Müntz Formula and Zero Free Regions for the Riemann Zeta Function

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Abstract. A formula first derived by Müntz which relates the Riemann zeta function ζ times the Mellin transform of a test function f and the Mellin transform of the theta transform of f is exploited, together with analytic techniques, to construct new zero free regions for ζ in the critical strip.

1. Introduction

Riemann's hypothesis [Ri] states that all zeroes of Riemann's ζ-function (defined for all $s \in \mathbb{C}$ by analytic continuation from $\sum\limits_{n\in\mathbb{N}} \frac{1}{n^s}$, $\mathrm{Re}s > 1$) lie on the critical line $\mathrm{Re}s \equiv u = \frac{1}{2}$, except for those exactly at $s \in -2\mathbb{N}$ (the latter are called trivial zeroes). There are well known connections of Riemann's hypothesis, the distribution of prime numbers and other problems of number theory and of other areas of mathematics, se, e.g. [Ri], [LaII], [Pa], [Ti], [Te], [Pra], [WeiI], [WeiII], [RuSa], [Iw], [IwSa], [FaPav], [Con], [Ed], [Ch], [Bo], [Ka], [Iv] and references therein.
A rigorous proof of Riemann's hypothesis has not yet been achieved, to the best of our knowledge. There are, however, a number of results establishing in a rigorous way zero free regions for ζ. A classical result by de la Vallée Poussin and Hadamard establishes that the non trivial zeroes of ζ are in the critical strip $o < u < 1$, see, e.g. [Ed], [Iv], [Ti], [Te], [Ja]. Using the functional equation and the symmetry $\overline{\zeta(s)} = \zeta(\bar{s})$ ($\bar{}$ meaning complex conjugate) it suffices then to look for zero free regions in $u \in [\frac{1}{2}, 1)$, $v \equiv \mathrm{Im}s > 0$. A result of Hardy (1914), see, e.g. [Ti], establishes that there are countable many zeroes on the critical line $u = \frac{1}{2}$, about their distribution there are several rigorous analytic results, see, e.g. [Ti], [Bo], [Ka], in addition to numerical results, see, e.g. [Ti], [Co], [O].
As for zero free region, for $u \in [\frac{1}{2}, 1)$ several results are known. They concern regions for small values of $v > 0$ and are of analytic nature, e.g. [LaI], yielding as zero free region $\Delta_{[La]} = \{u \in (\frac{1}{2}, 1), 0 < v, |s-1| \leq \sqrt{2}\}$, and [Ja], yielding as zero free region $\Delta_{[Ja]} = \{u \in [\frac{3}{4}, 1), 0 < v < \frac{5}{2}\}$.
Other, more implicit results, also by analytic methods, are in [ABrFS] and yield zero free regions of the form $u \in (\frac{1}{2}, 1)$, $v < v_0(u)$ for some $v_0(u) > 0$. There are also numerical results, using computers or supercomputers (and thus holding inasmuch as one trusts such methods) and yield zero free regions for $u \in (\frac{1}{2}, 1)$ of the form $0 < v < V_1$ for some large $V_1 > 0$ independent of u (e.g. [VadeLu],

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$V_1 = 545439823.215$; see also, e.g. [FoO], [O] (the first classical result in this direction is discussed in [Ti], and yields $V_1 \cong 14.13472$).
Let us also mention shortly other types of result, involving large values of v, e.g. the classical results established by Vinogradov's and Korobov's methods, involving estimates on trigonometric sums [Wa]. They yield zero free regions of the form $\{v \geq V_0, u > 1 - C(\ln v)^{-\frac{2}{3}}(\ln\ln v)^{\frac{1}{3}}\}$ [Te] (with non explicit constant) or, e.g. $\{v \geq 3, u \geq 1 - 0.003101(\ln v)^{-\frac{2}{3}}(\ln\ln v)^{\frac{1}{3}}\}$ (thus with explicit constants) [Che]. There are also results in the checking of numerical validity of the Riemann hypothesis, e.g. [Co], [O], [FoO], [Me], and results relating the Riemann hypothesis with other conjectures, see, e.g. [Bo], [Co], [Iv], [IwSa], [Pa], [Ti], [Kn], [Pu].
For a unified discussion of the Riemann hypothesis for the classical zeta function and related L-functions see, e.g. [IwSa], [APu].
In the present paper we use analytic methods, in particular a formula of Müntz [Mu] (see also [Ti]) relating the Riemann ζ-function with the quotient of the Mellin transform of a theta function $\Theta(f)$ associated with a test function f and the Mellin transform of the test function f itself, to establish rigorously new zero free regions for "small values of v and a discrete unbounded set of large ones". The main results are in Section 4 (Corollary 4.10 and Remark 4.11), and in Section 5 (Theorem 5.11. Our results contain as particular cases those mentioned above ([LaII], [Ka]). The main idea originate in unpublished work by Madrecki and exploits a certain symmetry under Fourier transforms, together with a Lemma on the sign of certain oscillatory integrals. Madrecki's method, as presented in our pages, might permit a further extension of the results, as we point out in several remarks. In particular we also point out a statement (Remark 4.13) which implies the validity of the Riemann hypothesis: this statement itself does not appear to be much stronger, but in fact it is, than the statements we have already proven.
Let us also note that the idea of using a formula of the type of Müntz' formula appears in discussions of local zeta functions, e.g. in [Ta], [Bu], [Bur], [BuNg].
The structure of the present work is as follows. In Section 2 we recall Müntz' formula and present it in a modified form, exploiting symmetry under Fourier transformations. In Section 3 we present a basic positivity result on oscillatory integrals. We specialize the choice of test functions in Section 4, yielding the first type of new results on zero free regions. In Section 5 an extension of the class of test functions permits to obtain new zero free regions (in particular a discrete set of upper unbounded ones). In Section 6 our results are compared with those using other related methods in the literature.

2. Müntz's formula and Fourier transforms

Proposition 2.1. *For any $f \in L^1_{loc}(\mathbb{R})$ such that $f(x) = O(x^{-1})$ for $x \to +\infty$, $f(x) = O(1)$ as $x \downarrow 0$, the Mellin transform*

$$M(f)(s) \equiv \int_0^{+\infty} x^{s-1} f(x)dx$$

exists for all s in the critical strip $\Gamma \equiv \{s \in \mathbb{C} | 0 < \text{Re } s < 1\}$ and is analytic in Γ.

Proof. See, e.g., [Ti], [FeGouDu]. $\square$

Corollary 2.2. *For $f \in \mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ being the Schwartz test function space) the Mellin transform $M(f)(s)$ exists for all $s \in \Gamma$ and is analytic in Γ.*

Proof. f obviously satisfies the assumptions in Proposition 2.1. $\square$

We call a function $f : \mathbb{R} \to \mathbb{R}$ even if $f(x) = f(-x)$ for all $x \in \mathbb{R}$.

Proposition 2.3. *For $f \in \mathcal{S}(\mathbb{R})$, f even and such that $\int\limits_0^{+\infty} f(x)dx = 0$, $f(0) = 0$ the following version of Poisson summation formula holds:*

$$\Theta(f)(x) = \frac{1}{x}\Theta(\mathcal{F}(f))(\frac{1}{x})$$

where

$$\Theta(f)(x) \equiv \sum_{n=1}^{\infty} f(nx),\ x > 0$$

and

$$\mathcal{F}(f)(x) \equiv \int f(y)e^{2\pi ixy}dy$$

.

Proof. See, e.g. [Zy] (p. 70, (13.14)). $\square$

Remark . *This also holds, e.g., for $f \in C^2(\mathbb{R})$, even, such that $\int_0^\infty f(x)dx = 0$, $f(0) = 0$, $|f(x)| \le \frac{c}{1+|x|^2}$ as $|x| \to \infty$.*

Proposition 2.4. *For any $f \in \mathcal{S}(\mathbb{R})$, f even, such that $\int\limits_0^{+\infty} f(x)dx = 0$, $f(0) = 0$, the theta transform $\Theta(f)(x) \equiv \begin{cases} \sum\limits_{n=1}^{\infty} f(nx), & x > 0 \\ 0, & x = 0 \end{cases}$ of f is such that its Mellin transform $M(\Theta(f))(s)$ exists and is analytic in $s \in \Gamma$.*
For any $f \in \mathcal{S}(\mathbb{R})$ we have $|\Theta(f)(x)| \le c \sum\limits_{n=1}^{\infty} \frac{1}{1+(nx)^2} \le \frac{c'}{x^2}$ as $x \to \infty$, for some positive constants c, c', hence, in particular, $\Theta(f)(x)$ is bounded by $\frac{c'}{x^2}$ for $x \to +\infty$. Since $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R})$, $\Theta(\mathcal{F}(f))\left(\frac{1}{x}\right)$ is also bounded by $c'x^2$ for $x \downarrow 0$.

Proof. From Proposition 2.3 we have

$$\Theta(f)(x) = \frac{1}{x}\Theta(\mathcal{F}(f))(\frac{1}{x}),\ x > 0.$$

Thus, in particular, the bound on $|\Theta(f)(x)|$ follows from $|f(x)| \le c\frac{1}{1+|x|^2}$, observing also that $\mathcal{F}(f) \in \mathcal{S}(\mathbb{R})$, $\begin{cases} \Theta(f)(x) = O(1) & \text{as } x \downarrow 0 \\ \Theta(f)(x) = O(\frac{1}{x}) & \text{as } x \to +\infty \end{cases}$
Moreover $\Theta(f) \in L^1_{\text{loc}}(0, +\infty)$ (cf, e.g. [Bur]), so that by Proposition 2.1 the integral in $M(\Theta(f))(s)$ is absolutely convergent and $M(\Theta(f))(s)$ is analytic for $s \in \Gamma$. $\square$

Proposition 2.5. *For any $\rho \in \mathcal{S}(\mathbb{R})$, ρ even, with $\rho(0) = 0$, $\mathcal{F}(\rho)(0) = 0$, we have that $f \equiv \rho - \mathcal{F}(\rho)$ satisfies the assumptions in Proposition 2.4. Also $\mathcal{F}(f) = -f$.*

Proof. It suffices to remark that $\mathcal{F}(\mathcal{F}(\rho))(x) = \rho(-x) = \rho(x)$, hence $\mathcal{F}(f) = -f$. □

Corollary 2.6. *For f as in Proposition 2.4:*

$$M(\Theta(f))(s) = \int_0^1 \Theta(f)(r) r^{s-1} dr + \int_1^{+\infty} \Theta(f)(r) r^{s-1} dr, \; s \in \Gamma.$$

Proof. We use Proposition 2.4 and the splitting $\int_0^{+\infty} = \int_0^1 + \int_1^{+\infty}$. □

Let $\zeta(s)$, $s \in \mathbb{C}$, be the classical Riemann's zeta function ([Ti]).

