{8.2}$$

where  $v = v_+ + v_-$  is the decomposition of  $v \in \mathcal{V}$  along the given fundamental decomposition of  $\mathcal{V}$ . We claim that the graph of  $P_+$  is closed, when  $\mathcal{V}$  is endowed with the topology  $\tau$ . Indeed, if  $(v_n)_{n \in \mathbb{N}}$  is a sequence converging (in the topology  $\tau$ ) to  $v \in \mathcal{V}$  and such that the sequence  $((v_n)_+)_{n \in \mathbb{N}}$  converges to  $z \in \mathcal{V}_+$  also in the topology  $\tau$ . Since the inner product is continuous with respect to  $\tau$  we have for  $w \in \mathcal{V}_+$

$$\begin{aligned} [z - v_+, w] &= \lim_{n \rightarrow \infty} [(v_n)_+, w] - [v_+, w] \\ &= \lim_{n \rightarrow \infty} [v_n, w] - [v_+, w] = [v, w] - [v_+, w] = [v - v_+, w] = 0 \end{aligned}$$

and so  $z = v_+$ . By the closed graph theorem (see Theorem 3.9)  $P_+$  is continuous. The same holds for the operator  $P_-v = v_-$  and so the operator

$$Jv = v_+ - v_- \tag{8.3}$$

is continuous from  $(\mathcal{V}, \tau)$  onto  $(\mathcal{V}, \tau)$ . Recall now that  $[Jv, v]$  is the square of the  $J$ -norm defining  $\tau_1$ . We have

$$[Jv, v] \leq k \|Jv\| \cdot \|v\|,$$

where  $\|\cdot\|$  denotes a norm defining  $\tau$ , and

$$[Jv, v] \leq k\|Jv\| \cdot \|v\| \leq k_1\|v\|^2$$

since  $J$  is continuous. It follows that the inclusion map is continuous from  $(\mathcal{V}, \tau)$  into  $(\mathcal{V}, \tau_1)$ , and so  $\tau_1 \leq \tau$ .  $\square$

**Proposition 8.4.** *Every decomposition majorant is a minimal majorant.*

*Proof.* A decomposition majorant is in particular a partial majorant and is normed (with associated  $J$ -norm  $\|u\|_J = [Ju, u]$ , where  $J$  is associated to the decomposition as in (8.3)). Thus, using Proposition 7.4, to prove the minimality it is enough to show that  $\|u\|_J$  is self-polar. That this holds follows from

$$\|u\|'_J = \sup_{\|v\|_J \leq 1} |[Ju, v]| = \sup_{\|v\|_J \leq 1} |[u, Jv]| = \|u\|_J.$$

$\square$

The question of uniqueness of a minimal majorant is considered in the next proposition.

**Proposition 8.5.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic inner-product space, admitting a decomposition*

$$\mathcal{V} = \mathcal{V}_+ [\oplus] \mathcal{V}_-, \quad (8.4)$$

where  $\mathcal{V}_+$  is positive definite and  $\mathcal{V}_-$  is negative definite. Assume that  $\mathcal{V}_+$  (resp.  $\mathcal{V}_-$ ) is intrinsically complete. Then, so is  $\mathcal{V}_-$  (resp.  $\mathcal{V}_+$ ). Then  $(\mathcal{V}, [\cdot, \cdot])$  has a unique minimal majorant.

*Proof.* The topology  $\tau$  defines a fundamental decomposition, and an associated minimal majorant  $\|\cdot\|_J$ . See Proposition 8.4. Let  $\tau$  be another minimal majorant. By Proposition 7.4 it is normed and self-polar and so there is a norm  $\|\cdot\|$  and  $k_1 > 0$  such that

$$\|v_+\| \leq k_1 \sup_{\substack{y \in \mathcal{V}_+ \\ \|y\| \leq 1}} |[v_+, y]|.$$

Using the uniform boundedness we find  $k_2 > 0$  such that

$$|[v_+, y]| \leq k_2[v_+, v_+], \quad \forall y \text{ such that } \|y\| \leq 1.$$

