

**The Music of the Primes,
Harmonic Music between Red and White Noise,
Subatomic Particles**

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The Berry-Keating conjecture [BeM] is about some evidence that the eigenvalues are energy levels, that is eigenvalues of a Hermitian quantum ("Riemann") operator associated with the classical Hamiltonian

$$H(x,p)=xp,$$

where x is the (one-dimensional) position coordinate and p the conjugate momentum.

"The non-trivial zeros of Riemann's zeta function arise from inquiries into the distribution of prime numbers. The eigenvalues of a random Hermitian matrix arise from inquiries into the behavior of systems of subatomic particles under the laws of quantum mechanics. What on earth does the distribution of prime numbers have to do with the behavior of subatomic particles?"

This question is addressed in the tab "QUANTUM GRAVITY" with an alternatively proposed ground state energy model for the 1-dimensional harmonic quantum oscillator. The reference to the current ground state energy model is given by the Hermite polynomials (proof P1). The baseline for the alternative model is a spectral *density* (generalized Fourier transform, not only discrete amplitudes) with respect to the inner product of a Hilbert space $H(-a)$, $a>0$ (proof P2), enabling an alternative (less regular than current) energy Hilbert space with an additional continuous spectrum.

The Hilbert space framework of proof P2 is also related to the spectral density of music melodies ([VoR], [VoR1], [SaM]). The relationship to the framework of proof P2 is basically given by the fact, that the spectral density of music melodies is exactly "in the middle" (in the sense of related scale numbers of corresponding Hilbert space domains) of the spectral densities of red and white noise. With respect to Brownian motion and Mandelbrot fractals we recall from [PeH] p. 40, [VoR] [VoR1]:

[PeH] p. 40: *"One of the most exciting discoveries was that almost all musical melodies mimic 1/f-noise. Music has the same blend of randomness and predictability that is found in 1/f-noise. ... 1/f-noise, an intermediate but very commonly found type of fluctuation in nature, its origin is, as yet, a mystery".*

[SaM] p. 192, *"Es war wie der erste Entwurf einer unvollendeten Symphonie. Die endgültige Komposition verdankt viel der mathematischen Virtuosität von Siegel.... der ... die musikalische Formel entlockten konnte. heute als Riemann-Siegel-Formel*

[SaM] p. 125, **"Der Klang der Primzahlen** war also kein weisses Rauschen. Die Punkte auf Meereshöhe (the zeros of the Zeta function) waren isolierte Punkte, also mussten sie auch isolierte Töne erzeugen. Mit anderen Worten: Irgendwo in den Primzahlen hatte die Natur die Musik eines mathematischen Orchesters versteckt."

References

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[VoR] Voss R. F., Clarke J., 1/f noise in music and speech, Nature 258 (1975) 317-318

[VoR1] Voss R. F., Clarke J., 1/f noise in Music, Music from 1/f Noise, J. Accouns. Soc. Am. 63 (1978) 258-262

Harmonic music “noise” signals between red (Brownian) and white noise

Internet information

Brownian noise or red noise is the kind of signal noise produced by Brownian motion. A Brownian motion (i.e. a Wiener process) is a continuous stationary stochastic process having independent increments, i.e. $B(t) - B(0)$ is a normal random variable with mean μt and variance $\sigma^2 t$, μ, σ^2 constant real numbers. The density function of a Brownian motion is given by

$$f_{B(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}$$

The sample paths of Brownian motion are not differentiable, a mathematical fact explaining the highly irregular motions of small particles. The total variation of Brownian motion over a finite interval $[0, T]$ is infinite. It holds

$$\text{Var}\left[\frac{B(t)}{t}\right] = \frac{\sigma^2}{t}.$$

If $W(t)$ is a Wiener process on the interval $[0, \infty)$, then, as well the process

$$W^*(t) := \begin{cases} tW(1/t), & t > 0 \\ 0, & t = 0 \end{cases}.$$

White noise can be defined as the derivative of a Brownian motion (i.e. a Wiener process) in the framework of infinite dimensional distribution theory, as the derivative $B'(t)$ of $B(t)$, does not exist in the ordinary sense. Not only $B'(t)$, but also all derivatives of Brownian motion are generalized functions on the same space. For each t , the white noise $B'(t)$ is defined as a generalized function (distribution) on an infinite dimensional space. A Brownian motion is obtained as the integral of a white noise signal $dB(t)$, i.e.

$$B(t) = \int_0^t dB(\tau)$$

meaning that Brownian motion is the integral of the white noise $dB(t)$ whose power spectral density is flat

$$S_0 = |\text{Fourier}[B'](\omega)|^2 = \text{const}.$$

This means, that the spectral density S_0 for white noise is flat, i.e. $S_0 / \omega^0 = c$ i.e. it is inversely proportional to ω^0 . It holds $\text{Fourier}[B'](\omega) = i\omega \text{Fourier}[B](\omega)$. Therefore the power spectrum of Brownian noise is given by

$$S(\omega) = |\text{Fourier}[B](\omega)|^2 = \frac{S_0}{\omega^2}.$$

This means, that the spectral density of Brownian (red) noise is S_0 / ω^2 , i.e. it is inversely proportional to ω^2 , meaning it has more energy at lower frequencies, even more so than pink noise. The proposed framework for proof P2 is indicating a spectral density for harmonic music “noise” signals given by

$$S^*(\omega) = |\text{Fourier}[B^*](\omega)|^2 = \frac{S_0}{\omega}.$$