Proposition 2.7. *(Müntz formula) For f as in Proposition 2.4:*

$$M(f)(s)\zeta(s) = M(\Theta(f))(s)$$

for all $s \in \Gamma$.

Proof. This follows for Re $s > 1$ from above Corollary 2.6, the definition of $\zeta(s)$ for Re $s > 1$ and interchange of $\sum$ and integrals, see, e.g. [Ti], [Bur]. By analytic continuation of both sides the relation in Proposition 2.7 is then also valid in Γ, both sides being well defined and analytic in Γ (by Propositions 2.3, 2.4, resp. well known analyticity of ζ in Γ, see [Ti]). □

Remark . *The formula in Proposition 2.7 was first derived by Müntz* [Mu]

Corollary 2.8. *For f as in Proposition 2.4*

$$M(f)(s)\zeta(s) = \int_0^1 x^{s-1} \Theta(f)(x) dx + \int_1^{+\infty} x^{s-1} \Theta(f)(x) dx, \; s \in \Gamma.$$

Proof. From Proposition 2.7 and Corollary 2.6. □

Proposition 2.9. *Let f be as in Proposition 2.4. Then*

$$\int_0^1 x^{s-1} \Theta(f)(x) dx = \int_1^{+\infty} x^{-s} \Theta(\mathcal{F}(f))(x) dx$$

for all $s \in \Gamma$.

Proof. From Proposition 2.3 we have, for $x > 0$:

$$\Theta(f)(x) = \frac{1}{x} \Theta(\mathcal{F}(f))(\frac{1}{x}).$$

Setting $y = \frac{1}{x}$ in the l.h.s. of Proposition 2.9 we have, for all $\varepsilon > 0$:

$$\begin{aligned}
\int_{\varepsilon}^{1} x^{s-1}\Theta(f)(x)dx &= \int_{1/\varepsilon}^{1} (\frac{1}{y})^{s-1} y\Theta(\mathcal{F}(f))(y)\left(-\frac{dy}{y^2}\right) \\
&= -\int_{1/\varepsilon}^{1} y^{-s}\Theta(\mathcal{F}(f))(y)dy \\
&= \int_{1}^{1/\varepsilon} y^{-s}\Theta(\mathcal{F}(f))(y)dy
\end{aligned}$$

The l.h.s. has a limit as $\varepsilon \downarrow 0$ (by Corollary 2.6), hence the r.h.s. also has a limit as $\epsilon \downarrow 0$ and taking the limit Proposition 2.9 is proven. □

Corollary 2.10. *For f as in Proposition 2.4 we have*

$$\text{Im } [M(f)(s)\zeta(s)] = \int_{1}^{+\infty} \left[x^{u-1}\Theta(f)(x) - x^{-u}\Theta(\mathcal{F}(f))(x)\right] \sin(v \ln x)dx,$$

with $s = u + iv$, $0 < u < 1$, $v \in \mathbb{R}$.

Proof. From Proposition 2.9 and $\text{Im } x^{s-1} = x^{u-1}\sin(v \ln x)$. □

3. Positivity properties of integrals

Proposition 3.1. *For any $h \geq 0$ on $(0, +\infty), h \in L^1_{loc}(0, +\infty)$, h decreasing and strictly decreasing on some open sub-interval of the form $(\frac{k\pi}{v}, \frac{(k+1)\pi}{v})$ for some $k \in \mathbb{N}_0$, and any $v > 0$ we have:*

$$\int_{0}^{+\infty} h(r)\sin(vr)dr > 0.$$

Moreover

$$\int_{0}^{T} h(r)\sin(vr)dr > 0$$

for any $T > 0$ if h satisfies the above assumptions with $(0, \infty)$ replaced by $(0, T)$.

Proof.

$$\begin{aligned}
\int_{0}^{+\infty} h(r)\sin(vr)dr &= \sum_{k=0}^{+\infty} \int_{k\pi/v}^{(k+1)\pi/v} h(r)\sin(vr)dr \\
&= \frac{1}{v}\sum_{k=0}^{\infty} \int_{k\pi}^{(k+1)\pi} h(\frac{r'}{v})\sin r' dr'.
\end{aligned}$$

But

$$\int_{k\pi}^{(k+1)\pi} h(\frac{r'}{v}) \sin r' dr' \underset{r''=r'-k\pi}{=} \int_0^{\pi} h(\frac{r''+k\pi}{v}) \sin(r''+k\pi) dr''$$
$$= (-1)^k \int_0^{\pi} h(\frac{r''+k\pi}{v}) \sin r'' dr''.$$

But $0 < a_k \equiv \int_0^{\pi} h(\frac{r''+k\pi}{v}) \sin r'' dr''$ and $a_{k+1} > a_k$ since $\sin r'' > 0$ for $r'' \in (0, \pi)$, $h \geq 0$ and h is decreasing. Strict inequality holds at least for one k, since h is strictly decreasing in some interval $(\frac{k\pi}{v}, \frac{(k+1)\pi}{v})$. By Leibniz Lemma on summation of alternating series, we then have the first statement. The moreover part follows by replacing h by $\chi_{[0,T]}h$ and remarking that $\chi_{[0,T]}h$ is also decreasing on $[0, T]$ and the interval $[T, +\infty]$ gives a zero contribution to the integral. $\square$

Corollary 3.2. *Let $v > 0$ and let $\tilde{x}_{v,k} \geq 1$ be such that $v \ln \tilde{x}_{v,k} = 2\pi k$, for some $k \in \mathbb{N}$. Let $h \geq 0$ on $(\ln \tilde{x}_{v,k}, +\infty)$, $h \in L^1_{loc}(0, +\infty)$, h decreasing, strictly decreasing on some interval of the form $(\frac{j\pi}{v}, \frac{(j+1)\pi}{v})$ for some $j \in \mathbb{N}$ such that $j\frac{\pi}{v} \geq \ln \tilde{x}_{v,k}$. Then $\int_{\ln \tilde{x}_{v,k}}^{+\infty} h(r) \sin(vr) dr > 0$. Moreover*

$$\int_{\ln \tilde{x}_{v,k}}^{T} h(r) \sin(vr) dr > 0$$

for any $T > \ln \tilde{x}_{v,k}$ if $T > (j+1)\frac{\pi}{v} \geq \ln \tilde{x}_{v,k}$.

Proof. With a corresponding argument as in the proof of Proposition 3.1 we see that (replacing r by $r' = r - \ln \tilde{x}_{v,k}$:

$$\int_{\ln \tilde{x}_{v,k}}^{+\infty} h(r) \sin(vr) dr = \int_0^{+\infty} h(r' + \ln \tilde{x}_{v,k}) sin(vr' + v \ln \tilde{x}_{v,k}) dr'$$
$$= \int_0^{+\infty} h(r' + \ln \tilde{x}_{v,k}) sin(vr' + 2\pi k) dr'$$
$$= \int_0^{+\infty} h(r' + \ln \tilde{x}_{v,k}) sin(vr') dr'$$

But $\check{h}(r') \equiv h(r' + \ln \tilde{x}_{v,k})$ satisfies the assumptions of Proposition 3.1. Applying Proposition 3.1 to the above integral we see that

$$\int_{\ln \tilde{x}_{v,k}}^{+\infty} h(r)\sin(vr)dr = \int_0^{+\infty} \check{h}(r')\sin(vr')dr' > 0.$$

The moreover part is proven by the same observation as in Proposition 3.1. ☐

Remark 3.3. *Given $v > 0$ we can take a $\Delta = e^{\frac{2\pi}{v}} \geq 1$ such that $\int_{\ln\Delta}^{+\infty} h(r)\sin(vr)dr > 0$ (with h having above properties on $(\ln\Delta, +\infty)$).*

Corollary 3.4. *Let v, $\tilde{x}_{v,k}$ be as in Corollary 3.2 and let $g \geq 0$ on $[\tilde{x}_{v,k}, +\infty)$, $g \in L^1_{loc}[\tilde{x}_{v,k}, +\infty)$, g such that $x \mapsto xg(x)$ is decreasing on $[\tilde{x}_{v,k}, +\infty)$, strictly decreasing on $(\ln\frac{j\pi}{v}, \ln\frac{(j+1)\pi}{v})$ for some $j \in \mathbb{N}$. Then $\int_{\tilde{x}_{v,k}}^{+\infty} g(x)\sin(v\ln x)dx > 0$. Moreover $\int_{\tilde{x}_{v,n}}^{T} g(x)\sin(v\ln x)dx > 0$ for any $T > \tilde{x}_{v,n}$, whenever $xg(x)$ is decreasing in $[\tilde{x}_{v,n}, T]$, strictly decreasing on $(\ln\frac{j\pi}{v}, \ln\frac{j+1\pi}{v})$ for some $j \in \mathbb{N}$ such that $\ln\frac{j\pi}{v} \geq \tilde{x}_{v,k}$ and $\ln\frac{j+1\pi}{v} \leq T$.*

Proof. It suffices to set $x = e^r$, use $\sin(v\ln x) = \sin(vr)$, $dx = e^r dr$, and realize that $h(x) \equiv xg(x)$ satisfies the assumptions in Corollary 3.2. ☐

Proposition 3.5. *For f as is Proposition 2.4, $0 < u < 1$ and f s.t. $f \geq 0$ for $x \geq x_c$, $\mathcal{F}(f)(x) \leq 0$ for $x \geq x_c$, for some $x_c \geq 1$, we have*

$$g_u(x) \equiv x^{u-1}\Theta(f)(x) - x^{-u}\Theta(\mathcal{F}(f))(x) \geq 0, \quad x \geq x_c.$$

One has strict inequality if either $f(x) > 0$ for some $x \in [x_c, +\infty)$ or $\mathcal{F}(f)(x) < 0$ for some $x \in [x_c, +\infty)$. $x \mapsto g_u(x)$ is continuous on $[1, +\infty)$.