Hence, with  $C = k_1 k_2$ ,

$$\|v_+\| \leq C[v_+, v_+], \quad \forall v_+ \in \mathcal{V}_+. \quad (8.5)$$

Let now  $v \in \mathcal{V}$  with decomposition  $v = v_+ + v_-$ , where  $v_\pm \in \mathcal{V}_\pm$ . Since  $\tau$  is a normed majorant, there exists  $C_1$  such that

$$\|v_+\|^2 \leq C[v_+, v_+] = C[v_+, v] \leq CC_1\|v_+\| \cdot \|v\|$$

Hence

$$\|v\|_J^2 = [Jv, v] \leq C_1\|Jv\| \cdot \|v\| = C_1\|2v_+ - v\| \cdot K\|v\|^2$$

for an appropriate  $K > 0$ . The identity map is there continuous from  $(\mathcal{V}, \tau)$  onto  $(\mathcal{V}, \|\cdot\|_J)$ . Since  $\tau$  is defined by a single norm, it follows that the identity map is also continuous from  $(\mathcal{V}, \|\cdot\|_J)$  onto  $(\mathcal{V}, \tau)$  and this ends the proof.  $\square$

**Proposition 8.6.** *Let  $(\mathcal{V}, [\cdot, \cdot])$  be a quaternionic inner-product space, admitting a decomposition of the form (8.4), and with associated fundamental symmetry  $J$ . Then:*

(a) *Let  $\mathcal{L}$  denote a positive subspace of  $\mathcal{V}$ . Then, the operator  $P_+|_{\mathcal{L}}$  and its inverse are  $\tau_J$  continuous.*

(b) *Given another decomposition of the form (8.4), the positive (resp. negative) components are simultaneously intrinsically complete.*

*Proof.* To prove the result we follow [10, pp. 93-94]. Let  $\mathcal{L}$  be a positive subspace of  $\mathcal{V}$  and let  $v \in \mathcal{L}$ . By recalling (5.5), (8.2) and setting  $P_-v = v_-$ , where  $v = v_+ + v_-$  is the decomposition of  $v$  with respect to the fundamental decomposition  $\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-$ , we have:

$$\|v\|_J^2 = \|P_+v\|_J^2 + \|P_-v\|_J^2.$$

Since  $\mathcal{V}_+$  and  $\mathcal{V}_-$  are  $J$ -orthogonal, see Remark 5.5, we then have

$$[v, v] = \|P_+v\|_J^2 - \|P_-v\|_J^2$$

and so, since  $\mathcal{L}$  is positive,

$$\|v\|_J^2 = 2\|P_+v\|_J^2 - [v, v] \leq 2\|P_+v\|_J^2.$$

It is immediate that  $\|P_+v\|_J^2 \leq \|v\|_J^2$  and so we conclude that both  $P_+$  and its inverse are  $\tau_J$  continuous as stated in (a).

To show (b), we assume that there is another fundamental decomposition  $\mathcal{V} = \mathcal{V}'_+[\oplus]\mathcal{V}'_-$ . If we suppose that  $\mathcal{V}'_+$  is intrinsically complete, then Proposition 8.5 implies that  $\mathcal{V}'_+$  is complete with respect to the decomposition majorant corresponding to the decomposition  $\mathcal{V} = \mathcal{V}_+[\oplus]\mathcal{V}_-$ . Part (a) of the statement implies that also  $P^+\mathcal{V}'_+$  is complete in this topology and so it is intrinsically complete. If  $P^+\mathcal{V}'_+ = \mathcal{V}_+$  there is nothing to prove. Otherwise there exists a non-zero  $\tilde{v} \in \mathcal{V}_+$  orthogonal to  $P^+\mathcal{V}'_+$  so  $\tilde{v}$  is orthogonal to  $\mathcal{V}'_+$ . Then the subspace  $\mathcal{U}$  spanned by  $\tilde{v}$  and  $\mathcal{V}'_+$  is positive. Indeed, for a generic nonzero element  $u = \tilde{v} + \tilde{v}'$  ( $\tilde{v}' \in \mathcal{V}'_+$ ) we have

$$[u, u] = [\tilde{v} + \tilde{v}', \tilde{v} + \tilde{v}'] = [\tilde{v}, \tilde{v}] + [\tilde{v}', \tilde{v}'] > 0.$$