Proof. The positivity follows from the assumption $f \geq 0$, and $\mathcal{F}(f)(x) \leq 0$, on $[x_c, +\infty)$. Strict positivity is immediate under the additional assumption $f(x) > 0$ resp. $\mathcal{F}(f)(x) < 0$ for some $x \in [x_c, +\infty)$. Continuity of g_u follows from the fact that $f(nx)$ is continuous in x and $\sum_{n=1}^{N} f(nx)$ converges absolutely for $x \geq x_c$ and uniformly since it is bounded by (cf. Proposition 2.4)

$$\sum_{n=1}^{N} \frac{c}{1+n^2x^2} \leq c\int_1^N \frac{1}{1+x^2}dx \leq c\int_1^\infty \frac{1}{1+y^2}dy < \infty$$

for some positive constant c, and corresponding statements hold for $\mathcal{F}(f)(\frac{1}{nx})$, together with Prop 2.3. ☐

Corollary 3.6. *If f is as in Proposition 3.5 and such that $\mathcal{F}(f)(x) = -\alpha f(\beta x)$ for $\alpha > 0, \beta \geq 1$ and $f \geq 0$ on $x \geq x_c$ then $g_u(x) \geq 0$, $x \geq x_c$, with g_u as in Proposition 3.5.*

Proof. $\mathcal{F}(f)$ satisfies the assumption $\mathcal{F}(f)(x) \leq 0$ on $x \geq x_c$. ☐

Proposition 3.7. *If $f = \rho - \mathcal{F}(\rho)$ with $\rho \in \mathcal{S}(\mathbb{R})$, ρ even, then $\mathcal{F}(f)(x) = -f(x)$, $x \in \mathbb{R}$. Moreover, if additionally $f(0) = 0$, f satisfies all assumptions in Proposition 3.5.*
Moreover $g_u(x) \equiv x^{u-1}\Theta(f)(x) - x^{-u}\Theta(\mathcal{F}(f))(x) = (x^{u-1} + x^{-u})\Theta(f)(x)$.
If $f(x) \geq 0$ for all $x \geq x_c$, for some $x_c \geq 1$, then $g_u(x) \geq 0 \quad \forall x \geq x_c$.
The strict inequality $g_u(x) > 0 \quad \forall x \geq x_c$ holds if $f(x) > 0 \quad \forall x \geq x_c$.

Proof. This follows easily from Proposition 2.5 and Proposition 3.5. The special form of g_u follows from $\mathcal{F}(f) = -f$. □

Proposition 3.8. *Let f, g_u be as in Proposition 3.7, with $x_c = \tilde{x}_{v,k}$ as in Corollary 3.4. Assume $f(x) > 0 \quad \forall x \geq x_c$ and that $xg_u(x)$ is strictly decreasing in $x \in [x_c, +\infty)$. Then $\int_{x_c}^{+\infty} g_u(x)\sin(v \ln x)dx > 0$.*

Proof. This is immediate consequence of Proposition 3.7 and Corollary 3.4. □

Corollary 3.9. *For f, g_u, x_c as in Proposition 3.8 and $s \equiv u + iv$, $0 < u < 1$, $v > 0$ we have*

$$Im[M(f)(s)\zeta(s)] > \int_1^{x_c} g_u(x)\sin(v \ln x)dx \tag{3.1}$$

Proof. This is immediately seen, by realizing that the integral on the r.h.s in Corollary 2.10 is equal

$$\int_1^{x_c} g_u(x)\sin(v \ln x)dx + \int_{x_c}^{+\infty} g_u(x)\sin(v \ln x)dx$$

and the latter integral is strictly positive by Proposition 3.8. □

Proposition 3.10. *Let f be as in Proposition 2.4. Let $g_{n,u}(x) \equiv x^{u-1}f(nx) - x^{-u}\mathcal{F}(f)(nx)$, $n \in \mathbb{N}$. For f such that $\mathcal{F}(f)(x) = -f(x)$ we have $g_{n,u}(x) = (x^{u-1} + x^{-u})f(nx)$.*
Then $g_{n,u} > 0$ for all $x \geq x_c$ if $f(x) > 0$ for all $x \geq x_c$.
A sufficient condition for $x \mapsto xg_{n,u}(x)$ to be decreasing for $x \geq max(x_c, x_c')$ is

$$\gamma_u(x) \equiv \frac{1}{x}\frac{ux^{2u-1} + 1 - u}{x^{2u-1} + 1} \leq -n\frac{f'(nx)}{f(nx)} \tag{3.2}$$

for $x \geq \max(x_c, x_c')$.

Proof. We have $(xg_{n,u}(x))' = xg_{n,u}'(x) + g_{n,u}(x)$. But $g_{n,u}'(x) = [(u-1)x^{u-2} - ux^{-u-1}]f(nx) + [x^{u-1} + x^{-u}]nf'(nx)$. Thus

$$\begin{aligned}(xg_{n,u}(x))' &= x[(u-1)x^{u-2} - ux^{-u-1}]f(nx) + x[x^{u-1} + x^{-u}]nf(nx) + \\ &\quad +[x^{u-1} + x^{-u}]f'(nx) \\ &= [ux^{u-1} + (1-u)x^{-u}]f(nx) + x[x^{u-1} + x^{-u}]nf'(nx)\end{aligned}$$

Thus

$$(xg_{n,u}(x))' \leq 0 \quad \Leftrightarrow \quad \frac{ux^{2(u-1)} + (1-u)x^{-1}}{x^{2u-1} + 1} \leq -\frac{nf'(nx)}{f(nx)}$$

□

Remark 3.11. *If $f(x) > 0$ for all $x \geq x_c$ and $f'(x) \leq 0$ for all $x \geq x'_c$ for some $x'_c \geq 1$, then (3.2) can be equivalently written in the form $\gamma_u(x) \leq n\frac{|f'(nx)|}{f(nx)}$.*

Proposition 3.12. *Let $f \geq 0$ on $[x_c, \infty)$, $x_c \geq 1$ and such that, for all $x \geq x'_c \geq 1$:*

$$\frac{|f'(x)|}{f(x)} \geq \gamma_u(x),$$

with γ_u as in Proposition 3.10, for some $u \in (0,1)$. Then, for all $n \in \mathbb{N}$

$$n\frac{|f'(nx)|}{f(nx)} \geq \gamma_u(x),$$

for all $x \geq max(x_c, x'_c)$.

Proof. We have, from the assumption $\frac{|f'(x)|}{f(x)} \geq \gamma_u(x)$, $x \geq max(x_c, x'_c) \equiv x''_c$, that $\frac{|f'(nx)|}{f(nx)} \geq \gamma_u(nx)$, $\forall x \geq \frac{x''_c}{n} \geq \frac{1}{n}$, $n \in \mathbb{N}$. Hence, in particular, $n\frac{|f'(nx)|}{f(nx)} \geq n\gamma_u(nx)$, $x \geq x''_c \geq \frac{x''_c}{n}$. So it remains to prove

$$n\gamma_u(nx) \geq \gamma_u(x) \qquad x \geq x''_c, n \in \mathbb{N}$$

But this is equivalent to

$$n\frac{1}{nx}\frac{u(nx)^{2u-1}+1-u}{(nx)^{2u-1}+1} \quad \geq \quad \frac{1}{x}\frac{ux^{2u-1}+1-u}{x^{2u-1}+1}$$

$$\Leftrightarrow \qquad un^{2u-1}x^{2(2u-1)} + x^{2u-1}(1-u) + un^{2u-1}x^{2u-1} + 1 - u \geq$$
$$\geq un^{2u-1}x^{2(2u-1)} + (1-u)n^{2u-1}x^{2u-1} + ux^{2u-1} + 1 - u$$

$$\Leftrightarrow \qquad q_u(x) := x^{2u-1}(1-u+un^{2u-1}-(1-u)n^{2u-1}-u) = x^{2u-1}(1-2u)(1-n^{2u-1}) \geq 0$$

And here, one has three cases:

- $u = \frac{1}{2} \qquad \Rightarrow q_u(x) \equiv 0$
- $0 \leq u \leq \frac{1}{2} \qquad \Rightarrow 1-2u > 0, 1-n^{2u-1} > 0 \Rightarrow q_u(x) > 0$
- $\frac{1}{2} \leq u \leq 1 \qquad \Rightarrow 1-2u < 0, 1-n^{2u-1} < 0 \Rightarrow q_u(x) > 0$

□

Corollary 3.13. *Let f be as in Proposition 2.4 and such that $\mathcal{F}(f)(x) = -f(x)$ (as in Proposition 3.10) and such that $f(x) \geq 0$ and $\frac{|f'(x)|}{f(x)} \geq \gamma_u(x)$ for all $x \geq x''_c \geq 1$. Then $x \mapsto xg_{n,u}(x)$ is decreasing for $x \geq x''_c$ and all $n \in \mathbb{N}$ (with $g_{n,u}$ as in Proposition 3.10).*

Proof. This is an immediate consequence of Proposition 3.12 and Proposition 3.10.