This implies that  $\mathcal{U}$  is a proper extension of  $\mathcal{V}'_+$  which is absurd by Proposition 5.3. This completes the proof.  $\square$

## 9. QUATERNIONIC KREIN SPACES

In this section we will study quaternionic Krein spaces following [10, Chapter V]. As in the classical case, they are characterized by the fact that they are inner product spaces non-degenerate, decomposable and complete. We will show that the scalar product associated to the decomposition gives a norm, and so a topology, which does not depend on the chosen decomposition. We will also study ortho-complemented subspaces of a Krein space and we will prove that they are closed subspaces which are Krein spaces themselves.

**Definition 9.1.** *If a quaternionic inner product space  $\mathcal{K}$  has a fundamental decomposition*

$$\mathcal{K} = \mathcal{K}_+[\oplus]\mathcal{K}_-, \tag{9.1}$$

*where  $\mathcal{K}_+$  is a strictly positive subspace while  $\mathcal{K}_-$  is strictly negative and if  $\mathcal{K}_+$  and  $\mathcal{K}_-$  are intrinsically complete, then we say that  $\mathcal{K}$  is a Krein space.*

The decomposition of a Krein space is obviously not unique when one of the components is not trivial. Both the spaces  $\mathcal{K}_+$  and  $\mathcal{K}_-$  are Hilbert spaces and they can be, in particular, of finite dimension. The Krein space is then called a Pontryagin space when  $\mathcal{V}_-$  is finite dimensional.

**Proposition 9.2.** *A Krein space is non-degenerate and decomposable. Each fundamental decomposition has intrinsically complete components  $\mathcal{K}_\pm$ .*

*Proof.* A Krein space is obviously decomposable by its definition and non-degenerate by Proposition 5.1.

By Theorem 8.6 (b), given (9.1) and any other fundamental decomposition  $\mathcal{K} = \mathcal{K}'_+[\oplus]\mathcal{K}'_-$  if  $\mathcal{K}_+$  is intrinsically complete so is  $\mathcal{K}'_+$  (and similarly for  $\mathcal{K}'_-$ ).  $\square$

**Proposition 9.3.** *A non-degenerate, decomposable, quaternionic inner product space  $\mathcal{K}$  is a Krein space if and only if for every associated fundamental symmetry  $J$ ,  $\mathcal{K}$  endowed with the inner product  $\langle v, w \rangle_J \stackrel{\text{def.}}{=} [v, Jw]$  is a Hilbert space.*

*Proof.* Let  $\mathcal{K}$  be a non-degenerate, decomposable, quaternionic inner product space, i.e.  $\mathcal{K} = \mathcal{K}'_+[\oplus]\mathcal{K}'_-$ . If  $\mathcal{K}$  is a Krein space then the associated fundamental symmetry  $J = P^+ - P^-$  makes it into a pre-Hilbert space, see Theorem 5.4. Completeness follows from the fact that  $\mathcal{K}_\pm$  are both complete. Conversely, assume that given a fundamental symmetry  $J$  the inner product  $\langle v, w \rangle_J \stackrel{\text{def.}}{=} [v, Jw]$  makes  $\mathcal{K}$  a Hilbert space. The intrinsic norm in  $\mathcal{K}_+$  is obtained by restricting the  $J$ -inner product to  $\mathcal{K}_+$ . Any Cauchy sequence in  $\mathcal{K}_+$  converges to an element in  $\mathcal{K}$  and it is immediate to verify that this element belongs to  $\mathcal{K}_+$ .  $\square$

**Theorem 9.4.** *Let  $\mathcal{K}$  be a quaternionic vector space with inner product  $[\cdot, \cdot]$ . Then  $\mathcal{K}$  is a Krein space if and only if :*

- (a)  $[\cdot, \cdot]$  has a Hilbert majorant  $\tau$  with associated inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$ ;
- (b) the Gram operator of  $[\cdot, \cdot]$  w.r.t.  $\langle \cdot, \cdot \rangle$ , i.e., the operator  $G$  which satisfies  $[v, w] = \langle v, Gw \rangle$ ,  $v, w \in \mathcal{K}$ , is boundedly invertible.