□

Corollary 3.14. *Let f be as in Corollary 3.13. Let $g_u(x)$ be as in Proposition 3.7. Let $v = \frac{2\pi k}{\ln x''_c} = \frac{2\pi\tilde{k}}{\ln \tilde{x}_c}$, $k, \tilde{k} \in \mathbb{N}$, $0 < u < 1$. Then*

$$\int_{x''_c \vee \tilde{x}_c}^{+\infty} g_u(x)\sin(v\ln x)dx > 0$$

and $Im[M(f)(s)\zeta(s)] > \int_1^{\tilde{x}_c \vee x''_c} g_u(x)\sin(v\ln x)dx.$

Proof. The statement follows immediately from Proposition 3.8 and Corollary 3.9. □

Proposition 3.15. *Let* $q:\mathbb{R}\to\mathbb{R}$*. Suppose* $xq(x)$ *is positive and strictly decreasing on* $[x''_c,+\infty]$ *with an* $x''_c \leq 2$*. Suppose moreover*

$$\int_1^y [x^{-u}+x^{u-1}]q(x)\sin(v\ln x)dx \geq 0 \qquad \forall \quad 0<u<1$$

for $v=\frac{2\pi k}{\ln y}$*, some* $k\in\mathbb{N}$ *and* $y>1$*. Then*

$$\int_1^y [x^{-u}+x^{u-1}]\Theta(q)(x)\sin(v\ln x)dx \geq 0 \qquad \forall \quad 0<u<1$$

Proof. Let $N\in\mathbb{N}$, then

$$\int_1^y [x^{-u}+x^{u-1}]\Theta_N(q)(x)\sin(v\ln x)dx = \sum_{n=1}^N I_n,$$

with $I_n \equiv \int_1^y [x^{-u}+x^{u-1}]q(nx)\sin(v\ln x)dx$, $\Theta_N(q)(x)\equiv\sum_{n=1}^N q(nx)$.
We have $I_1\geq 0$, by assumption. Moreover, we claim $I_n\geq 0 \quad \forall n\geq 2$.
In fact, with the following notation, $h(x):=x^{-u}+x^{u-1}$, $\tilde{h}(x):=\left(\frac{x}{n}\right)^{-u}+\left(\frac{x}{n}\right)^{u-1}=n^u x^{-u}+n^{1-u}x^{u-1}$, one obtains

$$\tilde{h}'(x) = -un^u x^{-u-1}+(u-1)n^{1-u}x^{u-2} \leq -ux^{-u-1}-(1-u)x^{u-2}=h'(x),$$

because $n^u\geq 1$, $n^{1-u}\geq 1$, $n\geq 2$, $0<u<1$, $x\geq 1$. Thus, $\tilde{h}'(x)\leq h'(x)$ and $\tilde{h}(x)\geq h(x)\geq 0$.
From assumption one has

$$(xh(x)q(x))' \leq 0 \qquad x\geq x''_c.$$

From this inequality, it follows that

$$\begin{aligned}
(x\tilde{h}q(x))' &= \tilde{h}(x)q(x)+x\tilde{h}'(x)q(x)+x\tilde{h}(x)q'(x)\\
&= x\tilde{h}(x)q(x)+\tilde{h}(x)(q(x)+xq'(x))\\
&\leq xh'(x)q(x)+h(x)(q(x)+xq'(x))\\
&= (xh(x)q(x))'\\
&\leq 0
\end{aligned}$$

Thus,

$$I_n \equiv \int_1^y [x^{-u}+x^{u-1}]q(nx)\sin(v\ln x)dx = \frac{1}{n}\int_n^{ny}\tilde{h}(x)q(x)\sin(v\ln\frac{x}{n})dx \geq 0$$

by Corollary 3.4 (using that in the latter integral we have $x \geq x_c''$).
This implies $\sum_{n=1}^{N} I_n \geq 0$. By the definition of $\Theta_N(q)$ we have

$$\Theta_N(q)(x) \to \Theta(q)(x), \quad N \to +\infty, \qquad \forall x \in [1, y]$$

But

$$|[x^{-u} + x^{u-1}]q(nx)\sin(v \ln x)| \leq q(x) \quad \forall n \in \mathbb{N}, \quad \forall x \in [1, y]$$

with $q \in L^1[1, y]$. By dominated convergence

$$0 \leq \sum_{n=1}^{+\infty} I_n = \int_1^y [x^{-u} + x^{u-1}]\Theta(q)(x)\sin(v \ln x)dx$$

□

4. A particular choice of test function and the application to the finding of zero-free regions

Proposition 4.1. *Let* $\rho(x) = (ax^2 + b)e^{-\frac{1}{2}(cx)^2}$, $a, b \in \mathbb{R}$, $c \in \mathbb{R}_+$. *Then*

$$\mathcal{F}(\rho)(x) = \frac{\sqrt{2\pi}}{c^3}\left(a\left(1 - \frac{(2\pi x)^2}{c^2}\right) + bc^2\right)e^{-\frac{2\pi^2 x^2}{c^2}}$$

For $a = bc^2\left(\frac{c}{\sqrt{2\pi}} - 1\right)$ *we have* $f(0) = 0$.
Set $f \equiv \rho - \mathcal{F}(\rho)$. *Then* f, *with the above choice of* a *as a function of* b, c, *satisfies the assumption in Propositions 2.4, 2.5.*

Proof. We have $\rho(x) = \frac{a}{4c^2}\psi_2(cx) + (b + \frac{a}{2c^2})\psi_0(cx)$, with $\psi_2(y) \equiv (4y^2 - 2)e^{-\frac{y^2}{2}}$ and $\psi_0(y) \equiv e^{-\frac{y^2}{2}}$ (so that ψ_k is the k-th Hermite function, in the normalization of, e.g., [Hida] p.311).
From, e.g. [Hida] p.312 we have

$$\begin{aligned}\mathcal{F}(\psi_2)(x) &= -\sqrt{2\pi}\psi_2(2\pi x)\\ \mathcal{F}(\psi_0)(x) &= \psi_0(2\pi x)\end{aligned}$$

hence, setting $\psi_k^c(x) \equiv \psi_k(cx)$, $k = 0, 2$:

$$\begin{aligned}\mathcal{F}(\psi_2^c)(x) &= -\frac{\sqrt{2\pi}}{c}\psi_2(2\pi\frac{x}{c})\\ \mathcal{F}(\psi_0^c)(x) &= \frac{\sqrt{2\pi}}{c}\psi_0(2\pi\frac{x}{c})\end{aligned}$$

Thus

$$\mathcal{F}(\rho) = \sqrt{2\pi}\frac{1}{c^3}\left(a\left(1 - \left(\frac{2\pi x}{c}\right)^2\right) + bc^2\right)e^{-\frac{1}{2}\left(\frac{2\pi x}{c}\right)^2}$$

The condition $f(0) = 0$ requires $a = b\left(\frac{c^3}{\sqrt{2\pi}} - c^2\right)$. That f, with the above choice of a as a function of b, c, satisfies the assumption of Proposition 2.4 is clear, since

$f \in \mathcal{S}(\mathbb{R})$, f is even and $\mathcal{F}(f)(x) = -f(x)$ is zero for $x = 0$, by construction, thus also

$$\int_0^{+\infty} f(x)dx = \frac{1}{2}\mathcal{F}(f)(0) = 0$$

□

Corollary 4.2. *Let*

$$\rho(x) := \left(ax^2 + 1\right) e^{-\frac{1}{2}c^2x^2},$$

with $a \equiv c^2\left(\frac{c}{\sqrt{2\pi}} - 1\right)$ *and* $c \in \mathbb{R}$. *Then* $\mathcal{F}(\rho)(0) = \rho(0) = 1$. *Let* f *be defined by* $f(x) = \rho(x) - \mathcal{F}(\rho)(x)$. *Then* $\mathcal{F}(f)(x) = -f(x)$ *and* $f(0) = \mathcal{F}(\rho)(0) = 0$.

Proof. From Proposition 4.1 it follows that

$$\mathcal{F}(\rho)(x) = \frac{\sqrt{2\pi}}{c^3}\left(a\left(1 - \frac{(2\pi)^2}{c^2}x^2\right) + c^2\right) e^{-\frac{2\pi^2}{c^2}x^2} \equiv (1 - bx^2)e^{-\frac{2\pi^2}{c^2}x^2},$$

with $b \equiv \frac{(2\pi)^{\frac{5}{2}}}{c^5}a$. This implies:

$$\mathcal{F}(\rho)(0) = \frac{\sqrt{2\pi}}{c^3}(a + c^2) = 1 = \rho(0)$$

The rest is immediate. □

Lemma 4.3. *Let* f *be defined as*

$$f(x) = (ax^2 + 1)e^{-\frac{1}{2}c^2x^2} + (bx^2 - 1)e^{-\frac{2\pi^2}{c^2}x^2},$$

with a, b *as in Corollary 4.2. Then the following inequality holds:*

$$f(x) > 0 \qquad \text{for} \quad x \in [\frac{1}{\sqrt{b}}, +\infty) \quad \text{and} \quad c = 3.$$

Moreover $f'(x) < 0$ *for* $x \in [2, +\infty)$ *and* $c = 3$.

Proof. One has, by assumption:

(a) $(ax^2 + 1)e^{-\frac{1}{2}c^2x^2} > 0 \quad \forall x \geq 0$, for $c \geq \sqrt{2\pi}$ (so that $a \geq 0$ by the above choice of a)

(b) $4b - 1 > 0$ (since $b \approx 0.72$ for our choice $c = 3$ and $a = 9(3/\sqrt{2\pi} - 1)$)

From (a) and (b) it follows that

$$f(x = 2) > 0 \tag{4.1}$$

One has

$$f(x) = 0 \Leftrightarrow (ax^2 + 1)e^{-\frac{1}{2}c^2x^2} = (1 - bx^2)e^{-\frac{2\pi^2}{c^2}x^2}. \tag{4.2}$$

In (4.2) $(ax^2+1)e^{-\frac{1}{2}c^2x^2}$ is strictly positive for all $x \geq 0$. Thus, a necessary condition for (4.2) to be satisfied is

$$\begin{aligned} 1 - bx^2 &> 0 \\ &\Leftrightarrow bx^2 < 1 \\ &\Leftrightarrow x < \frac{1}{\sqrt{b}} \qquad \text{for} \quad x \geq 0 \quad \text{and} \quad b > 0 \end{aligned}$$

With a suitable choice of c, for example $c = 3$, one has $a > 0$ and $b > 0$, moreover $\frac{1}{\sqrt{b}} \approx 1.18$.
This implies that $f(x)$ has in particular no zeros for $x \in [\frac{1}{\sqrt{b}}, +\infty)$. From (4.1) and (4.2) it follows finally that $f(x) > 0$ for $x \geq \frac{1}{\sqrt{b}}$.
That $f'(x) < 0$ for $x \geq 2$, $c = 3$, can be seen by a similar computation. □

Remark 4.4. *In fact we have also shown, that $f(1/\sqrt{b}) > 0$ and f has no zeros in $[1/\sqrt{b}, +\infty)$. Moreover we observe that (4.2) is equivalent, for $x \neq \frac{1}{\sqrt{b}}$ to:*

$$\frac{ax^2+1}{1-bx^2} = e^{\frac{1}{2}\left(c^2 - \frac{2\pi^2}{c^2}\right)x^2}$$

An easy graphical representation shows that, for $c = 3$, there exists exactly one solution of this equation, hence exactly one zero of $f(x)$ in $(0, 1/\sqrt{b}]$.

Corollary 4.5. *Let f be defined as in Lemma 4.3 (resp. in Remark 4.4). Then the following inequality holds for all $x \geq 2$ (or even for all $x \geq \frac{1}{\sqrt{b}}$):*

$$g_{1,u}(x) := (x^{-u} + x^{u-1})f(x) > 0$$

(see Proposition 3.10 for corresponding definition of $g_{n,u}(x)$.)

Proof. The assertion follows immediately from Proposition 3.7, Lemma 4.3 and Remark 4.4. □

Lemma 4.6. *Let the function f be defined as in Lemma 4.3 (and Corollary 4.5). Then the following inequality holds for all $x \geq 2$:*

$$\frac{|f'(x)|}{f(x)} \geq \gamma_u(x).$$

Hence the conclusion of Proposition 3.12 holds with $max(x_c, x_c')$ replaced by 2.

Proof. It has to be shown that:

$$|f'(x)|x(x^{2u-1}+1) > f(x)(ux^{2u-1}+1-u)$$

From the assumption $u \in (0, 1)$ one has

$$(x^{2u-1}+1) > ux^{2u-1}+1-u,$$

hence, it is sufficient to show:

$$\begin{aligned} |f'(x)|(ux^{2u-1}+1-u)x &> f(x)(ux^{2u-1}+1-u) \\ \Leftrightarrow \qquad x|f'(x)| &> f(x) \end{aligned}$$

But for all $x \in [2, +\infty)$ one has $f'(x) < 0$ (for the f as in Lemma 4.3) so we can write $|f'(x)| \equiv -f'(x)$. Thus it suffices to show that $-xf'(x) > f(x)$. But f is as in Lemma 4.3 and

$$-xf'(x) = x^2\left((ac^2x^2 - 2a + c^2)e^{-\frac{1}{2}c^2x^2} + (b(\frac{2\pi}{c})^2x^2 - 2b - (\frac{2\pi}{c})^2)e^{-\frac{2\pi^2}{c^2}x^2}\right)$$

Because $x \geq 1$, it suffices thus to show

$$\begin{aligned} ac^2x^2 - 2a + c^2 &> ax^2 + 1 \quad \wedge \\ b(\frac{2\pi}{c})^2x^2 - 2b - (\frac{2\pi}{c})^2 &> bx^2 - 1 \\ \Leftrightarrow \qquad x^2 &> \frac{2a+1-c^2}{a(c^2-1)} \quad \wedge \\ x^2 &> \frac{2b+(\frac{2\pi}{c})^2-1}{b((\frac{2\pi}{c})^2-1)} \end{aligned}$$

The right hand side of the first inequality is negative, the right hand side of the second is approximately equal to $\frac{2b+(\frac{2\pi}{c})^2-1}{b((\frac{2\pi}{c})^2-1)} \approx 2$ (since $c = 3$). Both inequalities are thus fullfilled for all $x \geq 2$. □

Lemma 4.7. *Let $f(x) := (ax^2+1)e^{-\frac{1}{2}c^2x^2}+(bx^2-1)e^{-\frac{2\pi^2}{c^2}x^2}$, with a, b as in Corollary 4.2, $c = 3$.*
Then $\int\limits_1^2 h(x)dx := \int\limits_1^2 g_{1,u}(x)\sin(\frac{2\pi}{\ln 2}\ln x)dx > 0$ (with (as before) $g_{1,u}(x) \equiv (x^{-u} + x^{u-1})f(x)$).

Proof. The function $h(x) \equiv g_{1,u}(x)\sin(\frac{2\pi}{\ln 2}\ln x)$ has four zeros in the interval $[1, 2]$, namely $h(1) = 0$, $h(\sqrt{2}) = 0$, $h(2) = 0$ and $h(\xi) = 0$, where ξ is a zero of the function f, $1 < \xi < 1.01$ (that ξ fullfills these inequalities can be seen numerically, by realizing that ξ satisfies $\frac{a\xi^2+1}{1-b\xi^2} = e^{c^2/2\xi^2}e^{2\pi^2/c^2\xi^2}$ and $c = 3$). One has: $h(x) < 0$ in $(1, \xi)$, $h(x) > 0$ in $(\xi, \sqrt{2})$ and $h(x) < 0$ in $(\sqrt{2}, 2)$. Thus, it is reasonable to divide the interval of integration:

$$\int\limits_1^2 h(x)dx = \int\limits_1^{\xi} h(x)dx + \int\limits_{\xi}^{\sqrt{2}} h(x)dx + \int\limits_{\sqrt{2}}^{2} h(x)dx =: I_1 + I_2 + I_3.$$

For the first part one obtains (using that $x^{-u} + x^{u-1} = \frac{x^{2u-1}+1}{x^u} \leq 2$ the maximum being taken for $u = 1$)

$$\begin{aligned} -I_1 &= -\int\limits_1^{\xi} (x^{-u} + x^{u-1})f(x)\sin(\frac{2\pi}{\ln 2}\ln x)dx \\ &\leq -2\int\limits_1^{\xi} f(x)\sin(\frac{2\pi}{\ln 2}\ln x)dx \\ &\leq -2f(1)\int\limits_1^{1.01} \sin(\frac{2\pi}{\ln 2}\ln x)dx \\ &\leq 2.628566 \cdot 10^{-7} \qquad \forall u \in [0, 1] \end{aligned}$$

As a function of the parameter u, $h(x,u)$ has a minimum at $u = 0.5$ and two maxima, at $u = 0$ resp. $u = 1$ in $u \in [0,1]$. Additionally, $h(x,1) = h(x,0) \forall x \geq 1$. For the second part one has then the estimation

$$I_2 \geq \sum_{i=1}^{s-1}(x_{i+1} - x_i) \cdot min\{h(x_i, u = 0.5), h(x_{i+1}, u = 0.5)\},$$

for any $s \geq 2$ and $1.01 \leq x_1 \leq \cdots \leq x_s \leq \sqrt{2}$, and for the third part

$$I_3 \geq \sum_{i=s+1}^{r}(x_{i+1} - x_i) \cdot min\{h(x_i, u = 1), f(x_{i+1}, u = 1)\},$$

for any $s \geq 2$, $r \geq s+2$, $1.01 \leq x_{s+1} \leq \sqrt{2} \leq \cdots \leq x_{r+1} \leq 2$, for all $u \in [0,1]$. Using the following sampling points:

$$x_1 = 1.01, x_2 = 1.05, x_3 = 1.10, x_4 = 1.15, \ldots, x_s = 1.40,$$
$$x_{s+1} = 1.41, x_{s+2} = 1.45, x_{s+3} = 1.50, \ldots, x_r = 1.95$$

it can be computed numerically that $I_2 \geq 2.328 \cdot 10^{-3}$ and $-I_3 \leq 1.642 \cdot 10^{-3}$. Thus,

$$\sum_{i=1}^{3} I_i > 0 \qquad \forall u \in [0,1]$$

□

Lemma 4.8. *Let f be as in Lemma 4.3. Then for all $\Delta \geq 2$ and $u \in (0,1)$ one has*

$$\int_1^\Delta g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx > 0,$$

with $g_{1,u}(x) = (x^{-u} + x^{u-1})f(x)$.