*Proof.* We follow the proof of Theorem V, 1.3 in [10], by repeating the main arguments. Assume that  $\mathcal{K}$  is a Krein space and denote by  $J$  the fundamental symmetry associated to the chosen decomposition (9.1). Define a norm using the  $J$ -inner product  $\langle \cdot, \cdot \rangle_J$  and let  $\tau_J$  be the corresponding topology which is a decomposition majorant by Proposition 8.5 and a Hilbert majorant. Since

$$[v, w] = [v, J^2w] = \langle v, Jw \rangle_J,$$

the Gram operator of  $[\cdot, \cdot]$  with respect to  $\langle \cdot, \cdot \rangle_J$  is  $J$  and  $J$  is boundedly invertible. We now prove part (b) of the statement. By Theorem 6.9 there is only one Hilbert majorant, thus if there are two positive inner products  $\langle \cdot, \cdot \rangle_1, \langle \cdot, \cdot \rangle_2$  whose associated norm define the Hilbert majorant, then the two norm must be equivalent. Reasoning as in [10], the two Gram operators  $G_j$   $j = 1, 2$  of  $[\cdot, \cdot]$  with respect to  $\langle \cdot, \cdot \rangle_j$ ,  $j = 1, 2$  are both boundedly invertible if and only if one of them is so. Since we have previously shown that (b) holds for  $G_1 = J$  then (b) holds for any other Gram operator.

Let us show the converse and assume that (a) and (b) hold. Then by Theorem 8.2,  $\mathcal{K}$  is decomposable and non-degenerate thus, by Proposition 5.1, it admits a decomposition

of the form (9.1). By Proposition 9.3,  $\mathcal{K}$  is a Krein space if for every chosen decomposition the  $J$ -inner product makes  $\mathcal{K}$  a Hilbert space or, equivalently, if  $\tau_J$  coincides with  $J$ . First of all we observe that since  $G$  is boundedly invertible, by the closed graph theorem we have that the Mackey topology coincides with  $\tau$ . By Theorem 7.7 we deduce that  $\tau$  is an admissible majorant and by Theorem 7.5  $\tau$  is also a minimal majorant and so  $\tau \leq \tau_J$ . However we know from Proposition 8.3 that  $\tau_J \leq \tau$  and the conclusion follows.  $\square$

**Remark 9.5.** Proposition 8.5 says that in a Krein space all the decomposition majorants are equivalent, in other words, all the  $J$ -norms are equivalent and will be called natural norms on  $\mathcal{K}$ . They define a Hilbert majorant called the strong topology of  $\mathcal{K}$ .

As an immediate consequence of the previous theorem we have:

**Corollary 9.6.** *The strong topology of  $\mathcal{K}$  equals the Mackey topology.*

In the sequel we will always consider a Krein space  $\mathcal{K}$  endowed with the strong topology  $\tau_M(\mathcal{K})$ .

**Proposition 9.7.** *The strong topology  $\tau_M(\mathcal{K})$  of the Krein space  $\mathcal{K}$  is an admissible majorant.*

*Proof.* By Proposition 7.7 that the Mackey topology is admissible and the fact that it is an admissible majorant is ensured by (5.7).  $\square$

**Theorem 9.8.** *Let  $\mathcal{K}$  be a quaternionic Krein space. A subspace  $\mathcal{L}$  of  $\mathcal{K}$  is ortho-complemented if and only if it is closed and it is a Krein space itself.*