Proof. Let $\int_1^\Delta g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx =: J(\Delta)$.
For $\Delta \geq 2$ one has then

$$\int_1^\Delta g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx =$$
$$\int_1^\xi g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx + \int_\xi^\Delta g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx =: J_1(\Delta) + J_2(\Delta),$$

where ξ denotes the only zero of the function f in $(1, +\infty)$, then one has that $f(x) < 0$ for all $x \in [1, \xi)$ and $f > 0$ for all $x > \xi$. Additionally we know about f

that it is strictly decreasing for all $x \geq \sqrt{2}$. Now, we can state for the first integral:

$$\begin{aligned}
|J_1(\Delta)| &\equiv |\int_1^{\xi} g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx| \leq |\int_1^{1.01} g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx| \\
&\leq 2|\int_1^{1.01} f(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx| \leq 2|f(1)|\int_1^{1.01}\sin(\frac{2\pi}{\ln\Delta}\ln x)dx \\
&\leq 2|f(1)|\int_1^{1.01}\sin(\frac{2\pi}{\ln 2}\ln x)dx =: |\tilde{J}_1(2)| \qquad \forall\Delta\geq 2
\end{aligned}$$

For the second integral it can be stated that

$$\begin{aligned}
J_2(\Delta) &\equiv \int_{\xi}^{\Delta} g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx \\
&\geq \int_{1.01}^{\Delta} g_{1,u}(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx \\
&\geq \int_{1.01}^{\Delta} f(x)\sin(\frac{2\pi}{\ln\Delta}\ln x)dx
\end{aligned}$$

Now consider the function f as a weight function for the positive (on $[1.01, \sqrt{\Delta})$) and negative (on $(\sqrt{\Delta}, \Delta)$) part of the sinus function and define:

$$\mu(t) := \frac{\int_{1.01}^{t} f(x)dx}{\int_{1.01}^{\infty} f(x)dx} < \frac{\int_{1.01}^{t} f(x)dx}{\int_{1.01}^{t^2} f(x)dx} := \mu_\Delta(t) \quad \text{for} \quad t \geq 1.01$$

One has

$$\mu'(t) = f(t) > 0 \quad \forall t \geq 1.01$$

$\Rightarrow$ $\mu(t)$ is strictly increasing on $(1.01, +\infty)$ and thus $\mu_\Delta(t) > \mu_\Delta(\sqrt{2}) > \frac{1}{2}$ for all $t \geq \sqrt{2}$. Set now $t = \sqrt{\Delta}$. Then one has $J_2(\Delta) \geq J_2(2)$ for all $\Delta \geq 2$. $\Rightarrow$ $J_(\Delta) = J_1(\Delta) + J_2(\Delta) \geq \tilde{J}_1(2) + J_2(2) > 0 \quad \forall\Delta \geq 2$, where the last inequality follows from Lemma 4.7. □

Corollary 4.9. *For f as in Lemma 4.7, $0 < v < \frac{2\pi}{\ln 2}$*

$$Im[M(f)(s)\zeta(s)] > 0$$

Proof. From Lemma 4.6, Corollary 3.13 and Proposition 3.15 we have $Im[M(f)(s)\zeta(s)] > \int_1^{\Delta} g_u(x)\sin(v\ln x)dx$, with $v = \frac{2\pi}{\ln\Delta}$, $\Delta > 1$. For $\Delta \geq 2$ we have shown in Lemma 4.7 that the latter integral is strictly positive. Hence $Im[M(f)(s)\zeta(s)] > 0$ for $v = \frac{2\pi}{\ln\Delta}$ for all $\Delta \geq 2$, i.e., $0 < v < \frac{2\pi}{\ln 2}$. □

Corollary 4.10. *There are no zeros of the ζ-function of the form $s = u + iv$, $\forall u \in (0, 1)$, $0 \le |v| < \frac{2\pi}{\ln 2}$.*

Proof. For $k \in \mathbb{N}$ this is immediate from Corollary 4.9, since $M(f)(s)$ is finite for such s. For $v = 0$ this is well known (see, eg. [Ti]). For $v < 0$ this follows from the already proven absence of zeros for $\frac{2\pi}{\ln 2} v > 0$, using the well known symmetry

$$\zeta(\bar{s}) = \overline{\zeta(s)},$$

for the latter see, eg. [Ti]. □

Remark 4.11. *With the same method as in Lemma 4.8 (using a refined partitioning of the interval $[1, \Delta]$) it can be shown that the maximal value v^* of $v = \frac{2\pi}{\ln \Delta}$, $1 < \Delta < +\infty$ such that*

$$\int_1^{\Delta} (x^{-u} + x^{u-1}) f(x) \sin(\frac{2\pi}{\ln \Delta} \ln x) dx \ge 0$$

lies within the interval $[\frac{2\pi}{\ln(1.74)}, \frac{2\pi}{\ln(1.73)}]$ for all $u \in (0, 1)$. This permits to extend the zero free region of Corollary 4.10 to the region $\Gamma_{max} = \{s = u + iv | u \in (0, 1), 0 \le |v| < v^*\}$ for $v^* \in [\frac{2\pi}{\ln(1.74)}, \frac{2\pi}{1.73}]$ (i.e. $v^* = \frac{2\pi}{\ln \Delta^*}$, with Δ^* in the closed interval $[1.73, 1.74]$).

Remark 4.12. *Given $s = u+iv$, $u \in (0, 1)$, $v > 0$, be the considerations of Corollary 3.14 Lemma 4.7 and Remark 4.11 the problem of showing $Im[M(f)\zeta(s)] > 0$ (and thus no zeros of the ζ-function for $s = u + iv$, $u \in (0, 1)$, $v > 0$) is reduced to the problem of showing*

$$\int_1^{\Delta_k} (x^{-u} + x^{u-1}) \Theta(f)(x) \sin(\frac{2\pi k}{\ln \Delta_k} \ln x) dx \ge 0,$$

with $\Delta_k = e^{2\pi k/v} \ge 2$, for some $k \in \mathbb{N}$. Due to Proposition 3.15 for this it is enough to show

$$\int_1^{\Delta_k} (x^{-u} + x^{u-1}) f(x) \sin(\frac{2\pi k}{\ln \Delta_k} \ln x) dx \ge 0, \quad \text{for some } k \in \mathbb{N}$$

Remark 4.13. *The choice of function f in Section 4 might not be optimal. In fact, if we fix $0 < u < 1$ and define F_u to be a class of functions g, which satisfy the following properties:*
$g \in \mathrm{F}_u \Leftrightarrow$

(a) $g \in C^2(\mathbb{R})$,
(b) $|g(x)| \le \frac{c}{1+|x|^2}$ *as* $|x| \to \infty$,
(c) $g(0) = 0$,
(d) $\mathcal{F}(g) = -g$,
(e) g *even,*
(f) $g(x) \ge 0$ *for all* $x \ge x_c \ge 1$ *for some* $x_c \ge 1$,

(g) $|g'(x)|/g(x) \geq \gamma_u(x)$ *for* $x \geq x_c$, *(with* γ_u *defined in Proposition 3.10)* $xg(x)$ *strictly decreasing for* x *in some open interval of* $[x_c, \infty]$,

then we have that $f \in \mathrm{F}_u$ *for all* $0 < u < 1$. *From Proposition 3.8, reasoning as in Corollary 3.9, we see that the Riemann hypothesis is implied by the existence of elements* g_u *in* F_u, *for which* $x_c = 1$, *for every* $u \in (0,1)$ *such that* $u \neq \frac{1}{2}$. *By our choice of the form of* f *we have* $x_c = 2$. *Better choices of* f*'s are probably possible. Numerically, however, it seems impossible to have* f*'s such that* $x_c = 1$.

5. Extension to a larger class of test functions. Zero free zones for large values of the imaginary part

Lemma 5.1. *Let a function* $\rho(x)$ *be defined as in Proposition 4.1 (and with the same choice of the parameter* $c = 3$*). Let* $\alpha > 0$ *and* $f_\alpha(x)$ *be defined as follows:*

$$f_\alpha(x) := \rho(x) - \sqrt{\alpha}\mathcal{F}(\rho)(\alpha x),$$

where $\mathcal{F}$ *denotes the Fourier transform. Then one has that*

$$\mathcal{F}(f_\alpha)(\sqrt{\alpha}x) = -\frac{1}{\sqrt{\alpha}} f_\alpha(\frac{x}{\sqrt{\alpha}})$$

Proof. One has

$$\mathcal{F}(\rho)(\alpha x) = \int \rho(y) e^{-2\pi i \alpha x y} dy \overset{z=\alpha x}{\equiv} \frac{1}{\alpha} \int \rho(\frac{z}{\alpha}) e^{-2\pi i x z} dz = \mathcal{F}(\tilde{\rho})(x)$$

with $\tilde{\rho}(x) \equiv \alpha\rho(\alpha x)$. From this one has that

$$\begin{aligned} \mathcal{F}(f_\alpha)(x) &= \mathcal{F}(\rho)(x) - \sqrt{\alpha}\tilde{\rho}(x) \\ &= -\frac{1}{\sqrt{\alpha}}(\rho(\alpha x) - \sqrt{\alpha}\mathcal{F}(\rho)(x)) = -\frac{1}{\sqrt{\alpha}} f_\alpha(\frac{x}{\alpha}) \end{aligned}$$

With the transformation $x = \sqrt{\alpha} y$ one obtains finally the assertion. □

Lemma 5.2. *For any* $\alpha > 0$, *for any* f *as in Proposition 2.7 we have*

$$\alpha^{-s} M(f)(s)\zeta(s) = M(\Theta^{(\alpha)}(f))(s)$$

with $M(f)(s) \equiv \int_0^{+\infty} x^{s-1} f(x) dx$ *and* $\Theta^{(\alpha)}(f)(x) \equiv \sum_{n\in\mathbb{N}} f(n\alpha x)$.

Proof. We recall Müntz' formula (from Proposition 2.7):

$$M(f)(s)\zeta(s) = M(\Theta(f))(s)$$

with $\Theta(f)(x) \equiv \sum_{n\in\mathbb{N}} f(nx)$. On the other hand

$$\begin{aligned}
M(\Theta(f))(s) &= \int_0^{+\infty} x^{s-1}\Theta(f)(x)dx \\
&= \int_0^{+\infty} x^{s-1}\sum_{n\in\mathbb{N}} f(nx)dx \\
&\underset{y=x/\alpha}{=} \alpha\alpha^{s-1}\int_0^{+\infty} y^{s-1}\sum_{n\in\mathbb{N}} f(n\alpha y)dy \\
&= \alpha^s M(\Theta^{(\alpha)}(f))(s)
\end{aligned}$$

$\square$

Lemma 5.3. *Let f be even, $f \in \mathcal{S}(\mathbb{R})$, $f(0) = 0$ and $\alpha > 0$. Then one has*

$$\alpha^{-s}(M(f)\zeta)(s) = \int_1^{\infty}(x^{s-1}\Theta(f)(\alpha x) + \frac{1}{\alpha}x^{-s}\Theta(\mathcal{F}(f))(\frac{x}{\alpha}))dx.$$

Proof. Using the definition of $\Theta^{(\alpha)}(f)$ one has

$$\begin{aligned}
M(\Theta^{(\alpha)}(f))(s) &= \int_1^{\infty} x^{s-1}\Theta^{(\alpha)}(f)dx + \frac{1}{\alpha}\int_1^{\infty} x^{-s}\Theta(\mathcal{F}(f))(\frac{x}{\alpha})dx \\
&\underset{y=\alpha x}{=} \int_1^{\infty} \alpha^{1-s}y^{s-1}\Theta^{(\alpha)}(f)\frac{dy}{\alpha} + \frac{1}{\alpha}\int_1^{\infty} x^{-s}\Theta(\mathcal{F}(f))(\frac{x}{\alpha})dx \\
&= \alpha^{-s}\int_1^{\infty} y^{s-1}\Theta(f)(y)dy + \frac{1}{\alpha}\int_1^{\infty} x^{-s}\Theta(\mathcal{F}(f))(\frac{x}{\alpha})dx \\
&\underset{x=\frac{y}{\alpha}}{=} \int_1^{\infty} x^{s-1}\Theta(f)(\alpha x)dx + \frac{1}{\alpha}\int_1^{\infty} x^{-s}\Theta(\mathcal{F}(f))(\frac{x}{\alpha})dx.
\end{aligned}$$

Together with Lemma 5.2 one obtains the assertion. $\square$

Lemma 5.4. *Let f_α be even, $f_\alpha \in \mathcal{S}(\mathbb{R})$, $f_\alpha(0) = 0$, $\alpha > 0$ be such that $\mathcal{F}(f_\alpha)(\sqrt{\alpha}x) = -\frac{1}{\sqrt{\alpha}}f_\alpha(\frac{x}{\sqrt{\alpha}})$ for all $x \in \mathbb{R}$. Then*

$$(M(f_\alpha)\zeta)(s) = \int_{\frac{1}{\sqrt{\alpha}}}^{\infty}(y^{s-1} - \alpha^{\frac{1}{2}-s}y^{-s})\Theta(f_\alpha)(y)dy.$$

Proof. From above Lemma 5.3 and with the transformation $\alpha \mapsto \frac{1}{\sqrt{\alpha}}$ we have

$$\begin{aligned}(M(f_\alpha)\zeta)(s) &= (\sqrt{\alpha})^{-s}[\int_1^{+\infty} x^{s-1}\Theta(f_\alpha)(\frac{x}{\sqrt{\alpha}})dx + \\ &+\sqrt{\alpha}\int_1^{+\infty} x^{-s}\Theta(\mathcal{F}(f_\alpha))(\sqrt{\alpha}x)]dx = \\ &= (\sqrt{\alpha})^{-s}[\int_1^{+\infty} x^{s-1}\Theta(f_\alpha)(\frac{x}{\sqrt{\alpha}})dx - \\ &-\int_1^{+\infty} x^{-s}\Theta(f_\alpha)(\frac{x}{\sqrt{\alpha}})]dx.\end{aligned}$$

Thus

$$\begin{aligned}(M(f_\alpha)\zeta)(s) &= (\sqrt{\alpha})^{-s}\int_1^{+\infty}(x^{s-1} - x^{-s})\Theta(f_\alpha)(\frac{x}{\sqrt{\alpha}})dx \\ &\underset{y=x/\sqrt{\alpha}}{=} \int_{\frac{1}{\sqrt{\alpha}}}^{\infty}(y^{s-1} - \sqrt{\alpha}\alpha^{-s}y^{-s})\Theta(f_\alpha)(y)dy\end{aligned}$$

□

Lemma 5.5. *Let us assume that f_α, defined as in Lemma 5.4, satisfies the assumption that $xf_\alpha(x)$ is positive and strictly decreasing on $[x''_c, +\infty)$, with an $x''_c \le 2$. Suppose moreover $\int_{\frac{1}{\sqrt{\alpha}}}^{y} f_\alpha(x)\sin(v\ln x)dx \ge 0$ for all $0 < u < 1$, for $v = \frac{2\pi k}{\ln\alpha}$, some $k \in \mathbb{N}$, $y > \frac{1}{\alpha}$. Then $\int_{\frac{1}{\sqrt{\alpha}}}^{y} \Theta(f_\alpha)(x)\sin(v\ln x)dx \ge 0$ for all $0 < u < 1$.*

Proof. This follows from Proposition 3.15. □

Corollary 5.6. *In order to discuss the positivity of $Im((M(f_\alpha)\zeta)(s))$ it is sufficient to consider the first term of the sum $\Theta(f_\alpha)(y)$. One has then, with $s = u + iv$, to consider the following integral*

$$\begin{aligned}I_\alpha^{im} &:= \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} f_\alpha(y)\sin(v\ln y)(y^{u-1} + \alpha^{\frac{1}{2}-u}y^{-u}\cos(v\ln\alpha))dy + \\ &+\int_{\frac{1}{\sqrt{\alpha}}}^{\infty} f_\alpha(y)\cos(v\ln y)\alpha^{\frac{1}{2}-u}y^{-u}\sin(v\ln\alpha)dy\end{aligned}$$

With a particular restriction to the values of the imaginary part v, $v = \frac{2\pi k}{\ln \alpha}$, $k \in \mathbb{N}$, $\alpha > 0$, $\alpha \neq 1$, one obtains

$$I_\alpha^{im} = \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} f_\alpha(y) \sin(\frac{2\pi k}{\ln \alpha} \ln y)(y^{u-1} + \alpha^{\frac{1}{2}-u} y^{-u}) dy.$$

Proof. For the first term of the sum $\Theta(f_\alpha)(y)$ one has

$$I_\alpha := \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} (y^{s-1} - \alpha^{\frac{1}{2}-s} y^{-s}) f_\alpha(y) dy.$$

With $s = u + iv$ one has
$Im(y^{s-1}) = y^{u-1} \sin(v \ln y)$,
$Im(\alpha^{\frac{1}{2}-s} y{-}s) = -\alpha^{-\frac{1}{2}-u} y^{-u} \sin(v \ln y + v \ln \alpha)$.
From this one has

$$\begin{aligned} I_\alpha^{im} &= \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \Big(y^{u-1} \sin(v \ln y) + \alpha^{\frac{1}{2}-u} y^{-u} \sin(v \ln y + v \ln \alpha) \Big) f(y) dy \\ &= \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \Big(y^{u-1} \sin(v \ln y) + \alpha^{\frac{1}{2}-u} y^{-u} (\sin(v \ln y) \cos(v \ln \alpha) + \cos(v \ln y) \sin(v \ln \alpha) \Big) f(y) dy \\ &= \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} \Big(y^{u-1} + \alpha^{\frac{1}{2}-u} y^{-u} \cos(v \ln \alpha) \Big) f(y) \sin(v \ln y) dy \\ &\quad + \int_{\frac{1}{\sqrt{\alpha}}}^{\infty} y^{-u} \alpha^{\frac{1}{2}-u} \sin(v \ln \alpha) f(y) \cos(v \ln y) dy \end{aligned}$$

With the choice $v = \frac{2\pi k}{\ln \alpha}$, $k \in \mathbb{N}$, $\alpha \neq 0$, one obtains the assertion. □

Lemma 5.7. *Let the function f_α be defined as in the Lemma 5.1 with the concrete form*

$$f_\alpha(x) = (ax^2 + 1) e^{-\frac{1}{2} c^2 x^2} + \sqrt{\alpha} (b \alpha^2 x^2 - \frac{1}{\sqrt{\alpha}}) e^{-\frac{2\pi^2}{c^2} \alpha^2 x^2}$$

with $a = c^2 \left(\frac{c}{\sqrt{2\pi\alpha}} - 1 \right)$, $b = \frac{(2\pi)^{\frac{5}{2}}}{c^5} a$. Let $\alpha = 1.1$. Then one has the following properties for f_α:

(a) $f_\alpha(x) > 0$ *for all* $x > 1.5$,
(b) $(x f_\alpha(x))' < 0$ *for all* $x > 1.5$.

Proof. (a) Similar to the computation in Lemma 4.3 one obtains the following inequality as a necessary condition for f_α to have zeroes:

$$\begin{aligned} \frac{1}{\sqrt{\alpha}} - b\alpha^2 x^2 &> 0 \\ \Leftrightarrow x &< \frac{1}{\sqrt{\alpha^{5/2} b}} \approx 1.23 \end{aligned}$$

From this f_α does not have zeros for $x \in [\frac{1}{\sqrt{\alpha^{5/2} b}}, +\infty)$. Together with the fact that $f_\alpha(2) > 0$ one has that $f_\alpha(x) > 0$ in particular for all $x > 1.5$.

(b) For the first derivative of f_α one obtains

$$f'_\alpha(x) = x\left(\underbrace{(2a - c^2) - ac^2x^2}_{<0 \forall x>0}\right) e^{-\frac{1}{2}c^2x^2} + x\left(\alpha^{\frac{5}{2}}\left(2b + \frac{(2\pi)^2}{c^2\sqrt{\alpha}}\right) - \alpha^{\frac{9}{2}} b \left(\frac{2\pi}{c}\right)^2 x^2\right) e^{-\frac{2\pi^2}{c^2}\alpha^2 x^2}$$

The necessary condition for the existence of zeroes is then

$$\begin{aligned} \alpha^2 b \left(\frac{2\pi}{c}\right)^2 x^2 &< 2b + \left(\frac{2\pi}{c}\right)^2 \frac{1}{\sqrt{\alpha}} \\ \Leftrightarrow x &< \frac{\sqrt{2b + \left(\frac{2\pi}{c}\right)^2 \frac{1}{\sqrt{\alpha}}}}{\alpha\sqrt{b}\frac{2\pi}{c}} \approx 1.3 \end{aligned}$$

Together with the fact that $f'_\alpha(2) < 0$ one has that in particular $f'_\alpha(x) < 0$ for all $x > 1.5$.