*Proof.* We assume that  $\mathcal{L}$  is ortho-complemented. Then Corollary 6.7 shows that  $\mathcal{L}$  is closed. By Theorem 9.4,  $\mathcal{K}$  has a Hilbert majorant and thus we can use the condition given in Proposition 7.8 (b) to say whether  $\mathcal{L}$  is ortho-complemented. To this end, let us denote by  $G_{\mathcal{L}}$  the Gram operator defined by  $[v, w] = \langle v, G_{\mathcal{L}}w \rangle_J$ , for  $v, w \in \mathcal{L}$ , where  $J$  denotes the fundamental symmetry of  $\mathcal{K}$  associated with the chosen decomposition. By Theorem 9.4, the Gram operator  $G$  is boundedly invertible and thus, by Proposition 7.8 (b)  $\mathcal{L}$  is ortho-complemented if and only if  $\text{Ran}(G_{\mathcal{L}}) = \mathcal{L}$  but, since  $G_{\mathcal{L}}$  is  $J$ -symmetric, this is equivalent to  $G_{\mathcal{L}}$  boundedly invertible and so, again by Theorem 9.4 to the fact that  $\mathcal{L}$  is a Krein space.  $\square$

Given a definite subspace  $\mathcal{L}$  of a Krein space  $\mathcal{K}$ , it is clear that the intrinsic topology  $\tau_{int}(\mathcal{L})$  is weaker than the topology induced by the strong topology  $\tau_M(\mathcal{K})$  induces on  $\mathcal{L}$ . Thus we give the following definition:

**Definition 9.9.** *A subspace  $\mathcal{L}$  of a Krein space  $\mathcal{K}$  is said to be uniformly positive (resp. negative) if  $\mathcal{L}$  is positive definite (resp. negative definite) and  $\tau_{int}(\mathcal{L}) = \tau_M(\mathcal{K})|_{\mathcal{L}}$ .*

Note that the second condition amounts to require that  $\mathcal{L}$  is uniformly positive if  $[v, v] \geq c\|v\|_J^2$  for  $v \in \mathcal{L}$  (resp.  $\mathcal{L}$  is uniformly negative if  $[v, v] \leq -c\|v\|_J^2$  for  $v \in \mathcal{L}$ ) where  $c$  is a positive constant.

**Theorem 9.10.** *Let  $\mathcal{K}$  be a Krein space.*

- (a) *A closed definite subspace  $\mathcal{L}$  of  $\mathcal{K}$  is intrinsically complete if and only if it is uniformly definite.*

(b) A semi-definite subspace  $\mathcal{L}$  of  $\mathcal{K}$  is ortho-complemented if and only if it is closed and uniformly definite (either positive or negative).

*Proof.* The first statement follows from the fact that Proposition 9.3 and the closed graph theorem imply that a closed and definite subspace  $\mathcal{L}$  is intrinsically complete if and only if  $\tau_{int}(\mathcal{L}) = \tau_M(\mathcal{K})|_{\mathcal{L}}$  i.e. if and only if  $\mathcal{L}$  is uniformly definite.

By Proposition 9.2 and Theorem 9.8, a subspace  $\mathcal{L}$  is ortho-complemented if and only if it is closed, definite and intrinsically complete, i.e. if and only if  $\mathcal{L}$  is uniformly definite (either positive or negative). This completes the proof.  $\square$

**Remark 9.11.** From the definition of uniformly positive (resp. negative) subspace, it follows that a subspace of  $\mathcal{K}$  is uniformly positive (resp. negative) if so is its closure.

Theorem 9.10 and the previous remark immediately give the following:

**Corollary 9.12.** A semi-definite subspace of  $\mathcal{K}$  is uniformly definite if and only if its closure is ortho-complemented.

As a consequence of Theorems 9.8 and 9.10 we also have the following result, which was the main motivation for the present paper:

**Theorem 9.13.** Let  $\mathcal{K}$  denote a quaternionic Krein space, and let  $\mathcal{M}$  be a closed uniformly positive subspace of  $\mathcal{K}$ . Then,  $\mathcal{M}$  is a Hilbert space and is ortho-complemented in  $\mathcal{K}$ : One can write

$$\mathcal{K} = \mathcal{M}[\oplus]\mathcal{M}^{[\perp]},$$

and  $\mathcal{M}^{[\perp]}$  is a Krein subspace of  $\mathcal{K}$ .

*Proof.* The space is a Hilbert space by (a) of Theorem 9.10. That it is orthocomplemented follows then from Theorem 9.8.  $\square$

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