By a similar computation one obtains that $(xf'_\alpha(x)) < 0$ for all $x > 1.5$. □

Lemma 5.8. *Let $\Delta > 1.5$. Then one has, together with Proposition 3.1 and Corollary 3.2*

$$\int_{\Delta}^{\infty} f_\alpha(y) \sin(\frac{2\pi k}{\ln \alpha} \ln y)(y^{u-1} + \alpha^{\frac{1}{2}-u} y^{-u}) dy > 0$$

for all $\Delta = \alpha^n$, $n \in \mathbb{N}$, such that $\alpha^n > 1.5$.

Proof. Because of the properties of f_α in Lemma 5.7 and the choice $\Delta = \alpha^n$, the assumptions of Proposition 3.1 and Corollary 3.2 are fullfilled. Then the assertion of the Lemma 5.8 follows immediately. □

The preceding Lemma 5.8 implies that it remains to show that

$$I_\Delta(v) \equiv \int_{\frac{1}{\sqrt{\alpha}}}^{\Delta} f_\alpha(y) \sin(\frac{2\pi k}{\ln \alpha} \ln y)(y^{u-1} + \alpha^{\frac{1}{2}-u} y^{-u}) dy > 0 \tag{5.1}$$

for some $\Delta = \alpha^n$, $n \in \mathbb{N}$, $\alpha^n > 1.5$ and some $k \in \mathbb{N}$.

Lemma 5.9. *Consider for f such that $f^{(j)} \in L^1[a,b]$ for all $j \in \mathbb{N}_0$, $a < b$, $a, b \in \mathbb{R}$, for some $n \in \mathbb{N}_0$, the following integral:*

$$A_q \equiv \int_a^b f(x)\sin(qx)dx.$$

Then the following asymptotic expansion holds as $q \to \infty$:

$$A_q = \sum_{j=0}^{N}(-1)^{j+1}\frac{1}{q^{j+1}}\left[f^{(j)}(b)\chi_j(qb) - f^{(j)}(a)\chi_j(qa)\right] + o_N(\frac{1}{q^{N+2}})$$

with $q^{N+2}o_N(\frac{1}{q^{N+2}}) \overset{q\to\infty}{\longrightarrow} 0$ and $\chi_j = cos$ if j is odd, $\chi_j = sin$ if j is even, $f^{(j)}$ is the j-th derivative of f.

Proof. The Lemma 5.9 is easily obtained by integration by parts.

$$A_q = \int_a^b f(x)\sin(qx)dx = -\frac{1}{q}[f(b)\cos(qb) - f(a)\cos(qa)] + \frac{1}{q}\int_a^b f^{(1)}(x)\cos(qx)dx$$

By Riemann-Lebesgue Lemma one has

$$\int_a^b f^{(1)}(x)\cos(qx)dx \overset{q\to\infty}{\longrightarrow} 0$$

as $f^{(1)} \in L^1([a,b])$ by assumption. Hence $A_q = -\frac{1}{q}[f(b)\cos(qb) - f(a)\sin(qa)] + o(q)$ and similarly for iterations of the procedure. □

Remark 5.10. *In the case of the integral 5.1 Lemma 5.9 implies the following relation, where* $v = \frac{4\pi k}{\ln \alpha}$ *and* $x = \ln y$*:*

$$\begin{aligned}
I_\Delta(v) &= \int\limits_{\ln(\frac{1}{\sqrt{\alpha}})}^{\ln \Delta} e^x f_\alpha(e^x) \sin(\frac{2\pi k}{\ln \alpha} x)(e^{x(u-1)} + \alpha^{\frac{1}{2}-u} e^{-ux}) dx \\
&= \int\limits_{\ln(\frac{1}{\sqrt{\alpha}})}^{\ln \Delta} e^x f_\alpha(e^x) \sin(\frac{2\pi k}{\ln \alpha} x)(e^{x(u-1)} + \alpha^{\frac{1}{2}-u} e^{-ux}) dx \\
&= \int\limits_{\ln(\frac{1}{\sqrt{\alpha}})}^{\ln \Delta} e^x f_\alpha(e^x) \sin(vx)(e^{x(u-1)} + \alpha^{\frac{1}{2}-u} e^{-ux}) dx \\
&=: \int\limits_{\ln(\frac{1}{\sqrt{\alpha}})}^{\ln \Delta} \Psi_\alpha(x) \sin(vx) dx \\
&= -\frac{1}{v}\left(\Psi_\alpha(\ln \Delta)\cos(v \ln \Delta) - \Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}}))\cos(\frac{1}{2} v \ln \alpha)\right) + \frac{o(1)}{v} \\
&= -\frac{1}{v}\left(\Psi_\alpha(\ln \Delta)\cos(v \ln \Delta) - \Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}}))\right) + \frac{o(1)}{v} \\
&\underset{\Delta=\alpha^n}{=} -\frac{1}{v}\left(\Psi_\alpha(\ln \Delta)\cos(4\pi k n) - \Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}}))\right) + \frac{o(1)}{v} \\
&= -\frac{1}{v}\left(\Psi_\alpha(\ln \Delta) - \Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}}))\right) + \frac{o(1)}{v}.
\end{aligned}$$

From the properties of the function f_α *one has that*

(a) $\Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}})) < 0$ *for the particular choice of* $\alpha = 1.1$

(b) $\Psi_\alpha(x) \underset{x\to\infty}{\longrightarrow} 0$ *for all* α.

Thus, there exists an $n \in \mathbb{N}$ *such that* $\Delta = \alpha^n$ *and* $\Psi_\alpha(\ln \Delta) < |\Psi_\alpha(\ln(\frac{1}{\sqrt{\alpha}}))|$*. Hence there exists a sequence* (v_k)*,* $v = \frac{2\pi k}{\ln \alpha}$*,* $k \in \mathbb{N}$ *such that* $\lim\limits_{k\to\infty} I_\Delta(v_k) > 0$.

We formulate the main result of this section as follows:

Theorem 5.11. *There is a subsequence* Σ *of integers* $k \to \infty$ *such that there are no zeroes of the Riemann zeta function in the region* $R = R_0 \bigcup\limits_{k\in\Sigma} A_k$*, where* $R_0 \equiv \{0 < u < 1 | 0 \le v \le v^*\}$*,* $A_k = \{0 < u < 1 | v = \frac{2\pi k}{\ln \alpha}, \alpha = 1.1\}$*,* $k \in \Sigma$ *and* v^* *is defined in Remark 4.11.*

Proof. Theorem 5.11 follows immediately from Remark 5.10 and the results of Section 4. □

Remark 5.12. *The way the subsequence should be chosen is described in Remark 5.10.*

6. Comparison with results obtained by other methods and conclusions

We have presented a new analytic method to derive zero free regions for Riemann's Zeta function. It is based on Müntz' formula which involves a test function. By a suitable choice of the test function we have obtained in section 4 for $v = \mathrm{Im} s$ a zero free region for $\zeta(s)$ of the form

$$\Gamma_0 \equiv \{s = u + iv | u \in (0, 1), 0 < v < \frac{2\pi}{\ln 2}\},$$

(Corollary 4.10, resp. Γ_{max}, with Γ_{max} as in Remark 4.11).
This analytic result is already better than other known analytic results, like the one discussed in [Ja] (which yields the region $\Gamma_1 = \{s = u+iv | u \in [\frac{3}{4}, 1], 0 < v < \frac{5}{2}\}$ and the one discussed in [LaI] (which yields the region $\Gamma_2 = \{s = u + iv || s - 1| \leq \sqrt{2}\}$).
It is possible that the method presented in Section 4 can be refined to yield better results in several directions. In particular we pointed out that a better choice of f, keeping however the properties of the f's considered Section 3, i.e. $f \in \mathcal{F}$ with $\mathcal{F}$ as in Remark 4.13 might permit to push the region of excluded positive v into a considerable part of the region $v \geq \frac{2\pi}{\ln 1.73} \cong 11.46$ (we recall that the "lowest zero" of ζ on $u = \frac{1}{2}$ is for $v \cong 14.13472$, [Ti]), respectively to include additional zero-free regions of the form $(\frac{2\pi k}{\ln \Delta_k} - \epsilon, \frac{2\pi k}{\ln \Delta_k} + \epsilon)$ for $k \in \mathbb{N}$, $\epsilon > 0$, and a certain range of $\Delta_k \geq 2$.
In Section 5 we extended our method in another direction, obtaining an upper unbounded zero free region on the form $R = R_0 \bigcup_{k \in \Sigma} A_k$, $R_0 \equiv \{0 < u < 1 | 0 \leq v \leq v^*\}$. $(A_k)_{k \in \Sigma}$ is a sequence of segments $\{0 < u < 1, v = v_k\}$, with $v_k \nearrow \infty$ as $k \in \Sigma$, $k \nearrow \infty$.
Further improvements would come from being able to differentiate between the case $u = \frac{1}{2}$ and the case $u \neq \frac{1}{2}$, establishing zero free regions involving continuous ranges of v depending on u.
In fact, computational methods (the latest results involving supercomputers) yield a zero free region for $v > 0$ of the form $\Gamma_3 = \{s = u + iv | 0 < v < 545439823.215\}$ [VadeLu] (hence much larger than the parts Γ_0, R_0 of the regions found by our analytic methods, as they stand; however our regions $\bigcup_{k \in \Sigma} A_k$ being upper unbounded are clearly outside the range of such computational methods).
Completely different methods are the ones, going back to de la Vallée Poussin and Hadamard, and/or exploiting methods of exponential sums leading to Vinogradov's type zero free regions, e.g. $\Gamma_4 = \{s = u + iv | v > 0, u \geq 1 - 0.00101(\ln v)^{-\frac{2}{3}}\}$ [Che].
These methods involve estimates for a continuum of large values of v, however depending on u, which in our approach could only be reached by a choice of test function $f = f_u$ with a point $x_c^{(u)}$ (in the sense of, e.g. Corollary 3.9) approaching 1, for $u \neq \frac{1}{2}$. Whether this is possible and in general whether it is possible by our methods to obtain results for zero free regions involving a continuum of large values of v remains to be seen.

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Verzeichnis der erschienenen Preprints ab No. 220

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